



The ring of modular forms of degree two in characteristic three

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Abstract

We determine the structure of the ring of Siegel modular forms of degree 2 in characteristic 3.

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1 Introduction

Let \mathcal{A}_g be the moduli space of principally polarized abelian varieties of dimension g . It is a Deligne-Mumford stack over \mathbb{Z} . It carries a natural vector bundle of rank g , the Hodge bundle \mathbb{E}_g . We write L for its determinant line bundle. The vector bundle \mathbb{E}_g extends in a natural way over any compactification $\tilde{\mathcal{A}}_g$ of Faltings-Chai type and we will denote the extension of \mathbb{E}_g and L again by the same symbols. Sections of $L^{\otimes k}$ over $\tilde{\mathcal{A}}_g$ are called modular forms of weight k . It is known that for $g \geq 2$ any section of L^k over \mathcal{A}_g extends to a section of L^k over $\tilde{\mathcal{A}}_g$, a fact usually referred to as the Koecher principle, see [7, Prop. 1.5, p. 140].

If $\mathbb{F} = \mathbb{Z}$ or \mathbb{Z}_p or a field one has the graded ring

$$\mathcal{R}_g(\mathbb{F}) = \bigoplus_k H^0(\tilde{\mathcal{A}}_g \otimes \mathbb{F}, L^k).$$

It is known by [7] that it is a finitely generated \mathbb{F} -algebra.

In the case of $\mathbb{F} = \mathbb{C}$ the ring $\mathcal{R}_g(\mathbb{C})$ is the ring of scalar-valued Siegel modular forms of degree g . It is well-known that $\mathcal{R}_1(\mathbb{C}) = \mathbb{C}[E_4, E_6]$ is freely generated over \mathbb{C} by the Eisenstein series E_4 and E_6 of weights 4 and 6. In the 1960s Igusa [11] determined the structure of $\mathcal{R}_2(\mathbb{C})$:

$$\mathcal{R}_2(\mathbb{C}) = \mathbb{C}[\psi_4, \psi_6, \chi_{10}, \chi_{12}, \chi_{35}] / (\chi_{35}^2 - P),$$

where the indices of the generators indicate the weights and P is a polynomial in $\psi_4, \psi_6, \chi_{10}$ and χ_{12} . Moreover, the ideal of cusp forms is generated by χ_{10}, χ_{12} and χ_{35} . For $g = 3$,

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Tsuyumine showed in [20] that $\mathcal{R}_3(\mathbb{C})$ is generated by 34 elements; recently the number of generators was reduced to 19 by Lercier and Ritzenthaler [14].

For $\mathbb{F} = \mathbb{F}_p$, a finite field with p elements, the ring $R_1(\mathbb{F}_p)$ was described by Deligne [5]. Besides giving the structure of the ring over \mathbb{Z}

$$\mathcal{R}_1(\mathbb{Z}) = \mathbb{Z}[c_4, c_6, \Delta] / (c_4^3 - c_6^2 - 1728 \Delta),$$

he showed that

$$\mathcal{R}_1(\mathbb{F}_2) = \mathbb{F}_2[a_1, \Delta] \quad \text{and} \quad \mathcal{R}_1(\mathbb{F}_3) = \mathbb{F}_3[b_2, \Delta],$$

where Δ is of weight 12 and a_1 (resp b_2) is of weight 1 (resp. 2). For $p \geq 5$ we have $\mathcal{R}_1(\mathbb{F}_p) = \mathbb{F}_p[c_4, c_6]$.

For $g = 2$, Igusa determined in [13] also the ring of modular forms over \mathbb{Z} ; it is generated by elements of weight

$$4, 6, 10, 12, 12, 16, 18, 24, 28, 30, 35, 36, 40, 42, 48.$$

For finite fields the structure of $\mathcal{R}_2(\mathbb{F}_p)$ is known for $p \geq 5$. For this we refer to Ichikawa's paper [10]. For $p \geq 5$ the ring is just as in characteristic zero generated by modular forms $\psi_4, \psi_6, \chi_{10}, \chi_{12}$ and χ_{35} with χ_{35} satisfying a relation $\chi_{35}^2 = P(\psi_4, \psi_6, \chi_{10}, \chi_{12})$. Moreover for $p \geq 5$ the reduction map $\mathcal{R}_2(\mathbb{Z}_p) \rightarrow \mathcal{R}_2(\mathbb{F}_p)$ is surjective. Nagaoka studied the image of the reduction map in [17, 18], see also [1].

