

DIVISORS ON HURWITZ SPACES: AN APPENDIX TO ‘THE CYCLE CLASSES OF DIVISORIAL MARONI LOCI’

GERARD VAN DER GEER AND ALEXIS KOUVIDAKIS

ABSTRACT. The Maroni stratification on the Hurwitz space of degree d covers of genus g has a stratum that is a divisor only if $d - 1$ divides g . Here we construct a stratification on the Hurwitz space that is analogous to the Maroni stratification, but has a divisor for all pairs (d, g) with $d \leq g$ with a few exceptions and we calculate the divisor class of an extension of these divisors to the compactified Hurwitz space.

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1. INTRODUCTION

The Hurwitz space $\mathcal{H}_{d,g}$ of simply-branched covers of genus g and degree d carries a stratification named after Maroni [7], which is defined as follows. If $\gamma: C \rightarrow \mathbb{P}^1$ is a simply-branched cover one takes the dual of the cokernel of the natural map

$$\mathcal{O}_{\mathbb{P}^1} \rightarrow \gamma_* \mathcal{O}_C,$$

which is a vector bundle of rank $d - 1$ on the projective line, hence is isomorphic to $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{d-1})$ for some $(d - 1)$ -tuple $\alpha = (a_1, \dots, a_{d-1})$, where we assume that the a_i are non-decreasing. The loci of covers $\gamma: C \rightarrow \mathbb{P}^1$ with fixed α are the strata.

It is known (see [1]) that for general $\gamma: C \rightarrow \mathbb{P}^1$ of genus

$$g = k(d - 1) + s \quad \text{with } 0 \leq s \leq d - 2$$

the tuple α takes the form $(k + 1, \dots, k + 1, k + 2, \dots, k + 2)$ with s entries equal to $k + 2$. Only the case with $s = 0$ yields a Maroni stratum that is a divisor (see [3] and [9, Thm. 1.15]).

In this paper we show how to define for the case that the genus g is not divisible by $d - 1$ a stratification that has a stratum that is a divisor for $g \geq d$ under exclusion of a few cases. If $d - 1$ divides g then this reduces to the stratification of Maroni loci. It uses instead of the cokernel of $\mathcal{O}_{\mathbb{P}^1} \rightarrow \gamma_* \mathcal{O}_C$ the cokernel of a natural map

$$\mathcal{O}_{\mathbb{P}^1} \rightarrow \gamma_* \mathcal{O}_C(D),$$

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where D is an appropriately chosen divisor of degree s with support in the ramification locus of γ . The cycle classes of an extension of these divisors to the compactified Hurwitz space $\overline{\mathcal{H}}_{d,g}$ can be calculated by using a global-to-local evaluation map $p^*p_*V \rightarrow V$ of a vector bundle V on an extension of the \mathbb{P}^1 -fibration $p: \mathbb{P} \rightarrow \mathcal{H}_{d,g}$ to the compactified Hurwitz space that is trivial on the generic fibre of p . The calculation and the answer are completely analogous to case of the cycle classes of the Maroni divisors calculated in [4]. The cycle classes are given in terms of an explicit sum of boundary divisors.

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2. THE SETTING

We recall the setting from [4]. We denote by $\overline{\mathcal{H}}_{d,g}$ the compactified Hurwitz space of admissible covers of degree d and genus g . We have

$$\overline{\mathcal{H}}_{d,g} - \mathcal{H}_{d,g} = \bigcup_{j,\mu} S_{j,\mu},$$

where the $S_{j,\mu} = S_{b-j,\mu}$ are divisors indexed by $2 \leq j \leq b - 2$ and a partition $\mu = (m_1, \dots, m_n)$ of d . These divisors can be reducible, but a generic point corresponds to an admissible cover $\gamma: C \rightarrow P$, where P is a genus 0 curve consisting of two components P_1, P_2 of genus 0 intersecting in one point Q with $j_1 = j$ or $b - j$ branch points on P_1 (resp. $j_2 = b - j$ or j branch points on P_2) and the inverse image $\gamma^{-1}(Q)$ consists of n points Q_1, \dots, Q_n on C with ramification indices m_1, \dots, m_n . Since $\overline{\mathcal{H}}_{d,g}$ is not normal we normalize it and this results in a smooth stack $\tilde{\mathcal{H}}_{d,g}$. We then have a diagram

