Siegel Modular Forms of Degree Two and Three and Invariant Theory

Gerard van der Geer

Dedicated to Ciro Ciliberto on the occasion of his 70th birthday

Abstract This is a survey based on the construction of Siegel modular forms of degree 2 and 3 using invariant theory in joint work with Fabien Cléry and Carel Faber.

Keywords Modular form · Siegel modular form · Invariant theory · Algebraic curves

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1 Introduction

Modular forms are sections of naturally defined vector bundles on arithmetic quotients of bounded symmetric domains. Often such quotients can be interpreted as moduli spaces and sometimes this moduli interpretation allows a description as a stack quotient under the action of an algebraic group like GL_n. In such cases, classical invariant theory can be used for describing modular forms.

In the 1960s, Igusa used the close connection between the moduli of principally polarized complex abelian surfaces and the moduli of algebraic curves of genus two to describe the ring of scalar-valued Siegel modular forms of degree two (and level 1) in terms of invariants of the action of GL_2 acting on binary sextics, see [20, 21]. Igusa

G. van der Geer (✉)
Korteweg-de Vries Instituut, Universiteit van Amsterdam, Postbus 94248, 1090 GE, Amsterdam, The Netherlands
e-mail: geer@science.uva.nl

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used theta functions and a crucial step in Igusa's approach was provided by Thomae's formulas from the nineteenth century that link theta constants for hyperelliptic curves to the cross ratios of the branch points of the canonical map of the hyperelliptic curve to $\mathbb{P}^1$.

In the 1980s Tsuyumine, continuing the work of Igusa, used the connection between the moduli of abelian threefolds and curves of genus 3 to describe generators for the ring of scalar-valued Siegel modular forms of degree 3 (and level 1). He used the moduli of hyperelliptic curves of genus 3 as an intermediate step and used theta functions and the invariant theory of binary octics as developed by Shioda, see [26, 30].

The description of the moduli of curves of genus 2 (resp. 3) in terms of a stack quotient of GL_2 acting on binary sextics (resp. of GL_3 acting on ternary quartics) makes it possible to construct the modular forms directly from the stack quotient without the recourse to theta functions or cross ratios. This applies not only to scalar-valued modular forms, but to vector-valued modular forms as well. Covariants (or concomitants) yield explicit modular forms in an efficient way. This is in contrast to earlier and more laborious methods of constructing vector-valued Siegel modular forms of degree 2 and 3 that use theta functions.

In joint work with Fabien Cléry and Carel Faber [7–9] we exploited this for the construction of Siegel modular forms of degree 2 and 3. In degree 2 the universal binary sextic, the most basic covariant, defines a meromorphic Siegel modular form $\chi_{6,-2}$ of weight $(6, -2)$. Substituting the coordinates of $\chi_{6,-2}$ in a covariant produces a meromorphic modular form that becomes holomorphic after multiplication by an appropriate power of χ_{10}, a cusp form of weight 10 associated to the discriminant. For degree 3 we can play a similar game, now involving the universal ternary quartic and a meromorphic Teichmüller modular form $\chi_{4,0,-1}$ of weight $(4, 0, -1)$ that becomes a holomorphic Siegel modular form $\chi_{4,0,8}$ of weight $(4, 0, 8)$ after multiplication with χ_9, a Teichmüller form of weight 9 related to the discriminant.

With this approach it is easy to retrieve Igusa's result on the ring of scalar modular forms of degree 2. Another advantage of this direct approach is that one can treat modular forms in positive characteristic as well. Thus it enabled the determination of the rings of scalar-valued modular forms of degree 2 in characteristic 2 and 3, two cases that were unaccounted for so far, see [6, 33].

In this survey we sketch the approach and indicate how one constructs Siegel modular forms of degree 2 and 3. We show how to derive the results on the rings of scalar-valued modular forms of degree 2.

2 Siegel Modular Forms

Classically, Siegel modular forms are described as functions on the Siegel upper half space. We recall the definition.

For $g \in \mathbb{Z}_{\geq 0}$ we set $\mathcal{L} = \mathbb{Z}^{2g}$ with generators $e_1, \ldots, e_g, f_1, \ldots, f_g$ and define a symplectic form $\langle \, , \, \rangle$ on $\mathcal{L}$ via $\langle e_i, f_j \rangle = \delta_{ij}$. The Siegel modular group $\Gamma_g =$

$\mathrm{Aut}(\mathcal{L}, \langle\ ,\ \rangle)$ of degree g is the automorphism group of this symplectic lattice. Here we write an element $\gamma \in \Gamma_g$ as a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of four $g \times g$ blocks using the basis e_i and f_i; we often abbreviate this as $\gamma = (a, b; c, d)$. The group Γ_g acts on the Siegel upper half space

$$\mathfrak{H}_g = \{\tau \in \mathrm{Mat}(g \times g, \mathbb{C}) : \tau^t = \tau,\ \mathrm{Im}(\tau) > 0\}$$

via $\tau \mapsto \gamma(\tau) = (a\tau + b)(c\tau + d)^{-1}$.

A scalar-valued Siegel modular form of weight k and degree $g > 1$ is a holomorphic function $f : \mathfrak{H}_g \to \mathbb{C}$ satisfying $f(\gamma(\tau)) = \det(c\tau + d)^k f(\tau)$ for all $\gamma = (a, b; c, d) \in \Gamma_g$. If $\rho : \mathrm{GL}(g) \to \mathrm{GL}(V)$ is a complex representation of GL_g then a vector-valued Siegel modular form of weight ρ and degree $g > 1$ is a holomorphic map $f : \mathfrak{H}_g \to V$ satisfying

$$f(\gamma(\tau)) = \rho(c\tau + d) f(\tau) \qquad \text{for all}\ \ \gamma = (a, b; c, d) \in \Gamma_g \tag{1}$$

We may restrict to irreducible representations ρ. For $g = 1$ we have to require an additional growth condition for $y = \mathrm{Im}(\tau) \to \infty$.

However, for an algebraic geometer modular forms are sections of vector bundles. Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g. This is a Deligne-Mumford stack of relative dimension $g(g + 1)/2$ over $\mathbb{Z}$. It carries a universal principally polarized abelian variety $\pi : \mathcal{X}_g \to \mathcal{A}_g$. This provides $\mathcal{A}_g$ with a natural vector bundle $\mathbb{E} = \mathbb{E}^{(g)}$, the Hodge bundle, defined as

$$\mathbb{E} = \pi_*(\Omega^1_{\mathcal{X}_g/\mathcal{A}_g})\,.$$

Starting from $\mathbb{E}$ we can create new vector bundles. Each irreducible representation ρ of GL_g defines a vector bundle $\mathbb{E}_\rho$ by applying a Schur functor (or just by applying ρ to the transition functions of $\mathbb{E}$). In particular, we have the determinant line bundle $L = \det(\mathbb{E})$. Scalar-valued modular forms of weight k are sections of $L^{\otimes k}$ and these form a graded ring. In fact, for $g \geq 2$ and each commutative ring F we have the ring

$$R_g(F) = \oplus_k H^0(\mathcal{A}_g \otimes F, L^{\otimes k})\,.$$

The moduli space $\mathcal{A}_g$ can be compactified. There is the Satake compactification, in some sense a minimal compactification, based on the fact that L is an ample line bundle on $\mathcal{A}_g$. This compactification $\mathcal{A}_g^*$ is defined as $\mathrm{Proj}(R_g)$ and satisfies the inductive property

$$\mathcal{A}_g^* = \mathcal{A}_g \sqcup \mathcal{A}_{g-1}^*\,.$$

Restriction to the 'boundary' $\mathcal{A}_{g-1}^*$ induces a map called the Siegel operator

$$\Phi : R_g(F) \to R_{g-1}(F)\,.$$

We will also use (smooth) Faltings-Chai type compactifications $\tilde{\mathcal{A}}_g$ and over these the Hodge bundle extends [14]. We will denote the extension also by $\mathbb{E}$.