In this paper we consider the case $p = 3$ and determine the structure of $\mathcal{R}_2(\mathbb{F}_3)$. We use the close connection between the moduli space \mathcal{A}_2 and the moduli space \mathcal{M}_2 of curves of genus 2 via the Torelli map $\mathcal{M}_2 \hookrightarrow \mathcal{A}_2$ and the description of \mathcal{M}_2 as a quotient stack for the action of $\mathrm{GL}(2)$ on the space of binary sextics. In that way invariant theory can be used to construct modular forms. The relation between invariants and modular forms was already exploited by Igusa in [11], but he used theta functions and Thomae's formula to relate these to cross ratios of the zeros of a binary sextic. Here we use not only invariants but also covariants giving vector-valued modular forms as introduced in [2] to analyze the regularity of scalar-valued modular forms.

Our result is:

Theorem 1.1 *The subring $\mathcal{R}_2^{\mathrm{ev}}(\mathbb{F}_3)$ of modular forms of even weight is generated by forms of weights 2, 10, 12, 14 and 36 and has the form*

$$\mathcal{R}_2^{\mathrm{ev}}(\mathbb{F}_3) = \mathbb{F}_3[\psi_2, \chi_{10}, \psi_{12}, \chi_{14}, \chi_{36}] / J$$

with J the ideal generated by the relation

$$\psi_2^3 \chi_{36} - \chi_{10}^3 \psi_{12} - \psi_2^2 \chi_{10} \chi_{14}^2 + \chi_{14}^3.$$

Moreover, $\mathcal{R}_2(\mathbb{F}_3) = \mathcal{R}_2^{\mathrm{ev}}(\mathbb{F}_3)[\chi_{35}] / (\chi_{35}^2 - P)$ with P a polynomial in $\psi_2, \chi_{10}, \psi_{12}, \chi_{14}$ and χ_{36} . The ideal of cusp forms is generated by $\chi_{10}, \chi_{14}, \chi_{35}, \chi_{36}$.

The generator ψ_2 is the Hasse invariant that vanishes on the locus of non-ordinary abelian surfaces and χ_{10} is a form that vanishes on the locus of products of elliptic curves. The ring of modular forms of degree 2 in characteristic 2 is described in [4].

2 The proof of Theorem 1.1

Since for $g = 2$ the moduli stack $\mathcal{A}_g \otimes \mathbb{F}_3$ has a canonical compactification due to Igusa we will use this compactification $\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_3$. We will denote the space of sections of L^k on

$\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_3$ by $M_k(\Gamma_2)$ and we thus have $\mathcal{R}_2(\mathbb{F}_3) = \bigoplus_k M_k(\Gamma_2)$. We write $M_k(\Gamma_1)$ for the space $H^0(\tilde{\mathcal{A}}_1 \otimes \mathbb{F}_3, L^k)$. The Satake compactification is denoted by $\mathcal{A}_2^* \otimes \mathbb{F}_3$. We denote the first Chern class of L by λ_1 .

We begin by constructing generators of weight 2 and 10. The locus V_1 of abelian surfaces with p -rank ≤ 1 is a divisor in $\mathcal{A}_2 \otimes \mathbb{F}_p$ and its closure \bar{V}_1 in $\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_p$ has cycle class $(p-1)\lambda_1$ in the Chow group with \mathbb{Q} -coefficients, so $[\bar{V}_1] = 2\lambda_1$ for $p = 3$, see [6,22]. Therefore the effective divisor \bar{V}_1 is the divisor of a section of $L^{\otimes 2}$ and there is a modular form ψ_2 of weight 2 whose zero divisor is \bar{V}_1 . It is determined up to multiplication by a non-zero scalar. We will normalize it later. This form is known as the Hasse invariant. Multiplication by ψ_2 implies that $\dim M_k(\Gamma_2) \leq \dim M_{k+2}(\Gamma_2)$.