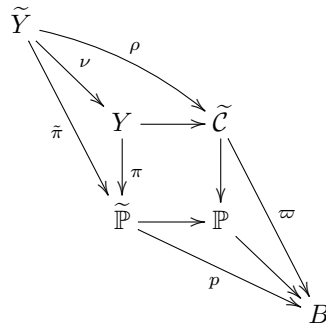
$$\begin{array}{ccc} \tilde{\mathcal{C}} & \xrightarrow{c} & \overline{\mathcal{M}}_{0,b+1} \\ \varpi \downarrow & & \downarrow \pi_{b+1} \\ \tilde{\mathcal{H}}_{d,g} & \xrightarrow{h} & \overline{\mathcal{M}}_{0,b}, \end{array}$$

where $\tilde{\mathcal{C}}$ is the universal curve and $\overline{\mathcal{M}}_{0,b}$ is the moduli space of stable b -pointed curves of genus 0 and π_{b+1} is the map that forgets the $(b + 1)$ st point. With \mathbb{P} the fibre product of $\overline{\mathcal{M}}_{0,b+1}$ and $\tilde{\mathcal{H}}_{d,g}$ over $\overline{\mathcal{M}}_{0,b}$ we have a diagram

$$\begin{array}{ccccc} \tilde{\mathcal{C}} & \xrightarrow{\alpha} & \mathbb{P} & \xrightarrow{c'} & \overline{\mathcal{M}}_{0,b+1} \\ & \searrow \varpi & \downarrow \varpi' & & \downarrow \pi_{b+1} \\ & & \tilde{\mathcal{H}}_{d,g} & \xrightarrow{h} & \overline{\mathcal{M}}_{0,b}. \end{array}$$

We now work over a base B (it can be $\mathcal{H}_{d,g}, \tilde{\mathcal{H}}_{d,g}$ or often a 1-dimensional base). But we shall suppress the index B . Note that normalization commutes with base change. In [4, Lemma 4.1] we showed that $\tilde{\mathcal{C}}$ and \mathbb{P} have only singularities of type A_k . We resolve the singularities of \mathbb{P} obtaining a space $\tilde{\mathbb{P}}$ and then let Y be the normalization of $\tilde{\mathcal{C}} \times_{\mathbb{P}} \tilde{\mathbb{P}}$ and let \tilde{Y} be the resolution of singularities of Y . We then find the basic diagram as in [4]:

Diagram 2.1.



We observe that the finite map $\pi: Y \rightarrow \tilde{\mathbb{P}}$ is flat as Y is Cohen–Macaulay and $\tilde{\mathbb{P}}$ is smooth. Actually, Y has rational singularities only.

3. CONSTRUCTING DIVISORS

Let \mathcal{D} be an effective divisor on Y of relative degree s over B , supported on the sections. Then \mathcal{D} intersects the generic fibre C of π in the ramification locus of π on that fibre C . This divisor \mathcal{D} is a Cartier divisor since the sections do not intersect the singular locus and so $\mathcal{O}(\mathcal{D})$ is a line bundle on Y . Therefore it follows that $\pi_*\mathcal{O}(\mathcal{D})$ is a locally free sheaf on $\tilde{\mathbb{P}}$. We denote by $\tilde{\mathcal{D}}$ the proper (and full) transform of \mathcal{D} under the resolution map ν . Since $\nu_*\mathcal{O}(\tilde{\mathcal{D}}) = \nu_*\mathcal{O}_{\tilde{Y}} \otimes \mathcal{O}(\mathcal{D})$, we conclude that $\tilde{\pi}_*\mathcal{O}(\tilde{\mathcal{D}}) = \pi_*\mathcal{O}(\mathcal{D})$. We can use the restriction of $\pi_*(\mathcal{O}(\mathcal{D}))$ to the open part over $\mathcal{H}_{d,g}$ to define a stratification by type of the bundle on \mathbb{P}^1 just as for the Maroni stratification. We are interested in the case we get a divisor.