For $g > 1$ the Koecher principle holds: sections of $\mathbb{E}_\rho$ over $\mathcal{A}_g$ extend to regular sections of the extension of $\mathbb{E}_\rho$ over $\tilde{\mathcal{A}}_g$, see [14, Prop 1.5, p. 140]. For $g = 1$ this does not hold since the boundary in $\mathcal{A}_1^*$ is a divisor, and we define modular forms of weight k as sections of $L^{\otimes k}$ over $\tilde{\mathcal{A}}_1$. If D denote the divisor added to $\tilde{\mathcal{A}}_g$ to compactify $\mathcal{A}_g$, then elements of $H^0(\tilde{\mathcal{A}}_g, \mathbb{E}_\rho \otimes \mathcal{O}(-D))$ are called cusp forms.

We will write $M_\rho(\Gamma_g)(F)$ for $H^0(\tilde{\mathcal{A}}_g \otimes F, \mathbb{E}_\rho)$ or simply $M_\rho(\Gamma_g)$ when F is clear. The space of cusp forms is denoted by $S_\rho(\Gamma_g)$. By the Koecher principle the spaces $M_\rho(\Gamma_g)(F)$ and $S_\rho(\Gamma_g)(F)$ do not depend on the choice of a Faltings-Chai compactification.

Over the complex numbers if $\rho : \mathrm{GL}(g) \to \mathrm{GL}(V)$ is an irreducible representation, elements of $H^0(\tilde{\mathcal{A}}_g \otimes \mathbb{C}, \mathbb{E}_\rho)$ correspond to holomorphic functions $f : \mathfrak{H}_g \to V$ satisfying (1). Such a function allows a Fourier expansion

$$f(\tau) = \sum_{n \geq 0} a(n)\, q^n ,$$

where the sum is over symmetric $g \times g$ half-integral matrices (meaning $2n$ is integral and even on the diagonal) which are positive semi-definite, $a(n) \in V$ and q^n is shorthand for $e^{2\pi i \mathrm{Tr}(n\tau)}$.

The definition

$$R_g(F) = \oplus_k H^0(\tilde{\mathcal{A}}_g \otimes F, L^{\otimes k})$$

for a commutative ring F allows one speak of modular forms in positive characteristic by taking $F = \mathbb{F}_p$. One cannot define such modular forms by Fourier series.

We summarize what is known about the rings $R_g(F)$. It is a classical result that the ring $R_1(\mathbb{C})$ is freely generated by two Eisenstein series E_4 and E_6 of weights 4 and 6. Deligne determined in [10] the ring $R_1(\mathbb{Z})$ and the rings $R_1(\mathbb{F}_p)$. He showed that

$$R_1(\mathbb{Z}) = \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\, \Delta) ,$$

where Δ is a cusp form of weight 12 and c_4 and c_6 are of weight 4 and 6. Reduction modulo p gives a surjection of $R_1(\mathbb{Z})$ to $R_1(\mathbb{F}_p)$ for $p \geq 5$. Moreover, Deligne showed that $R_1(\mathbb{F}_p)$ in characteristic 2 and 3 is given by

$$R_1(\mathbb{F}_2) = \mathbb{F}_2[a_1, \Delta], \quad R_1(\mathbb{F}_3) = \mathbb{F}_3[b_2, \Delta],$$

where in each case Δ is a cusp form of weight 12 and a_1 (resp. b_2) is a modular form of weight 1 (resp. 2).

In [20] Igusa determined the ring $R_2(\mathbb{C})$. He showed that the subring $R_2^{\mathrm{ev}}(\mathbb{C})$ of even weight modular forms is generated freely by modular forms of weight 4, 6, 10, and 12 and $R_2(\mathbb{C})$ is generated over $R_2^{\mathrm{ev}}(\mathbb{C})$ by a cusp form of weight 35 whose square lies in $R_2^{\mathrm{ev}}(\mathbb{C})$; see also [21]. Later [22] he also determined the ring $R_2(\mathbb{Z})$; it has 15 generators of weights ranging from 4 to 48.

In characteristic $p \geq 5$ the structure of the rings $R_2(\mathbb{F}_p)$ is similar to that of $R_2(\mathbb{C})$, see [2, 24]; these are generated by forms of weight 4, 6, 10, 12, and 35. The structure of $R_2(\mathbb{F}_p)$ for $p = 2$ and 3 was determined recently in [6, 33]. All these cases can be dealt with easily using the approach with invariant theory.

In degree 2 one can provide the R_2-module

$$M = \oplus_{j,k} M_{j,k}(\Gamma_2) \qquad \text{with } M_{j,k}(\Gamma_2) = H^0(\tilde{\mathcal{A}}_2, \mathrm{Sym}^j(\mathbb{E}) \otimes \det(\mathbb{E})^k)$$

with the structure of a ring using the projection of GL_2-representations $\mathrm{Sym}^m(V) \otimes \mathrm{Sym}^n(V) \to \mathrm{Sym}^{m+n}(V)$ with V the standard representation by interpreting $\mathrm{Sym}^j(V)$ as the space of homogeneous polynomials of degree j in two variables, say x_1, x_2 and performing multiplication of polynomials. The ring M is not finitely generated as Grundh showed, see [3, p. 234]. The dimensions of the spaces $S_{j,k}(\Gamma_2)(\mathbb{C})$ are known by Tsushima [29] for $k \geq 4$; for $k = 3$ they were obtained independently by Petersen and Taïbi [25, 28].

For fixed j the $R_2^{\mathrm{ev}}(\mathbb{C})$-modules

$$\oplus_k M_{j,2k}(\Gamma_2)(\mathbb{C}) \quad \text{and} \quad \oplus_k M_{j,2k+1}(\Gamma_2)(\mathbb{C})$$

are finitely generated modules and their structure has been determined in a number of cases by Satoh, Ibukiyama and others, see the references in [8]. Invariant theory makes it easier to obtain such results.

For $g = 3$ the results are less complete. Tsuyumine showed in 1985 [30] that the ring $R_3(\mathbb{C})$ is generated by 34 generators. Recently Lercier and Ritzenthaler showed in [23] that 19 generators suffice.