The divisor of products of elliptic curves $H_1 := \mathcal{A}_{1,1} \otimes \mathbb{F}_3$ gives rise to a second modular form. (The notation refers to the fact that H_1 is the Humbert surface of discriminant 1.) In the Chow group of codimension 1 of $\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_3$ (resp. $\mathcal{A}_2^* \otimes \mathbb{F}_3$) we have the relation (cf. e.g. [16, p. 317])

$$2[\bar{H}_1] + [D] = 10\lambda_1 \quad (\text{resp. } 2[\bar{H}_1] = 10\lambda_1),$$

with D the divisor at infinity, hence there exists a modular form of weight 10 vanishing with multiplicity 2 on H_1 . We call this form χ_{10} (up to a normalization to be determined later). The automorphism group of a generic product of elliptic curves has an extra involution (when compared with the automorphism group of a generic principally polarized abelian surface) and it acts by -1 on L , hence every modular form of even weight vanishes with even multiplicity along H_1 .

Restriction to H_1 yields for even k an exact sequence

$$0 \rightarrow H^0(\mathcal{A}_2 \otimes \mathbb{F}_3, L^k \otimes \mathcal{O}(-2H_1)) \rightarrow H^0(\mathcal{A}_2 \otimes \mathbb{F}_3, L^k) \rightarrow H^0(H_1, L^k_{|H_1})$$

and in view of the degree 2 morphism $\mathcal{A}_1 \times \mathcal{A}_1 \rightarrow \mathcal{A}_{1,1}$ induced by interchanging the two factors, we can identify this with

$$0 \rightarrow M_{k-10}(\Gamma_2) \rightarrow M_k(\Gamma_2) \rightarrow \text{Sym}^2(M_k(\Gamma_1)), \tag{2.1}$$

where the second arrow is multiplication by χ_{10} . Moreover $M_{k-10}(\Gamma_2) = (0)$ for $k < 8$ since L is ample on $\mathcal{A}_2^* \otimes \mathbb{F}_3$. The exact sequence (2.1) and the fact that we know $M_k(\Gamma_1)$ implies that $\dim M_k(\Gamma_2) = 1$ for $k = 2, 4, 6, 8$ and $\dim M_{10}(\Gamma_2) = 2$ and $M_{10}(\Gamma_2)$ is generated by ψ_2^5 and χ_{10} .

We now turn to the construction of the other generators. We use the ideas of [2]. The Torelli map defines an embedding $\mathcal{M}_2 \otimes \mathbb{F}_3 \rightarrow \mathcal{A}_2 \otimes \mathbb{F}_3$. A smooth projective curve of genus 2 can be given by an equation

$$y^2 = f(x) \quad \text{with } f = \sum_{i=0}^6 a_i x^{6-i}. \tag{2.2}$$

We let $V = \langle x_1, x_2 \rangle$ be the \mathbb{F}_3 -vector space generated by x_1, x_2 and write f as a homogeneous polynomial $\sum_{i=0}^6 a_i x_1^{6-i} x_2^i$. Note that a curve as in (2.2) comes with a basis of the space of regular differentials, viz. $dx/y, xdx/y$.

We have a description of $\mathcal{M}_2 \otimes \mathbb{F}_3$ as the stack quotient $[\mathcal{X}^0/\text{GL}(V)]$ with $\mathcal{X}^0 \subset \mathcal{X} = \text{Sym}^6(V) \otimes \det(V)^{-2}$ the locus given by the non-vanishing of the discriminant, see [3, Section 3, p. 3].

The pullback to \mathcal{X}^0 of the Hodge bundle under the composition of $\mathcal{X}^0 \rightarrow \mathcal{M}_2$ with the Torelli map $\mathcal{M}_2 \hookrightarrow \mathcal{A}_2$ is the equivariant bundle V on \mathcal{X}^0 as the basis $dx/y, xdx/y$ of the

space of regular differentials on the curve $y^2 = f(x)$ shows. The pullback of L is $\det(V)$. As a consequence pulling back defines a homomorphism