For a divisor \mathcal{D} we have an inclusion

$$\iota_{\mathcal{D}} : \mathcal{O}_{\tilde{\mathbb{P}}} \rightarrow \tilde{\pi}_*\mathcal{O}(\tilde{\mathcal{D}}).$$

Note that the image $\iota_{\mathcal{D}}(1)$ of the section 1 is a nowhere vanishing section of $\tilde{\pi}_*\mathcal{O}(\tilde{\mathcal{D}})$.

We now introduce the vector bundle that we use to define a stratification.

Definition 3.1. We let $\mathcal{K}_{\mathcal{D}}$ be the cokernel of $\iota_{\mathcal{D}}$. We define $V_{\mathcal{D}} := \mathcal{K}_{\mathcal{D}}^{\vee}$ as the dual $\mathcal{O}_{\tilde{\mathbb{P}}}$ -module. Since $\iota_{\mathcal{D}}(1)$ is a nowhere vanishing section of $\tilde{\pi}_*\mathcal{O}(\tilde{\mathcal{D}})$ the sheaf $\mathcal{K}_{\mathcal{D}}$ is locally free of rank $d - 1$ on $\tilde{\mathbb{P}}$ and therefore so is $V_{\mathcal{D}}$.

We start with a lemma which follows immediately from the Riemann-Roch theorem.

Lemma 3.2. Let $U = \bigoplus_{i=1}^r \mathcal{O}(a_i)$ be a vector bundle of rank r and degree n on \mathbb{P}^1 . Suppose that $-1 \leq a_1 \leq \dots \leq a_r$. Then $h^0(U) = r + n$. Moreover, this is the minimum dimension for the space of sections of a vector bundle of rank r and degree n on \mathbb{P}^1 .

We recall that we write $g = k(d - 1) + s$ with $0 \leq s \leq d - 2$. Given a cover $\gamma: C \rightarrow \mathbb{P}^1$ and an effective divisor D of degree s supported on the ramification divisor of γ , we write

$$\gamma_*\mathcal{O}_C(D) \cong \mathcal{O}_{\mathbb{P}^1}(-a_0) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(-a_{d-1}),$$

with $a_0 \leq a_1 \leq \dots \leq a_{d-1}$. Note that $\sum_{i=1}^{d-1} a_i = (k + 1)(d - 1)$.

We let $\gamma: C \rightarrow \mathbb{P}^1$ represent a general point of $\mathcal{H}_{d,g}$. We want to determine the numbers a_i . We know by results of Ballico [1] that

$$\gamma_* \mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-(k+1))^{\oplus d-1-s} \oplus \mathcal{O}_{\mathbb{P}^1}(-(k+2))^{\oplus s}.$$

Thus we get

$$\gamma_* \mathcal{O} \otimes \mathcal{O}(k) \cong \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus d-1-s} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus s},$$

so that $h^0(kg_d^1) = k + 1$ and $h^1(kg_d^1) = s$.