3 Moduli of Curves of Genus Two as a Stack Quotient

We start with $g = 2$. Let F be a field of characteristic $\neq 2$ and $V = \langle x_1, x_2 \rangle$ the F-vector space with basis x_1, x_2. The algebraic group GL_2 acts on V via $(x_1, x_2) \mapsto (ax_1 + bx_2, cx_1 + dx_2)$ for $(a, b; c, d) \in \mathrm{GL}_2(F)$. We will write $V_{j,k} = \mathrm{Sym}^j(V) \otimes \det(V)^{\otimes k}$ for $j \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}$. This is an irreducible representation of GL_2. The underlying vector space can be identified with the space of homogeneous polynomials of degree j in x_1, x_2. We will denote by $V_{j,k}^0$ the open subspace of polynomials with non-vanishing discriminant.

The moduli space $\mathcal{M}_2$ of smooth projective curves of genus 2 over F allows a presentation as an algebraic stack

$$\mathcal{M}_2 \xrightarrow{\sim} [V_{6,-2}^0/\mathrm{GL}_2]$$

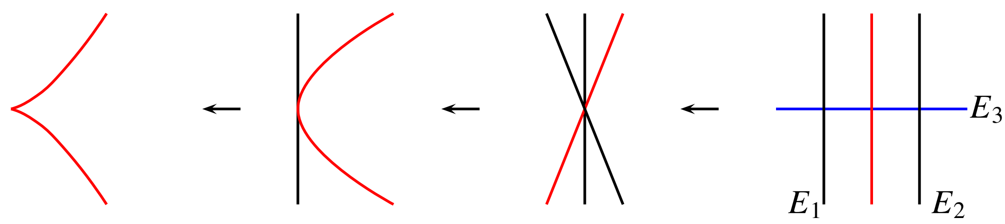
Here the action of $(a, b; c, d) \in \mathrm{GL}_2(F)$ is by $f(x_1, x_2) \mapsto (ad - bc)^{-2} f(ax_1 + bx_2, cx_1 + dx_2)$.

Indeed, if C is a curve of genus 2 the choice of a basis ω_1, ω_2 of $H^0(C, K)$ with $K = \Omega^1_C$ defines a canonical map $C \to \mathbb{P}^1$. Let ι denote the hyperelliptic involution of C. Choosing a non-zero element $\eta \in H^0(C, K^3)^{\iota=-1}$ yields eight elements η^2, $\omega_1^6, \omega_1^5\omega_2, \ldots, \omega_2^6$ in the 7-dimensional space $H^0(C, K^6)^{\iota=1}$ and thus a non-trivial relation.

In inhomogeneous terms, this gives us an equation $y^2 = f$ with $f \in F[x]$ of degree 6 with non-vanishing discriminant. The space $H^0(C, K)$ has a basis $x\,dx/y, dx/y$. If we let GL_2 act on (x, y) via $(x, y) \mapsto ((ax + b)/(cx + d), y(ad - bc)/(cx + d)^3)$ then this action preserves the form of the equation $y^2 = f$ if we take f in $V_{6,-2}$. Then $\lambda\,\mathrm{Id}_V$ acts via λ^2 on $V_{6,-2}$. Thus the stabilizer of a generic element f is of order 2. Moreover $-\mathrm{Id}_V$ acts by $y \mapsto -y$ on y and the action of GL_2 on the differentials is by the standard representation.

Conclusion 3.1 The pull back of the Hodge bundle $\mathbb{E}$ on $\mathcal{M}_2$ under the composition $V^0_{6,-2} \to [V^0_{6,-2}/\mathrm{GL}_2] \xrightarrow{\sim} \mathcal{M}_2$ is the equivariant bundle V.

The moduli space $\overline{\mathcal{M}}_2$ can be constructed from the projectivized space $\mathbb{P}(V_{6,-2})$ of binary sextics. The discriminant defines a hypersurface $\mathbb{D}$ whose singular locus has codimension 1 in $\mathbb{D}$. The locus of binary sectics with three coinciding roots forms an irreducible component $\mathbb{D}'$ of the singular locus. To illustrate the relation between $\mathbb{P}(V_{6,-2})$ at a general point of $\mathbb{D}'$ and $\overline{\mathcal{M}}_2$ at a point of the locus δ_1 in $\overline{\mathcal{M}}_2$ of stable curves whose Jacobian is a product of two elliptic curves, we reproduce the picture of [12, p. 80].



Here we look at a plane Π intersecting $\mathbb{D}$ transversally at a general point of $\mathbb{D}'$. One blows up three times, starting at $\Pi \cap \mathbb{D}'$, and then blows down the exceptional divisors E_1 and E_2; after that E_3 corresponds to the locus δ_1 in $\overline{\mathcal{M}}_2$; in $\mathcal{A}_2$ this corresponds to the locus $\mathcal{A}_{1,1}$ of products of elliptic curves.

4 Invariant Theory of Binary Sextics

We review the invariant theory of GL_2 acting on binary sextics. Let $V = \langle x_1, x_2 \rangle$ be a 2-dimensional vector space over a field F. By definition an invariant for the action of GL_2 acting on the space $\mathrm{Sym}^6(V)$ of binary sextics is an element invariant under $\mathrm{SL}_2(F) \subset \mathrm{GL}_2(F)$. If we write

$$f = \sum_{i=0}^{6} a_i x_1^{6-i} x_2^i \tag{2}$$

for an element of $\mathrm{Sym}^6(V)$ and thus take $(a_0, \dots, a_6)$ as coordinates on $\mathrm{Sym}^6(V)$, then an invariant is a polynomial in $a_0, \dots, a_6$ invariant under $\mathrm{SL}_2(F)$. The discriminant of a binary sextic, a polynomial of degree 10 in the a_i, is an example.

For $F = \mathbb{C}$ the ring of invariants was determined by Clebsch, Bolza and others in the 19th century. It is generated by invariants A, B, C, D, E of degrees 2, 4, 6, 10 and 15 in the a_i. Also for $F = \mathbb{F}_p$ we have generators of these degrees. We refer to [15, 20].

A covariant for the action of GL_2 on binary sextics is an element of $V \oplus \mathrm{Sym}^6(V)$ invariant under the action of SL_2. Such an element is a polynomial in $a_0, \dots, a_6$ and x_1, x_2. One way to make such covariants is to consider equivariant embeddings of an irreducible GL_2-representation U into $\mathrm{Sym}^d(\mathrm{Sym}^6(V))$. Equivalently, we consider an equivariant embedding

$$\varphi : \mathbb{C} \hookrightarrow \mathrm{Sym}^d(\mathrm{Sym}^6(V)) \otimes U^\vee .$$

Then $\Phi = \varphi(1)$ is a covariant. If U has highest weight $(\lambda_1 \geq \lambda_2)$ then Φ is homogeneous of degree d in $a_0, \dots, a_6$ and degree $\lambda_1 - \lambda_2$ in x_1, x_2. We say that Φ has degree d and order $\lambda_1 - \lambda_2$.