$$\mu : \mathcal{R}_2(\mathbb{F}_3) \rightarrow I \tag{2.3}$$

with I the ring of invariants of the action of $GL(V)$ on $\text{Sym}^6(V)$. Here an invariant is a polynomial in a_0, \dots, a_6 , the coefficients of f that is invariant under $SL(V)$. Since the image of \mathcal{M}_2 in \mathcal{A}_2 is a Zariski open part with complement H_1 , not every invariant corresponds to a modular form; but every invariant corresponds to a rational modular form that is regular outside H_1 . In particular, it becomes regular on all of \mathcal{A}_2 when multiplied with a sufficiently high power of χ_{10} . This provides us with homomorphisms

$$\mathcal{R}_2(\mathbb{F}_3) \xrightarrow{\mu} I \xrightarrow{\nu} \mathcal{R}_2(\mathbb{F}_3)_{\chi_{10}},$$

where $\mathcal{R}_2(\mathbb{F}_3)_{\chi_{10}}$ is obtained from $\mathcal{R}_2(\mathbb{F}_3)$ by allowing powers of χ_{10} in the denominator. We have $\nu \circ \mu = \text{id}$.

This generalizes as follows to vector-valued modular forms. For each finite dimensional irreducible representation ρ of $GL(2)$ there is a vector bundle \mathbb{E}_2^ρ obtained from \mathbb{E}_2 by applying a Schur functor. Such a ρ is of the form $\text{Sym}^j(\text{St}) \otimes \det^k(\text{St})$ with St the standard representation of $GL(V)$. A section of $\text{Sym}^j(\mathbb{E}_2) \otimes \det(\mathbb{E}_2)^k$ over \mathcal{A}_2 is called a modular form of degree 2 and weight (j, k) . The Koecher principle also applies to these modular forms: sections of \mathbb{E}_2^ρ over \mathcal{A}_2 extend over $\tilde{\mathcal{A}}_2$, see [7, Prop. 1.5, p. 140]. We write

$$M_{j,k}(\Gamma_2) = H^0(\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_3, \text{Sym}^j(\mathbb{E}_2) \otimes \det(\mathbb{E}_2)^k)$$

and we consider the $\mathcal{R}_2(\mathbb{F}_3)$ -module

$$M = \bigoplus_{j,k} M_{j,k}(\Gamma_2).$$

It is even a ring. The map (2.3) can be extended to a map from M to the ring of covariants. Here a covariant can be described as an invariant for the action of $GL(V)$ on $V \oplus \text{Sym}^6(V)$. Alternatively, covariants can be obtained by taking an equivariant embedding of an irreducible $GL(V)$ -representation $U \rightarrow \text{Sym}^d(\text{Sym}^6(V))$, or equivalently, an equivariant map

$$\varphi : \mathbb{F}_3 \rightarrow \text{Sym}^d(\text{Sym}^6(V)) \otimes U^\vee$$

and then $\Phi = \varphi(1)$ is a covariant. If U is an irreducible representation of highest weight (w_1, w_2) then one may view Φ as a homogeneous form in a_0, \dots, a_6 of degree d and in x_1, x_2 of degree $w_1 - w_2$, see [2,9,19]. For example, taking $U = \text{Sym}^6(V)$ and $d = 1$ yields the covariant $\Phi = f$, the universal binary sextic. Covariants form a ring \mathcal{C} that was much studied in the 19th and early 20th century. Grace and Young determined generators of this ring in [9].

The maps $\mathcal{R}_2(\mathbb{F}_3) \rightarrow I \rightarrow \mathcal{R}_2(\mathbb{F}_3)_{\chi_{10}}$ now extend to

$$M \xrightarrow{\mu} \mathcal{C} \xrightarrow{\nu} M_{\chi_{10}},$$

where $M_{\chi_{10}}$ is obtained from M by admitting powers of χ_{10} as denominators. We have $\nu \circ \mu = \text{id}_M$.

The image under ν of the covariant f , the universal binary sextic, is a rational modular form $\chi_{6,-2}$, that is, a rational section of $\text{Sym}^6(\mathbb{E}_2) \otimes \det(\mathbb{E}_2)^{-2}$ that is regular after multiplication by an appropriate power of χ_{10} . The power -2 comes from the twisting used in the description of the stack quotient $[\mathcal{X}^0/GL(V)]$, where $\mathcal{X}^0 \subset \text{Sym}^6(V) \otimes \det(V)^{-2}$, see [3, Section 3, p. 3].