Proposition 3.3. *For $\gamma: C \rightarrow \mathbb{P}^1$ a general point of $\mathcal{H}_{d,g}$ we can choose an effective divisor D of degree s supported on the ramification locus R of γ satisfying*

$$h^0(D) = 1 \quad \text{and} \quad h^0(kg_d^1 + D) = k + 1.$$

Proof. Note that since for a general γ we have $h^0(kg_d^1) = k + 1$, the first condition is implied by the second. This is because for any effective divisors D_1, D_2 on C we have that $h^0(D_1 + D_2) \geq h^0(D_1) + h^0(D_2) - 1$. Observe that $b - (2g - 2) = 2d$. We consider the linear system $|K_C - kg_d^1|$. For reasons of degree we can choose a ramification point p_1 which is not a base point of $|K_C - kg_d^1|$ and this gives $h^0(K_C - kg_d^1 - p_1) = s - 1$. Then by the same degree argument we can find a ramification point p_2 such that it is not a base point of $|K_C - kg_d^1 - p_1|$ such that $h^0(K_C - kg_d^1 - p_1 - p_2) = s - 2$. Repeating the argument we arrive at a divisor of degree s supported on the ramification locus such that $h^0(K_C - kg_d^1 - D) = 0$, hence by duality $h^1(kg_d^1 + D) = 0$. By Riemann–Roch we have $h^0(kg_d^1 + D) = k + 1$. \square

Now if we choose D as in Proposition 3.3 we have $h^0(D) = 1$ and therefore $a_0 = 0$ and $a_i > 0$ for $i \geq 1$. Moreover $h^0(kg_d^1 + D) = k + 1$ and since

$$\gamma_* \mathcal{O}_C(D) \otimes \mathcal{O}(k) \cong \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(k - a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(k - a_{d-1})$$

this implies that $a_i \geq k + 1$ for all $i = 1, \dots, d - 1$. Since the a_i add up to $(k + 1)(d - 1)$ we conclude that all a_i are equal to $k + 1$.

Conclusion 3.4. *If we choose γ and D as in Proposition 3.3 then the dual of the cokernel of $\mathcal{O}_{\mathbb{P}^1} \rightarrow \gamma_* \mathcal{O}_C(D)$ has type $\mathcal{O}(k + 1)^{\oplus d-1}$.*

We now want to see that our degeneracy locus is non-empty in the open Hurwitz space $\mathcal{H}_{d,g}$. According to Ohbuchi [8] only so-called acceptable $(d - 1)$ -tuples (a_1, \dots, a_{d-1}) can occur as the indices of the dual of the cokernel of $\mathcal{O}_{\mathbb{P}^1} \rightarrow \gamma_* \mathcal{O}_C$; here acceptable is defined as follows, see [9].

Definition 3.5. A non-decreasing $(d - 1)$ -tuple of natural numbers (a_1, \dots, a_{d-1}) with $\sum_{i=1}^{d-1} a_i = b/2$ is said to be acceptable for (d, g) if the a_i satisfy

- (1) $a_1 \geq b/d(d - 1)$;
- (2) $a_{d-1} \leq b/d$;
- (3) $a_{i+1} - a_1 \leq a_1$.

Now we consider the unique acceptable $(d - 1)$ -tuple α with $a_1 = k$ and for which the sum

$$\sum_{i=1}^{d-1} (d - i)a_i$$

is maximal. This means that this α is the most balanced $(d - 1)$ -tuple with $a_1 = k$. In [2] Coppens proved the following existence result, see also [9].

Theorem 3.6. *Let a_1 be an integer satisfying (2) of Definition 3.5. If α is the unique acceptable $(d - 1)$ -tuple (a_1, \dots, a_{d-1}) for which the sum $\sum_{i=1}^{d-1} (d - i)a_i$ is maximal, then the locus in $\mathcal{H}_{d,g}$ of covers $\gamma: C \rightarrow \mathbb{P}^1$ with invariant α is not empty.*

We apply this in the case $a_1 = k$ and $\sum_{i=1}^{d-1} (d - i)a_i = (k + 1)(d - 1) + s$. We find the following maximizing $(d - 1)$ -tuples for the a_i of $V = \bigoplus_{i=1}^{d-1} \mathcal{O}(a_i)$ with $a_1 = k$.