The simplest example is the universal binary sextic f given by (2); it corresponds to taking $U = \mathrm{Sym}^6(V)$.

Another example is the Hessian of f. Indeed, we decompose in irreducible representations

$$\mathrm{Sym}^2(\mathrm{Sym}^6(V)) = V[12, 0] \oplus V[10, 2] \oplus V[8, 4] \oplus V[6, 6] ,$$

where $V[a, b] = \mathrm{Sym}^{a-b}(V) \otimes \det(V)^b$ is the irreducible representation of highest weight (a, b). By taking $U = V[12, 0]$ we find the covariant $\Phi = f^2$ and by taking $U = V[10, 2]$ we get the Hessian; $U = V[6, 6]$ gives the invariant A.

The covariants form a ring $\mathcal{C}$ and the invariants form a subring $I = I(2, 6)$. The ring of covariants $\mathcal{C}$ was studied intensively at the end of the nineteenth century and the beginning of the twentieth century. The ring $\mathcal{C}$ is finitely generated and Grace and Young presented 26 generators for the ring $\mathcal{C}$, see [16]. These 26 covariants are constructed as transvectants by differentiating in a way similar to the construction of the Hessian. The kth transvectant of two forms $g \in \mathrm{Sym}^m(V)$, $h \in \mathrm{Sym}^n(V)$ is defined as

$$(g,h)_k = \frac{(m-k)!(n-k)!}{m!\,n!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{\partial^k g}{\partial x_1^{k-j} \partial x_2^j} \frac{\partial^k h}{\partial x_1^j \partial x_2^{k-j}}$$

and the index k is usually omitted if $k = 1$. Examples of the generators are $C_{1,6} = f$, $C_{2,0} = (f, f)_6$, $C_{2,4} = (f, f)_4$, $C_{3,2} = (f, C_{2,4})_4$. We refer to [8] for a list of these 26 generators.

5 Covariants of Binary Sextics and Modular Forms

The Torelli morphism induces an embedding $\mathcal{M}_2 \hookrightarrow \mathcal{A}_2$. The complement of the image is the locus $\mathcal{A}_{1,1}$ of products of elliptic curves. As a compactification we can take $\tilde{\mathcal{A}}_2 = \overline{\mathcal{M}}_2$.

We now fix the field F to be $\mathbb{C}$ or a finite prime field $\mathbb{F}_p$.

In the Chow ring $\mathrm{CH}^*_{\mathbb{Q}}(\tilde{\mathcal{A}}_2) \otimes F$ we have the cycle relation

$$10\lambda_1 = 2[\mathcal{A}_{1,1}] + [D]$$

with $\lambda_1 = c_1(\mathbb{E})$ the first Chern class of $\mathbb{E}$ and D the divisor that compactifies $\mathcal{A}_2 \otimes F$. This implies that there exists a modular form of weight 10 with divisor $2\,\mathcal{A}_{1,1} + D$, hence a cusp form. It is well-defined up to a non-zero multiplicative constant. We will normalize it later. We denote it by $\chi_{10} \in R_2(F)$.

We let V be the F-vector space with basis x_1, x_2. The fact that the pullback of the Hodge bundle $\mathbb{E}$ under

$$V^0_{6,-2} \to [V^0_{6,-2}/\mathrm{GL}_2] \to \mathcal{M}_2 \otimes F \hookrightarrow \mathcal{A}_2 \otimes F \tag{3}$$

is the equivariant bundle V implies that a section of $L^k = \det(\mathbb{E})^k$ pulls back to an invariant of degree k. We thus get an embedding of the ring of scalar-valued modular forms of degree 2 into the ring of invariants

$$R_2(F) \hookrightarrow I(2,6)(F)\,.$$

Conversely, an invariant of degree d defines a section of L^d on $\mathcal{M}_2 \otimes F$, hence a rational (meromorphic) modular form of weight d that is holomorphic outside $\mathcal{A}_{1,1} \otimes F$. By multiplying it with an appropriate power of χ_{10} it becomes holomorphic on $\mathcal{A}_2 \otimes F$, hence on all of $\tilde{\mathcal{A}}_2 \otimes F$. We thus get maps

$$R_2(F) \hookrightarrow I(2,6)(F) \xrightarrow{\nu} R_2(F)[1/\chi_{10}] \tag{4}$$

the composition of which is the identity on $R_2(F)$.

From the description of the moduli $\overline{\mathcal{M}}_2$ given above one sees that the image of a cusp form is an invariant divisible by the discriminant D. The image of χ_{10} is a non-zero multiple of the discriminant D. We may fix χ_{10} by requiring that $\nu(D) = \chi_{10}$.

This extends to the case of vector-valued modular forms. Let

$$M(F) = \oplus_{j,k} M_{j,k}(\Gamma_2)(F)$$

denote the ring of vector-valued modular forms of degree 2.

Proposition 5.1 *Pullback via (3) defines homomorphisms*

$$M(F) \hookrightarrow \mathcal{C}(2,6)(F) \xrightarrow{\nu} M(F)[1/\chi_{10}]\,,$$

the composition of which is the identity.

A modular form of weight (j,k) corresponds to a covariant of degree $d = j/2 + k$ and order j. A covariant of degree d and order r gives rise to a meromorphic modular form of weight $(r, d - r/2)$.

The most basic covariant is the universal binary sextic f. By construction $\nu(f)$ is a meromorphic modular form of weight $(6, -2)$. Therefore the central question is: *Which rational modular form is* $\nu(f)$?

Let $\mathcal{A}_{1,1} \subset \mathcal{A}_2$ be the locus of products of elliptic curves. Under the map

$$\mathcal{A}_1 \times \mathcal{A}_1 \to \mathcal{A}_{1,1} \to \mathcal{A}_2$$

the pullback of the Hodge bundle $\mathbb{E} = \mathbb{E}^{(2)}$ is $p_1^*\mathbb{E}^{(1)} \oplus p_2^*\mathbb{E}^{(1)}$ with p_1 and p_2 the projections of $\mathcal{A}_1 \times \mathcal{A}_1$ on its factors. The pullback of an element $h \in M_{j,k}(\Gamma_2)$ thus can be identified with an element of

$$\bigoplus_{i=0}^{j} M_{k+j-i}(\Gamma_1) \otimes M_{k+i}(\Gamma_1)\,.$$

Near a point of $\mathcal{A}_{1,1}$ we can write such an element symbolically as

$$h = \sum_{i=0}^{j} \eta_j X_1^{j-i} X_2^i\,,$$

where the X_i are dummy variables to indicate the vector coordinates, and such that the coefficient η_j defines the element of $M_{k+j-i}(\Gamma_1) \otimes M_{k+i}(\Gamma_1)$.

In particular, we have

$$\nu(f) = \sum_{i=0}^{6} \alpha_i X_1^{6-i} X_2^i ,$$

where α_i are rational functions near a point of $\mathcal{A}_{1,1}$. By interchanging x_1 and x_2 (that corresponds to the element $\gamma \in \Gamma_2$ that interchanges e_1 and e_2) we see that $\alpha_{6-i} = \alpha_i$ for $i = 0, \ldots, 3$.