This construction was given in [2] in characteristic zero and yields a meromorphic modular form, here denoted $\varphi_{6,-2}$, that becomes holomorphic after multiplication by χ_{10} . The reduction of the characteristic zero rational modular form $\varphi_{6,-2}$ yields a rational modular form in characteristic 3. This implies that $\chi_{6,-2}$ becomes regular after multiplication by χ_{10} . We can write the form $\chi_{6,-2}$ locally on $\mathcal{A}_2 \otimes \mathbb{F}_3$ symbolically as

$$\chi_{6,-2} = \sum_{i=0}^6 \alpha_i X_1^{6-i} X_2^i, \tag{2.4}$$

where the monomials $X_1^{6-i} X_2^i$ are dummies to indicate the coordinates in the fibres of $\text{Sym}^6(\mathbb{E}_2) \otimes \det(\mathbb{E}_2)^{-2}$. Here we view α_i locally as a rational function on $\mathcal{A}_2 \otimes \mathbb{F}_3$. Using the local expression (2.4) one can give the image $\nu(T)$ of an invariant $T = T(a_0, \dots, a_6)$ locally by $T(\alpha_0, \dots, \alpha_6)$.

We note that interchanging X_1 and X_2 induces an involution replacing α_i by α_{6-i} .

Comparing with the characteristic 0 case and using semi-continuity we see that the orders of the rational functions α_i along the divisor H_1 are at least equal to the orders of their complex analogues along H_1 . The Fourier expansion in characteristic 0 given in [2, page 1658] implies the following inequalities for the orders of α_i along H_1 in characteristic 3:

$$\text{ord}_{H_1}(\alpha_0, \dots, \alpha_6) = (\geq 2, \geq 1, \geq 0, \geq -1, \geq 0, \geq 1, \geq 2). \tag{2.5}$$

Moreover, the symmetry that interchanges x_1 and x_2 implies that the orders of α_i and α_{6-i} along H_1 are equal. Another way to see the estimates for the orders is by developing $\chi_{6,8} = \chi_{6,-2}\chi_{10}$ along the locus $\mathcal{A}_{1,1} \otimes \mathbb{F}_3 \subset \mathcal{A}_2 \otimes \mathbb{F}_3$. Since the pullback of the Hodge bundle \mathbb{E}_2 to $\mathcal{A}_1 \times \mathcal{A}_1$ via $\mathcal{A}_1^2 \rightarrow \mathcal{A}_{1,1} \subset \mathcal{A}_2$ is $\bigoplus_{i=0}^6 p_1^*(\mathbb{E}_1)^i \otimes p_2^*(\mathbb{E}_1)^{6-i}$ the restriction of $\alpha_i \chi_{10}$ lies in $S_{14-i}(\Gamma_1) \otimes S_{8+i}(\Gamma_1)$ and this is zero. The next Taylor term in the Taylor development along $\mathcal{A}_{1,1}$ lies in $S_{15-i}(\Gamma_1) \otimes S_{9+i}(\Gamma_1)$ and this is zero for $i \neq 3$.

The ring of invariants I for the action of $\text{GL}(V)$ on $\text{Sym}^6(V)$ in characteristic 3 is generated by invariants A, B, C, D and E of degree 2, 4, 6, 10 and 15, see e.g. [11] or [8]. The invariants A, B, C, D that we use here can be expressed in the reductions modulo 3 of the invariants J_2, J_4, J_6 et J_{10} given in [15]: $A = -J_2(\text{mod}3)$, $B = -J_4(\text{mod}3)$, $C = -J_6 - A^3(\text{mod}3)$, $D = J_{10}(\text{mod}3)$. The invariant E can be found in [12, p. 848].

The invariant A has the form $A = a_1a_5 - a_2a_4$. We know of the existence of a modular form ψ_2 of weight 2. Under the map μ it must map to a non-zero multiple of A . We fix ψ_2 by requiring $\mu(\psi_2) = A$. The restriction to H_1 of the Hasse invariant ψ_2 is a non-zero multiple of $\text{Sym}^2(b_2)$, with b_2 the Hasse invariant for $g = 1$, hence ψ_2 does not vanish identically on H_1 .