Lemma 3.7. *If $a_1 = k$ we have the following maximizing acceptable $(d - 1)$ -tuples:*

- (1) *if $s \leq d - 4$ the maximizing sequence is $(k, (k + 1)^{d-s-3}, (k + 2)^{s+1})$;*
- (2) *if $s = d - 3$ the maximizing sequence is $(k, (k + 2)^{d-2})$, with $g \neq 2(d - 2)$;*
- (3) *if $s = d - 2$ the maximizing sequence is $(k, (k + 2)^{d-3}, k + 3)$, with $g \neq 2d - 3$ and $(d, g) \neq (3, 5)$.*

We consider the three cases of Lemma 3.7.

Lemma 3.8. *Let $\gamma: C \rightarrow \mathbb{P}^1$ be a general cover of type (1), (2) or (3) of Lemma 3.7. Then there exists a divisor of degree s with support in the ramification locus of γ such that $h^0(kg_d^1) = h^0(kg_d^1 + D)$ and $h^0(D) = 1$.*

Proof. In the first case we consider a general cover $\gamma: C \rightarrow \mathbb{P}^1$ of this type. Then we have by [1]

$$\gamma_*\mathcal{O}_C = \mathcal{O} \oplus \mathcal{O}(-k) \oplus \mathcal{O}(-(k + 1))^{\oplus d-s-3} \oplus \mathcal{O}(-(k + 2))^{\oplus s+1},$$

so that $h^0(kg_d^1) = k + 2$ and accordingly $h^1(kg_d^1) = s + 1$. We follow the argument of Proposition 3.3. The argument for the cases (2) and (3) is similar. \square

For this pair (γ, D) the conditions $h^0(D) = 1, h^0(kg_d^1 + D) = k + 2$ imply that for

$$\gamma_*\mathcal{O}(D) = \bigoplus_{i=0}^{d-1} \mathcal{O}(-a_i)$$

we must have $a_0 = 0, a_1 = k$ and $a_i \geq k + 1$ for $i = 2, \dots, d - 1$. Then the twisted (with $\mathcal{O}(-k - 1)$) dual of the cokernel of $\mathcal{O}_{\mathbb{P}^1} \rightarrow \gamma_*\mathcal{O}_C(D)$ has the form

$$\bigoplus_{i=1}^{d-1} \mathcal{O}(b_i) \quad \text{with } b_1 = -1 \text{ and } b_i \geq 0 \text{ for } i = 2, \dots, d - 1$$

and thus is not balanced with a space of sections of minimum dimension (see Lemma 3.2). The other two cases are dealt with similarly.

Conclusion 3.9. *Let $\gamma: C \rightarrow \mathbb{P}^1$ be a general cover such that the dual of the cokernel of $\mathcal{O}_{\mathbb{P}^1} \rightarrow \gamma_*\mathcal{O}_C$ has type as in Lemma 3.7. Then there exists a divisor D of degree s with support in the ramification divisor of γ such that the dual of the cokernel of $\mathcal{O}_{\mathbb{P}^1} \rightarrow \gamma_*\mathcal{O}_C(D)$ has unbalanced type as above.*

We define now our divisors that are analogous to the Maroni divisors. We consider the pullback of the diagram 2.1 to the open base $B = \mathcal{H}_{d,g}$ and choose on Y_B a reduced divisor \mathcal{D} consisting of any s sections of $\tilde{\pi}|_B$ and we let $V_{\mathcal{D}}$ be the dual

of the cokernel of the natural map $\iota_{\mathcal{D}}|_B: \mathcal{O}_{\tilde{Y}_B} \rightarrow \tilde{\pi}_*\mathcal{O}(\mathcal{D})$ and tensor $V_{\mathcal{D}}$ with a line bundle M on $\tilde{\mathbb{P}}_B$ such that $V'_{\mathcal{D}} = V_{\mathcal{D}} \otimes M$ has zero degree on the generic fibre of π . The sheaf $p_*V'_{\mathcal{D}}$ is reflexive (see [6]) and therefore it is a vector bundle on an open U of $\mathcal{H}_{d,g}$ with complement of codimension ≥ 3 .