Proposition 5.2 *If* $\mathrm{char}(F) \neq 2$ *and* $\neq 3$, *then* $\dim S_{6,8}(\Gamma_2)(F) = 1$ *and* $\chi_{10}\nu(f)$ *is a generator of* $S_{6,8}(\Gamma_2)(F)$.

Proof We shall use that $\dim S_{6,8}(\Gamma_2)(\mathbb{C}) \geq 1$. Indeed, we know an explicit cusp form of weight (6, 8), see below. (Alternatively, we know the dimensions of $S_{j,k}(\Gamma_2)(\mathbb{C})$ for $k \geq 4$, see [29]; in particular we know $\dim S_{6,8}(\Gamma_2)(\mathbb{C}) = 1$.) By semi-continuity this implies that $\dim S_{6,8}(\Gamma_2)(F) \geq 1$.

The restriction of an element of $S_{6,8}(\Gamma_2)(F)$ to the locus $\mathcal{A}_{1,1} \otimes F$ lands in

$$\bigoplus_{i=0}^{6} S_{8+6-i}(\Gamma_1)(F) \otimes S_{8+i}(\Gamma_1)(F) ,$$

and as we have $\dim S_k(\Gamma_1)(F) = 0$ for $k < 12$ it vanishes on $\mathcal{A}_{1,1} \otimes F$.

The tangent space to $\mathcal{A}_2$ at a point $[X = X_1 \times X_2]$ of $\mathcal{A}_{1,1}$, with X_i elliptic curves, can be identified with

$$\mathrm{Sym}^2(T_X) = \mathrm{Sym}^2(T_{X_1}) \oplus (T_{X_1} \otimes T_{X_2}) \oplus \mathrm{Sym}^2(T_{X_2})$$

with T_X (resp T_{X_i}) the tangent space at the origin of X (resp. X_i), and with the middle term corresponding to the normal space. Thus we see that the pullback of the conormal bundle of $\mathcal{A}_{1,1}$ to $\mathcal{A}_1 \times \mathcal{A}_1$ is the tensor product of the pullback of the Hodge bundles on the two factors $\mathcal{A}_1$.

Let $h \in S_{6,8}(\Gamma_2)(F)$ and write h as

$$h = \sum_{i=0}^{6} \eta_i \, X_1^{6-i} X_2^i$$

locally at a general point of $\mathcal{A}_{1,1} \otimes F$. If we consider the Taylor development in the normal direction of $\mathcal{A}_{1,1}$ of the form h that vanishes on $\mathcal{A}_{1,1} \otimes F$ then the first non-zero Taylor term of η_i, say the rth term, is an element of

$$S_{14-i+r}(\Gamma_1)(F) \otimes S_{8+i+r}(\Gamma_1)(F) .$$

Since $S_k(\Gamma_1)(F) = (0)$ for $k < 12$, a non-zero rth Taylor term of η_i can occur only for $14 - i + r \geq 12$ and $8 + i + r \geq 12$. We thus find:

$$\mathrm{ord}_{\mathcal{A}_{1,1}}(\eta_0, \ldots, \eta_6) \geq (4, 3, 2, 1, 2, 3, 4) .$$

Lemma 5.3 *We have* $\mathrm{ord}_{\mathcal{A}_{1,1}}(\eta_3) = 1$.

Proof If $\mathrm{ord}_{\mathcal{A}_{1,1}}(\eta_3) \geq 2$ then h/χ_{10} is a regular form in $S_{6,-2}(\Gamma_2)$ and we write it as $h/\chi_{10} = \sum_{i=0}^{6} \xi_i X_1^{6-i} X_2^i$ with $\xi_i = \eta_i/\chi_{10}$ regular. Then the invariant $A = 120\, a_0 a_6 - 20\, a_1 a_5 + 8\, a_2 a_4 - 3\, a_3^2$ defines a non-zero regular modular form

$$\nu(A) = 120\, \xi_0 \xi_6 - 20\, \xi_1 \xi_5 + 8\, \xi_2 \xi_4 - 3\, \xi_3^2$$

in $M_2(\Gamma_2)(F)$. But restriction to $\mathcal{A}_{1,1}$ gives for even k an exact sequence

$$0 \to M_{k-10}(\Gamma_2)(F) \to M_k(\Gamma_2)(F) \to \mathrm{Sym}^2(M_k(\Gamma_1)(F)) \qquad (5)$$

with the second arrow multiplication by χ_{10}. This implies that $\dim M_2(\Gamma_2)(F) = 0$ for $\mathrm{char}(F) \neq 2$ and $\neq 3$. This proves the lemma. □

The image of a non-zero element $\chi_{6,8}$ of $S_{6,8}$ in $\mathcal{C}(2,6)$ is a covariant of degree 11 and order 6. But since $\chi_{6,8}$ is a cusp form, this covariant is divisible by the discriminant which is of degree 10. Therefore, $\chi_{6,8}/\chi_{10}$ corresponds to a covariant of degree 1, hence is a non-zero multiple of f. This implies that $\dim S_{6,8}(\Gamma_2)(F) = 1$. □

Corollary 5.4 *If we write* $\nu(f) = \sum_{i=0}^{6} \alpha_i X_1^{6-i} X_2^i$ *then*

$$\mathrm{ord}_{\mathcal{A}_{1,1}}(\alpha_0, \ldots, \alpha_6) \geq (2, 1, 0, -1, 0, 1, 2)$$

and $\mathrm{ord}_{\mathcal{A}_{1,1}}(\alpha_3) = -1$.

6 Constructing Vector-Valued Modular Forms of Degree 2

Now that we know $\nu(f)$ by Proposition 5.2 we can describe the map $\nu : \mathcal{C}(2,6) \to M[1/\chi_{10}]$ explicitly. Recall that a covariant is a polynomial in $a_0, \ldots, a_6$ and x_1, x_2. We arrive at the following conclusion.

Proposition 6.1 *The map* $\nu : \mathcal{C}(2,6) \to M[1/\chi_{10}]$ *is substitution of* α_i *for* a_i *(and* X_i *for* x_i*).*

In order to efficiently apply the proposition we need to know the coordinates of a generator $\chi_{6,8}$ of $S_{6,8}$ very explicitly.

Remark 6.2 If $F \neq \mathbb{F}_2$ the moduli space $\mathcal{A}_2[2]$ of level 2 is a Galois cover of $\mathcal{A}_2$ with group $\mathrm{Sp}(2, \mathbb{Z}/2\mathbb{Z})$. This group is isomorphic to the symmetric group $\mathfrak{S}_6$. The sign character of $\mathfrak{S}_6$ defines a character ϵ of Γ_2. The pullback of χ_{10} under $\pi : \mathcal{A}_2[2] \to \mathcal{A}_2$ is a square χ_5^2 since the pullback of D under $\tilde{\mathcal{A}}_2[2] \to \tilde{\mathcal{A}}_2$ is divisible by 2 as a divisor. Thus χ_5 is a modular form of weight 5 with character ϵ.