By the inequalities (2.5) and the expression for A we see that $\text{ord}_{H_1}(\alpha_2) = 0 = \text{ord}_{H_1}(\alpha_4)$ and

$$\text{ord}_{H_1}(\alpha_0, \dots, \alpha_6) = (\geq 2, \geq 1, 0, \geq -1, 0, \geq 1, \geq 2).$$

In degree 4 we find another invariant B , not a multiple of A^2 :

$$\begin{aligned} B &= 2 a_0 a_1 a_5 a_6 + a_0 a_2 a_4 a_6 + 2 a_0 a_2 a_5^2 + 2 a_0 a_4^3 + 2 a_1^2 a_4 a_6 + 2 a_1 a_2 a_4 a_5 \\ &\quad + a_1 a_3^2 a_5 + a_1 a_3 a_4^2 + 2 a_2^3 a_6 + a_2^2 a_3 a_5 + a_2^2 a_4^2 + 2 a_2 a_3^2 a_4. \end{aligned}$$

Since we know $\dim M_4(\Gamma_2) = 1$ there cannot be a regular modular form in weight 4 that is not a multiple of ψ_2^2 . This implies that $\text{ord}_{H_1}(\alpha_3) < 0$ and hence $\text{ord}_{H_1}(\alpha_3) = -1$. Thus $B = (a_1a_5 - a_2a_4)a_3^2 + (a_1a_4^2 + a_2^2a_5)a_3 + \dots$ defines a rational modular form $\chi_B = \nu(B)$

of weight 4 with order -2 along H_1 . Since χ_{10} vanishes with multiplicity 2 along H_1 we thus find that

$$\chi_{14} := \chi_B \chi_{10}$$

is a regular modular form of weight 14.

The vector space of invariants of degree 6 is generated by A^3 , AB and an invariant C

$$C = 2a_3^6 + Aa_3^4 + 2(a_1a_4^2 + a_2^2a_5)a_3^3 + \dots$$

and we see that $\chi_C = \nu(C)$ has order -6 along H_1 . In degree 10 there is a new invariant

$$D = (a_1a_5)^3a_3^4 + (a_0a_2^3a_3^3 + a_1^3a_4^3a_6 + 2a_1^3a_4^2a_5^2 + 2a_1^2a_2^2a_5^3)a_3^3 + \dots$$

yielding a modular form that vanishes with multiplicity ≥ 2 on H_1 . Indeed, since $\alpha_1\alpha_5$ vanishes with multiplicity ≥ 2 the first term $(\alpha_1\alpha_5)^3\alpha_3^4$ vanishes with order ≥ 2 ; the next terms also vanish with order ≥ 2 as one easily checks. Therefore χ_D is regular and vanishes with multiplicity ≥ 2 . Since χ_D is not zero, it must be a multiple of χ_{10} and then vanishes on H_1 with multiplicity 2. This implies that the order of vanishing of α_1 and α_5 along H_1 is 1.

Corollary 1 *We have $\text{ord}_{H_1}(\alpha_0, \dots, \alpha_6) = (\geq 2, 1, 0, -1, 0, 1, \geq 2)$.*

We fix χ_{10} by setting it equal to $\chi_D = \nu(D)$. This fixes χ_{14} too.

In a similar manner one checks that the rational modular form $\psi_S = \nu(S)$ with S equal to

$$S = B^3 + A^3C - A^2B^2 = (a_1a_4^2 + a_2^2a_5)^3a_3^3 + \dots$$

is regular too. We put $\psi_{12} = \psi_S$. We thus find a 3-dimensional subspace of $M_{12}(\Gamma_2)$ generated by ψ_2^6 , $\psi_2\chi_{10}$ and ψ_{12} . From the fact that B and D are not divisible by A we see that χ_{14} does not lie in $\psi_2M_{12}(\Gamma_2)$. Therefore $\dim M_{12}(\Gamma_2) < \dim M_{14}(\Gamma_2)$. Since we know by (2.1) that $\dim M_{14}(\Gamma_2) \leq 4$ we conclude that $\dim M_{12}(\Gamma_2) = 3$.