Consider an open subset U' of U such that $p_*V'_{\mathcal{D}}$ is a trivial of rank r , that is, isomorphic to $\mathcal{O}_{U'}^r$. Choose r generating sections s_1, \dots, s_r of $p_*V'_{\mathcal{D}}$ on U' . If we consider their pullbacks under p and restrict these for $x \in U'$ to $H^0(p^{-1}(x), V_{\mathcal{D}}|_{p^{-1}(x)})$ then they generate the stalk of $V'_{\mathcal{D}}$ at a point of $p^{-1}(x)$ if and only if $V'_{\mathcal{D}}|_{p^{-1}(x)}$ is the trivial bundle of rank r on $p^{-1}(x) = \mathbb{P}^1$; indeed, $V'_{\mathcal{D}}|_{p^{-1}(x)} = \bigoplus_i \mathcal{O}(a_i)$ with $\sum_i a_i = 0$ and if some a_i are negative, then these sections cannot generate it since these do not see the $\mathcal{O}(a_i)$ with a_i negative; if $V'_{\mathcal{D}}|_{p^{-1}(x)} = \mathcal{O}_{\mathbb{P}^1}^r$ by Grauert's theorem they will generate it (see e.g. [5, III, Cor. 12.9]). Thus we arrive at the following result.

Theorem 3.10. *Suppose that $g = k(d - 1) + s$ with $0 \leq s \leq d - 2$ and $3 \leq d \leq g$, satisfying the conditions that $g \neq 2d - 3$, $g \neq 2d - 4$ and $(d, g) \neq (3, 5)$. Then the vanishing locus of the determinant of the evaluation map $ev : p^*p_*V'_{\mathcal{D}} \rightarrow V'_{\mathcal{D}}$ defines a non-empty divisor $\mathfrak{d}_{\mathcal{D}}$ on $\mathcal{H}_{d,g}$.*

Proof. We take a cover as in Conclusion 3.4. Then there is a point in $\mathcal{H}_{d,g}$ representing this cover and such that the divisor D of Conclusion 3.4 is given by the restriction of the above \mathcal{D} to the corresponding fiber. This is because the points of $\mathcal{H}_{d,g}$ parametrize simply branched coverings with ordered branch points, and hence the ramification points are ordered too. We conclude that $V'_{\mathcal{D}}$ is trivial on the generic fibre. But by Conclusion 3.9 above we see, for similar reasons, that the locus in $\mathcal{H}_{d,g}$, where $V'_{\mathcal{D}}$ restricted to a fibre is non-trivial is not empty. By Lemma 3.2 and Grauert's theorem both the above loci belong to U . Thus the evaluation map on $p^{-1}U$ is a map of vector bundles of the same rank with degeneracy locus which is not the whole space nor empty. Therefore it defines a divisor on $p^{-1}U$ which extends uniquely to a divisor on $\mathcal{H}_{d,g}$. \square

4. CYCLE CLASSES

In this section we indicate how to calculate the classes of closed divisors in $\overline{\mathcal{H}}_{d,g}$ that extend the above defined divisors $\mathfrak{d}_{\mathcal{D}}$. We proceed as in [4] for the Maroni case.

We take the standard line bundle M constructed in [4, Definition 6.2] and we let $V'_{\mathcal{D}} = V_{\mathcal{D}} \otimes M$. We then want to find a divisor $A_{\mathcal{D}}$ supported on the singular fibres of $p: \tilde{Y} \rightarrow B$ such that $c_1(V'_{\mathcal{D}}) - A_{\mathcal{D}}$ is a pullback under p from the base B . In [4, Section 6] we constructed such a divisor $A_{\mathcal{D}}$ in the case where the divisor \mathcal{D} is zero. This is done locally on the base B . We restrict to 1-dimensional bases B and consider the fibre of p over a points x where B intersects the boundary. The fibre there consists of a chain of rational curves. In the present case the degrees of V' , the bundle used in [4] and corresponding to $\mathcal{D} = 0$, and $V'_{\mathcal{D}}$ differ at a chain $R_0 = P_1, R_1, \dots, R_m = P_2$ only at R_0 and R_m . Adapting the result gives the following analogue of [4, Conclusion 6.5].