Let now $F = \mathbb{C}$. Recall that $\chi_{6,8}$ vanishes on $\mathcal{A}_{1,1}$. Dividing $\chi_{6,8}$ by χ_5 provides a holomorphic vector-valued modular form $\chi_{6,3} \in M_{6,3}(\Gamma_2, \epsilon)(F)$ with character ϵ. Such a form can be constructed as follows.

We consider the six odd order two theta functions $\vartheta_i(\tau, z)$ with $(\tau, z) \in \mathfrak{H}_2 \times \mathbb{C}^2$. The gradient $G_i = (\partial\vartheta_i/\partial z_1, \partial\vartheta_i/\partial z_2)(\tau, 0)$ is a modular form of weight $(1, 1/2)$ on some congruence subgroup, but the product of the transposes of these six gradients defines a vector-valued modular form of weight $(6, 3)$ on Γ_2 with character ϵ. The product $\chi_{6,8} = \chi_5\chi_{6,3}$ is a cusp form of weight $(6, 8)$ on Γ_2. A non-zero multiple of its Fourier expansion starts with (with $q_1 = e^{2\pi i\tau_{11}}$, $q_2 = e^{2\pi i\tau_{22}}$ and $r = e^{2\pi i\tau_{12}}$)

$$\chi_{6,8}(\tau) = \begin{pmatrix} 0 \\ 0 \\ r^{-1}-2+r \\ 2(r-r^{-1}) \\ r^{-1}-2+r \\ 0 \\ 0 \end{pmatrix} q_1q_2 + \begin{pmatrix} 0 \\ 0 \\ -2(r^{-2}+8r^{-1}-18+8r+r^2) \\ 8(r^{-2}+4r^{-1}-4r-r^2) \\ -2(7r^{-2}-4r^{-1}-6-4r+7r^2) \\ 12(r^{-2}-2r^{-1}+2r-r^2) \\ -4(r^{-2}-4r^{-1}+6-4r+r^2) \end{pmatrix} q_1q_2^2$$
$$+ \begin{pmatrix} -4(r^{-2}-4r^{-1}+6-4r+r^2) \\ 12(r^{-2}-2r^{-1}+2r-r^2) \\ -2(7r^{-2}-4r^{-1}-6-4r+7r^2) \\ 8(r^{-2}+4r^{-1}-4r-r^2) \\ -2(r^{-2}+8r^{-1}-18+8r+r^2) \\ 0 \\ 0 \end{pmatrix} q_1^2q_2 + \begin{pmatrix} 16(r^{-3}-9r^{-1}+16-9r+r^3) \\ -72(r^{-3}-3r^{-1}+3r-r^3) \\ +128(r^{-3}-2+r^3) \\ -144(r^{-3}+5r^{-1}-5r-r^3) \\ +128(r^{-3}-2+r^3) \\ -72(r^{-3}-3r^{-1}+3r-r^3) \\ 16(r^{-3}-9r^{-1}+16-9r+r^3) \end{pmatrix} q_1^2q_2^2 + \dots$$

Proposition 6.1 provides an extremely effective way of constructing complex vector-valued Siegel modular forms of degree 2. Let us give a few examples. In the decomposition

$$\mathrm{Sym}^2(\mathrm{Sym}^6(V)) = V[12, 0] \oplus V[10, 2] \oplus V[8, 4] \oplus V[6, 6]$$

of $\mathrm{Sym}^2(\mathrm{Sym}^6(V))$ the covariant H defined by $V[10, 2]$ is the Hessian and by Corollary 5.4 gives rise to a form $\chi_{8,8} = \nu(H)\chi_{10} \in S_{8,8}(\Gamma_2)$ and using the Fourier expansion of $\chi_{6,8}$ we obtain the Fourier expansion of $\chi_{8,8}$. Similarly, the covariant corresponding to $V[8, 4]$ gives a form $\chi_{4,10}$ after multiplication with χ_{10}. Finally, the covariant defined by $V[6, 6]$ is the invariant A and defines the cusp form $\chi_{12} = \nu(A)\chi_{10}$. We refer to [7] for more details.

As an illustration of this we refer to the website [1] that gives the Fourier series for generators for all cases where $\dim S_{j,k}(\Gamma_2) = 1$.

Another illustration of the efficacity of the construction of modular forms appears when one considers the modules $\oplus_k M_{j,k}(\Gamma_2)$ and $\oplus_k M_{j,k}(\Gamma_2, \epsilon)$. Let R_2^{ev} be the ring of scalar-valued modular forms of even weight. The structure of the R_2^{ev}-modules

$$\oplus_k M_{j,2k}(\Gamma_2), \qquad \oplus_k M_{j,2k+1}(\Gamma_2)$$

has been determined for $j = 2, 4, 6, 8, 10$ by Satoh, Ibukiyama, Kiyuna, van Dorp and Takemori using various methods. Using covariants one can uniformly treat these cases and the cases of modular forms with character for the same values of j

$$\oplus_k M_{j,2k}(\Gamma_2, \epsilon), \qquad \oplus_k M_{j,2k+1}(\Gamma_2, \epsilon)\,.$$

For example, the R_2-module $\oplus_k M_{2,2k+1}(\Gamma_2, \epsilon)$ is free with generators of weight $(2, 9)$, $(2, 11)$, and $(2, 17)$ and the module $\oplus_k M_{10}^{10,2k}(\Gamma_2, \epsilon)$ is free with 10 generators. We refer to [8].

Yet another application of the construction of modular forms via covariants deals with small weights. It is known by Skoruppa [27] that $\dim S_{j,1}(\Gamma_2) = 0$. He proved this using Fourier-Jacobi forms. We conjecture $\dim S_{j,2}(\Gamma_2) = 0$ and proved this for $j \leq 52$ using covariants. We refer to [5].

As a final illustration, for $k = 3$ the smallest j such that $\dim S_{j,3}(\Gamma_2) \neq 0$ is 36. It is not difficult to construct a generator of $S_{36,3}(\Gamma_2)$ using covariants, see [5].

7 Rings of Scalar-Valued Modular Forms

The approach explained in the preceding section makes it easy to find generators for the rings $R_2(F) = \oplus_k M_k(\Gamma_2)(F)$ of modular forms of degree 2 for $F = \mathbb{C}$ or $F = \mathbb{F}_p$. We write ν_F for the map $I(2,6)(F) \to R_2(F)[1/\chi_{10}]$. We denote by $R_2^{\mathrm{ev}}(F)$ the subring of even weight modular forms.

The degree 2 invariant A of a binary sextic $f = \sum_{i=0}^{6} a_i x_1^{6-i} x_2^i$ can be written as

$$120\, a_0 a_6 - 20\, a_1 a_5 + 8\, a_2 a_4 - 3\, a_3^2 \,.$$

Corollary 5.4 implies that $\nu_F(A)$ cannot be regular for $F = \mathbb{C}$ or $\mathbb{F}_p$ with $p \geq 5$, but also that $\nu_F(AD)$ is a cusp form $\chi_{12} \in S_{12}(\Gamma_2)(F)$ of weight 12.