A further generator is

$$\chi_{36} = \nu(CD^3) = \chi_C\chi_{10}^3.$$

Since the orders of χ_C and χ_{10} along H_1 are -6 and 2 the modular form χ_{36} is regular and does not vanish identically on H_1 . The modular form χ_{36} is not contained in the subring generated by $\psi_2, \chi_{10}, \psi_{12}$ and χ_{14} as one sees by looking at the invariants. We have the identity

$$(B^3 + A^3C - A^2B^2)D^3 = B^3D^3 + A^3CD^3 - A^2DB^2D^2$$

by which we can express $\psi_{12}\chi_{10}^3$ in the other generators:

$$\psi_{12}\chi_{10}^3 = \chi_{14}^3 + \psi_2^3\chi_{36} - \psi_2^2\chi_{10}\chi_{14}^2. \tag{2.6}$$

Since A, B, C, D are generators of the ring of invariants and are algebraically independent the forms $\psi_2, \chi_{10}, \psi_{12}, \chi_{14}$ are algebraically independent. The form χ_{36} then satisfies the algebraic relation (2.6) and since there is no non-trivial relation of lower weight involving χ_{36} it implies that this relation generates the ideal of relations between the generators $\psi_2, \chi_{10}, \psi_{12}, \chi_{14}$ and χ_{36} .

The forms $\psi_2, \chi_{10}, \psi_{12}, \chi_{14}$ and χ_{36} generate a subring R^{ev} of the ring $\mathcal{R}_2^{\text{ev}}(\mathbb{F}_3)$ with generating function

$$G = \frac{(1 - t^{42})}{(1 - t^2)(1 - t^{10})(1 - t^{12})(1 - t^{14})(1 - t^{36})}.$$

and by the Riemann-Roch theorem we have $\dim M_k(\Gamma_2) = k^3/1080 + O(k^2)$ for even k . Note that

$$\frac{42}{2 \cdot 10 \cdot 12 \cdot 14 \cdot 36} = \frac{1}{2880}.$$

On the other hand we have $c_1(L)^3 = 1/2880$, see [22, p. 74]. We can use the degree of $\text{Proj}(\mathcal{R}_2^{\text{ev}}(\mathbb{F}_3))$ to show that there cannot be more generators of $\mathcal{R}_2^{\text{ev}}(\mathbb{F}_3)$, but one can see this also in a more elementary way as follows.

Let $d(k) = \dim M_k(\Gamma_2)$ and $r(k) = \dim R_k$ where $R_k = R^{\text{ev}} \cap M_k(\Gamma_2)$.

Proposition 1 *We have $d(k) = r(k)$ for even $k \geq 0$.*

Proof We know that $d(k) \geq r(k)$ for even k and $d(k) = r(k)$ for even $0 \leq k \leq 14$. Suppose by induction that $d(k) = r(k)$ for even $k \leq m$. The exact sequence (2.1) gives the upper bound $d(k) \leq r(k-10) + c(k)(c(k)+1)/2$ for $k \leq m+10$, where $c(k) = \dim M_k(\Gamma_1) = \lfloor k/12 \rfloor + 1$. Using the generating function G one sees that $r(k) - r(k-10) = c(k)(c(k)+1)/2$ for $k \not\equiv 0 \pmod{12}$ and $k \not\equiv 2 \pmod{12}$. Hence $d(k) = r(k)$ for even $k \leq m+10$ with $k \not\equiv 0, 2 \pmod{12}$. But we have

$$d(k+2) - d(k) \geq r(k+2) - r(k),$$

as we show in the next lemma. This proves $d(k) = r(k)$ for even $k \leq m+10$. Therefore we conclude the proof by induction. \square