Proposition 4.1. *For each irreducible component Σ of $S_{j,\mu}$ we define*

$$c_{\Sigma, \mathcal{D}} = d - n - 2(r - \mathcal{D} \cdot P_2).$$

If we let

$$A_{\mathcal{D}}^{\Sigma} = -\frac{1}{2} \sum_{i=0}^{m-1} ((m-i)c_{\Sigma, \mathcal{D}} - \delta_i) R_i^{\Sigma}$$

then the degree of $A_{\mathcal{D}}^{\Sigma}$ on R_i^{Σ} equals the degree of $V'_{\mathcal{D}}$ on R_i^{Σ} .

The formalism described in [4] shows that if Q denotes the degeneracy locus of the evaluation map $\text{ev}: p^*p_*V'_{\mathcal{D}} \rightarrow V'_{\mathcal{D}}$ we get that

$$c_1(Q) + p^*R^1p_*V'_{\mathcal{D}} + (-A_{\mathcal{D}})_{\text{sh}},$$

where the index sh denotes the shift as in [4, Definition 3.7], is an effective class on $\tilde{\mathbb{P}}$ that is a pullback under p , and pulling it back under a section of p gives us an effective class $\bar{\mathfrak{d}}_{\mathcal{D}}$ which extends the divisor $\mathfrak{d}_{\mathcal{D}}$.

To calculate the class $\bar{\mathfrak{d}}_{\mathcal{D}}$ of the degeneracy locus we wish to use the formula of [4, Theorem 3.10]. For this we need first a the following lemma (see [4, Proposition 10.1]).

Lemma 4.2. *Let \mathcal{L} be a line bundle on \tilde{Y} with first Chern class ℓ . If U denotes the ramification divisor of $\tilde{\pi}$ we have*

$$c_1(\tilde{\pi}_*\mathcal{L}) = \tilde{\pi}_*\ell - \frac{1}{2}\tilde{\pi}_*(U),$$

$$c_2(\tilde{\pi}_*\mathcal{L}) - c_2(\pi_*\mathcal{O}_{\tilde{Y}}) = \frac{1}{2}((\tilde{\pi}_*\ell)^2 - \tilde{\pi}_*(\ell^2)) - \frac{1}{2}(\tilde{\pi}_*U \cdot \tilde{\pi}_*\ell - \tilde{\pi}_*(U \cdot \ell)).$$

We denote the sections of p by σ_i ($i = 1, \dots, b$) and their images by Ξ_i ($i = 1, \dots, b$). These are divisors on $\tilde{\mathbb{P}}$. Over the section Ξ_i we have a ramification divisor U_i . If \mathcal{D}_1 is a reduced divisor with support on the sections $\cup_i \Xi_i$ we let \mathcal{D} be the corresponding divisor with support on the ramification divisor of $\tilde{\pi}$. Then we have

$$\tilde{\pi}_*\mathcal{D} = \mathcal{D}_1, \quad \tilde{\pi}^*\mathcal{D}_1 = 2\mathcal{D} + \Gamma_{\mathcal{D}},$$

with $\Gamma_{\mathcal{D}}$ disjoint from \mathcal{D} . Another relation that we will use with $W = \tilde{\pi}_*(U)$ is

$$U \cdot \mathcal{D} = \frac{1}{2}W \cdot \mathcal{D}_1 = \frac{1}{2}\mathcal{D}_1^2.$$