In degree 4 there is the invariant B given by

$$(81\, a_0 a_6 + 9\, a_1 a_5) a_3^2 - 3\,(15\, a_0 a_4 a_5 + 15\, a_1 a_2 a_6 + a_1 a_4^2 + a_2^2 a_5) a_3 + \cdots + a_2^2 a_4^2$$

and Corollary 5.4 implies that it defines a regular modular form $\psi_4 = \nu_F(B)$ of weight 4.

The invariant C of degree 6 is given by

$$18\,(9\, a_0 a_6 + 4\, a_1 a_5) a_3^4 - 6\,(33\, a_0 a_4 a_5 + 33\, a_1 a_2 a_6 + 4\, a_1 a_4^2 + 4 a_2^2 a_5) a_3^3 + \cdots$$

and in a similar way one sees that $AB - 3\,C$ starts with

$$1458\, a_0 a_6 a_3^4 - 486\,(a_0 a_4 a_5 + a_1 a_2 a_6) a_3^3 + \cdots$$

and defines a regular modular form $\psi_6 = \nu_F(AB - 3C)$ of weight 6.

The discriminant D starts as

$$729\, a_0^2 a_6^2\, a_3^6 - 54(9\, a_0^2 a_4 a_5 a_6 - 2\, a_0^2 a_5^3 + 9\, a_0 a_1 a_2 a_6^2 - 2\, a_1^3 a_6^2) a_3^5 + \cdots$$

and is seen to have order 2 along $\mathcal{A}_{1,1}$. It defines a cusp form that is a non-zero multiple of χ_{10}.

Proposition 7.1 *For $F = \mathbb{C}$ or $F = \mathbb{F}_p$ with $p \geq 5$ the modular forms ψ_4, ψ_6, χ_{10} and χ_{12} generate $R_2^{\rm ev}(F)$.*

Proof The algebraic independence of A, B, C, D shows that the generators are algebraically independent. Therefore $\psi_4, \psi_6, \chi_{10}, \chi_{12}$ generate a graded subring $T(F) \subseteq R_2^{\rm ev}(F)$ such that for even k we have

$$\dim T_k(F) = \frac{k^3}{17280} + O(k^2)\,.$$

Now by Riemann-Roch we have for even k

$$\dim M_k(\Gamma_2)(F) = \frac{c_1(L)^3}{3!} k^3 + O(k^2)$$

since $c_1(L)^3 = 1/2880$, [32, p. 72]. Therefore there cannot be more generators. Note that $4 \cdot 6 \cdot 10 \cdot 12 = 2880$. □

Remark 7.2 Restriction to $\mathcal{A}_{1,1}$ shows that $\psi_4, \psi_6, \chi_{10}, \chi_{12}$ generate $M_k(\Gamma_2)(F)$ for $k \leq 12$. Let $d(k) = \dim_F M_k(\Gamma_2)(F)$ and $t(k) = \dim_F T_k(F)$. Then $t(k) \leq d(k)$ and for even k the exact sequence (5) yields

$$d(k) \leq d(k-10) + \frac{c(k)(c(k)+1)}{2}$$

with $c(k) = \dim_F M_k(\Gamma_1)(F)$. Now one easily sees $t(k) - t(k-10) = c(k)(c(k)+1)/2$. Thus if we assume $d(k-10) = t(k-10)$ we get

$$t(k) \leq d(k) \leq d(k-10) + \frac{c(k)(c(k)+1)}{2} = t(k)$$

and this provides via induction another proof that $\psi_4, \psi_6, \chi_{10}$ and χ_{12} generate $R_2^{\rm ev}(F)$ for $F = \mathbb{C}$ or $\mathbb{F}_p$ with $p \geq 5$.

The odd degree invariant E (of degree 15) of binary sextics starts with

$$-729(a_0^2 a_5^3 - a_1^3 a_6) a_3^{10} + \ldots$$

and one checks that it has order -3 along $\mathcal{A}_{1,1}$. So $\chi_{10}^2 \nu_F(E)$ defines a regular cusp form $\chi_{35} \in S_{35}(\Gamma_2)(F)$ with order 1 along $\mathcal{A}_{1,1}$.

Let now $\mathrm{char}(F) \neq 2$. The locus in $\mathcal{A}_2 \otimes F$ of principally polarized abelian surfaces X with $\mathrm{Aut}(X)$ containing $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ consists of two irreducible divisors $H_1 = \mathcal{A}_{1,1}$ and H_4, the Humbert surface of degree 4 of abelian surfaces isogenous with a product by an isogeny of degree 4. In terms of moduli of curves, H_4 is the locus of curves that are double covers of elliptic curves. We know that the cycle class of $H_1 + H_4$ is $35\lambda_1$ in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{A}_2)$, see [31, p. 218].

Lemma 7.3 *Suppose that* $\mathrm{char}(F) \neq 2$. *A modular form* $f \in M_k(\Gamma_2)(F)$ *with* k *odd vanishes on* H_1 *and* H_4.

Proof An abelian surface $[X] \in H_1$ or $[X] \in H_4$ possesses an involution that acts by -1 on $H^0(X, \Omega_X^2)$. □

Corollary 7.4 *The form* χ_{35} *as a section of* $L^{\otimes 35}$ *has as divisor* $H_1 + H_4 + D$ *with* D *the divisor at infinity.*

We can now easily derive the results of Igusa and Nagaoka (see [20, 24], and also [19]).

Theorem 7.5 *Let* $F = \mathbb{C}$ *or* $F = \mathbb{F}_p$ *with* $p \geq 5$. *Then the ring* $R_2(\mathbb{F}_p)$ *is generated over* $R_2^{\mathrm{ev}}(\mathbb{F}_p) = F[\psi_4, \psi_6, \chi_{10}, \chi_{12}]$ *by the cusp form* χ_{35} *of weight* 35 *with* $\chi_{35}^2 \in R_2^{\mathrm{ev}}(F)$.

Proof Any odd weight modular form vanishes on H_1 and H_4, hence is divisible by χ_{35}. □

Remark 7.6 The same argument proves Theorem 7.5 for any commutative ring F in which 6 is invertible. It can also be used to obtain Igusa's result on the ring $R_2(\mathbb{Z})$.

Now positive characteristic sometimes allows more modular forms than characteristic zero. We know that the locus in $\mathcal{A}_g \otimes \mathbb{F}_p$ of abelian varieties of p-rank $< g$ has cycle class $(p-1)\lambda_1$, [13, 32]. This implies that there is a non-zero modular form of weight $p-1$ in characteristic p. This modular form is called the Hasse invariant of degree g and weight $p-1$. The image of the Hasse invariant of degree g under the Siegel operator is the Hasse invariant of degree $g-1$.