Lemma 1 *We have $d(k+2) - d(k) \geq r(k+2) - r(k)$ for even $k \geq 0$.*

Proof We can write $R_{k+2} = \psi_2 R_k \oplus N_{k+2}$ with N_{k+2} the subspace with basis the forms $\chi_{10}^a \psi_{12}^b \chi_{14}^c \chi_{36}^d$ with $a, b, c, d \geq 0$ and $c \leq 2$ in view of the relation (2.6). Then we have $\dim N_{k+2} = r(k+2) - r(k)$. The inequality $d(k+2) - d(k) \geq \dim N_{k+2}$ follows from the fact that $N_{k+2} \cap \psi_2 M_k(\Gamma_2) = (0)$. To see this fact, suppose that $f \in M_k(\Gamma_2)$ such that $f \notin R_k$ and $\psi_2 f \in R_{k+2}$. Then $\psi_2 f = P$ with P a sum of monomials $\chi_{10}^a \psi_{12}^b \chi_{14}^c \chi_{36}^d$ with $c \leq 2$. Then $P = v(Q)$ with Q a polynomial in

$$D, B^3 + A^3 C - A^2 B^2, BD, CD^3.$$

Since $P = \psi_2 f$ this polynomial must be divisible by A . But this implies that if $Q \neq 0$ then it must have at least one monomial with $c \geq 3$, but we excluded this. \square

The invariant E of degree 15 is of the form

$$E = (a_1 a_4^2 - a_2^2 a_5)^3 a_3^6 + \dots$$

and $v(E)$ has order -3 along H_1 . Therefore

$$\chi_{35} := v(ED^2)$$

is a regular modular form. It vanishes on H_1 and on the Humbert surface H_4 of discriminant 4, both with multiplicity 1. The surfaces H_1 and H_4 parametrize abelian surfaces that possess an extra involution. Locally near H_4 the extra automorphism corresponds to the symmetry that interchanges x_1 and x_2 .

We know that the cycle class of $2H_4$ on $\mathcal{A}_2^* \otimes \mathbb{F}_3$ is $60\lambda_1$, see [21, Prop. 3.3, p. 217]. Therefore the divisor of χ_{35} is $H_1 + H_4$ and since the closure of H_1 contains the 1-dimensional cusp χ_{35} is a cusp form. Then χ_{35}^2 is of even weight, hence can be expressed as a polynomial

in $\psi_2, \chi_{10}, \psi_{12}, \chi_{14}$ and χ_{36} . If ψ is an odd weight modular form then it must vanish on H_1 and H_4 , hence it will be divisible by χ_{35} .

The relation between the space of binary sextics and the moduli space $\overline{\mathcal{M}}_2$ (see for example [3, Section 4]) implies that a modular form χ is a cusp form if and only if the invariant $\mu(\chi)$ is divisible by the discriminant D in I . From the form of the generators one easily sees that $\chi_{10}, \chi_{14}, \chi_{36}$ and χ_{35} generate the ideal of cusp forms. This completes the proof.

Remark 1 One can use the knowledge of the dimensions of $M_k(\Gamma_2)$ to deduce non-vanishing of $H^1(\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_3, L^k)$ for certain values of k . The short exact sequence of sheaves on $\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_3$

$$0 \rightarrow L^k \otimes \mathcal{O}(-\bar{V}_1) \rightarrow L^k \rightarrow L^k_{|\bar{V}_1} \rightarrow 0$$

gives rise to a long exact sequence which can be identified with

$$0 \rightarrow M_{k-2}(\Gamma_2) \rightarrow M_k(\Gamma_2) \rightarrow H^0(\bar{V}_1, L^k) \rightarrow H^1(\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_3, L^{k-2}) \rightarrow \dots$$

For example, if $\dim M_{k-2}(\Gamma_2) = \dim M_k(\Gamma_2)$ we get an injection $H^0(\bar{V}_1, L^k) \rightarrow H^1(\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_3, L^{k-2})$ and if $k \equiv 0 \pmod{4}$ and $k \geq 0$ one can show that $H^0(\bar{V}_1, L^k) \neq 0$ by showing that $H^0(\bar{V}_1[2], L^k)^{\mathfrak{S}_6} \neq (0)$, the space of invariants under the symmetric group \mathfrak{S}_6 acting on $H^0(\bar{V}_1[2], L^k)$ with $V_1[2]$ the 3-rank ≤ 1 locus in the level 2 moduli space $\tilde{\mathcal{A}}_2[2]$. Thus for example, $H^1(\tilde{\mathcal{A}}_2 \otimes \mathbb{F}_3, L^{14}) \neq (0)$.

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