Using Lemma 4.2 we thus find

$$c_2(\tilde{\pi}_*\mathcal{O}_{\tilde{Y}}(\mathcal{D})) = c_2(\tilde{\pi}_*\mathcal{O}_{\tilde{Y}}), \quad c_1^2(\tilde{\pi}_*\mathcal{O}_{\tilde{Y}}(\mathcal{D})) = c_1^2(\tilde{\pi}_*\mathcal{O}_{\tilde{Y}}).$$

Therefore, when we apply [4, Theorem 3.10] in order to calculate the class $\bar{\mathfrak{d}}_{\mathcal{D}}$ the only thing that differs from the case treated there is the choice of the line bundle $A_{\mathcal{D}}$. This affects only the definition of $c_{\Sigma, \mathcal{D}}$ and therefore the class is calculated by the same formula given in [4, Theorem 8.3] with $c_{j,\mu}$ determined by $c_{\Sigma, \mathcal{D}}$.

We can twist the bundle $V'_{\mathcal{D}}$ by a line bundle N corresponding to a divisor on $\tilde{\mathbb{P}}$ with support on the boundary of $\tilde{\mathbb{P}}$. This results in a divisor class $\bar{\mathfrak{d}}_{\mathcal{D}, N}$ on $\tilde{\mathcal{H}}_{d,g}$ extending the divisor $\mathfrak{d}_{\mathcal{D}}$ on $\mathcal{H}_{d,g}$. We thus arrive as in [4, Section 9] at the following theorem.

Theorem 4.3. *Suppose that $3 \leq d \leq g$ and $g \neq 2d - 3$, $g \neq 2d - 4$ and $(d, g) \neq (3, 5)$. Let Σ be an irreducible component of the boundary $S_{j,\mu}$ of $\overline{\mathcal{H}}_{d,g}$. Then the coefficient $\sigma_{j,\mu}$ of Σ of the locus $\cap_N \overline{\mathfrak{d}}_{\mathcal{D},N}$ is equal to*

$$m(\mu) \left(\frac{1}{12} \left(d - \sum_{\nu=1}^{n(\mu)} \frac{1}{m_\nu} \right) + \frac{j(b-j)(d-2)}{8(b-1)(d-1)} \right) - \frac{1}{8(d-1)} \sum_{i=1}^{m(\mu)} (\delta_{i-1} - \delta_i)^2 - \frac{d-1}{2} \left(\frac{m(\mu)}{4} - \sum_{i=1}^{m(\mu)} (e_{i-1} - e_i)^2 \right).$$

For the numbers $m(\mu)$, δ_i and e_i we refer to [4]. Note that the rational numbers e_i that come from rounding off a rational solution to an integral one depend on the constants $c_{j,\mu}$ that occur above.

In a similar way one can obtain analogues of the theorems of Sections 10 and 11 of [4] by adding to \mathcal{D} an effective divisor Z supported on the boundary of \tilde{Y} . One can then define an effective class $\overline{\mathfrak{d}}_{\mathcal{D},N,Z}$ that extends the class of $\mathfrak{d}_{\mathcal{D}}$. One may ask whether

$$\bigcap_{N,Z} \overline{\mathfrak{d}}_{\mathcal{D},N,Z}$$

is the class of the Zariski closure of $\mathfrak{d}_{\mathcal{D}}$ on $\overline{\mathcal{H}}_{d,g}$.

The classes $\mathfrak{d}_{\mathcal{D}}$ depend on \mathcal{D} , but using the monodromy one sees that their images on the Hurwitz space with unordered branch points do not.

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KORTEWEG-DE VRIES INSTITUUT, UNIVERSITEIT VAN AMSTERDAM, POSTBUS 94248, 1090 GE AMSTERDAM, THE NETHERLANDS

E-mail address: G.B.M.vanderGeer@uva.nl

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS, UNIVERSITY OF CRETE, GR-70013 HERAKLION, GREECE

E-mail address: kouvid@uoc.gr