The Hasse invariants for degree 1 and characteristic 2 and 3 appear as the generators a_1 and b_2 in

$$R_1(\mathbb{F}_2) = \mathbb{F}_2[a_1, \Delta], \quad R_1(\mathbb{F}_3) = \mathbb{F}_3[b_2, \Delta]\,.$$

The degree 2 invariant A of binary sextics reduces to $a_1a_5 - a_2a_4$ modulo 3 and in view of Conclusion 5.4 defines a form $\nu_{\mathbb{F}_3}(A) \in M_2(\Gamma_2)(\mathbb{F}_3)$ and it must agree with the Hasse invariant (up to a non-zero multiplicative scalar) as there is only one invariant of degree 2 (up to multiplicative scalars). A careful analysis of the invariants in characteristic 3 leads to the description of the ring $R_2(\mathbb{F}_3)$ given in [33].

Theorem 7.7 *The subring* $\mathcal{R}_2^{\mathrm{ev}}(\mathbb{F}_3)$ *of modular forms of even weight is generated by forms of weights* 2, 10, 12, 14, *and* 36 *and has the form*

$$\mathcal{R}_2^{\rm ev}(\mathbb{F}_3) = \mathbb{F}_3[\psi_2, \chi_{10}, \psi_{12}, \chi_{14}, \chi_{36}]/J$$

with J the ideal generated by the relation $\psi_2^3\chi_{36} - \chi_{10}^3\psi_{12} - \psi_2^2\chi_{10}\chi_{14}^2 + \chi_{14}^3$. Moreover, $\mathcal{R}_2(\mathbb{F}_3)$ is generated over $\mathcal{R}_2^{\rm ev}$ by a form χ_{35} of weight 35 whose square lies in $R_2^{\rm ev}(\mathbb{F}_3)$. The ideal of cusp forms is generated by $\chi_{10}, \chi_{14}, \chi_{35}, \chi_{36}$.

The case of characteristic 2 was treated in joint work with Cléry in [5]. In the case of characteristic 2 a curve of genus 2 is not described by a binary sextic. Instead we find an equation

$$y^2 + a\,y + b = 0$$

with a (resp. b) in $k[x]$ of degree ≤ 3 (resp. ≤ 6) and the hyperelliptic involution is $y \mapsto y + a$. It comes with a basis $xdx/a, dx/a$ of regular differentials. In this case we look at pairs $(a, b) \in V_{3,-1} \times V_{6,-2}$ with $V_{n,m} = \mathrm{Sym}^n(V) \otimes \det(V)^m$. Let $\mathcal{V}^0 \subset V_{3,-1} \times V_{6,-2}$ be the open subset defining smooth hyperelliptic curves. Now we have an action of GL_2 and an action of $\mathrm{Sym}^3(V)$ via

$$(a, b) \mapsto (a, b + v^2 + va)$$

Together this defines a stack quotient

$$[\mathcal{V}^0/\mathrm{GL}_2 \ltimes V_{3,-1}]$$

Now by an invariant we mean a polynomial in the coefficients $a_0, \ldots, a_3$ and $b_0, \ldots, b_6$ that is invariant under $\mathrm{SL}(V) \ltimes \mathrm{Sym}^3(V)$. Let $\mathcal{K}$ be the ring of invariants. A first example is the square root of the discriminant of a:

$$K_1 = a_0a_3 + a_1a_2\,.$$

As an analog of (5) we now get homomorphisms

$$R_2(\mathbb{F}_2) \hookrightarrow \mathcal{K} \xrightarrow{\nu} R_2(\mathbb{F}_2)[1/\chi_{10}]$$

the composition of which is the identity.

In order to construct characteristic 2 invariants one can still use binary sextics as Igusa suggested in [20]. Indeed, one lifts the curve given by $y^2 + ay + b = 0$ to the Witt ring, say defined by $y^2 + \tilde{a}y + \tilde{b} = 0$ and takes an invariant of the binary sextic given by $\tilde{a}^2 + 4\tilde{b}$, then divides these by the appropriate power of 2 and reduces modulo 2.

For example, the degree 2 invariant of binary sextics yields in this way an invariant K_2 that equals K_1^2. A degree 4 invariant yields an invariant K_4 that turns out to be divisible by K_1. We thus find an invariant K_3 of degree 3.

The Hasse invariant ψ_1 must map to K_1. As in characteristic 3 a careful analysis gives the orders of a_i and b_i along $\mathcal{A}_{1,1}$ and we can deduce for an invariant K the order of $\nu(K)$ along $\mathcal{A}_{1,1}$. The ring $R_2(\mathbb{F}_2)$ was described in [5].

Theorem 7.8 *The ring $\mathcal{R}_2(\mathbb{F}_2)$ is generated by modular forms of weights* 1*,* 10*,* 12*,* 13*, and* 48 *satisfying one relation of weight* 52*:*

$$\mathcal{R}_2(\mathbb{F}_2) = \mathbb{F}_2[\psi_1, \chi_{10}, \psi_{12}, \chi_{13}, \chi_{48}]/(R)$$

with $R = \chi_{13}^4 + \psi_1^3 \chi_{10} \chi_{13}^3 + \psi_1^4 \chi_{48} + \chi_{10}^4 \psi_{12}$. The ideal of cusp forms is generated by χ_{10}, χ_{13} and χ_{48}.

8 Moduli of Curves of Genus Three and Invariant Theory of Ternary Quartics

Now we turn to genus 3 treated in [9] and consider the moduli space $\mathcal{M}_3^{\mathrm{nh}}$ of non-hyperelliptic curves of genus 3 over a field F. This is an open part of the moduli space $\mathcal{M}_3$ with as complement the divisor $\mathcal{H}_3$ of hyperelliptic curves. Let now $V = \langle x_0, x_1, x_2 \rangle$ be the 3-dimensional F-vector space with basis x_0, x_1, x_2. We let $V_{4,0,-1}$ be the irreducible representation $\mathrm{Sym}^4(V) \otimes \det(V)^{-1}$. The underlying space is the space of ternary quartics. It contains the open subset $V_{4,0,-1}^0$ of ternary quartics with non-vanishing discriminant; that is, the ternary quartics that define smooth plane quartic curves.

It is known that $\mathcal{M}_3^{\mathrm{nh}}$ has a description as stack quotient

$$\mathcal{M}_3^{\mathrm{nh}} \xrightarrow{\sim} [V_{4,0,-1}^0/\mathrm{GL}_3]$$

Indeed, if C is a non-hyperelliptic curve of genus 3 then a choice of basis of $H^0(C, K)$ defines an embedding of C into $\mathbb{P}^2$ and the image satisfies an equation $f(x_0, x_1, x_2) = 0$ with f homogeneous of degree 4. In order that the action on the space of differentials with basis

$$x_i\, (x_0 dx_1 - x_1 dx_0)/(\partial f/\partial x_2), \qquad i = 0, 1, 2$$

is the standard representation V we need to twist $\mathrm{Sym}^4(V)$ by $\det(V)^{-1}$. Then $\lambda \mathrm{Id} \in \mathrm{GL}_3(F)$ acts by λ on $V_{4,0,-1}$ and we arrive at the familiar stack quotient $[Q/\mathrm{PGL}_3]$ with Q the space of smooth projective curves of degree 4 in $\mathbb{P}^2$ by first dividing by the multiplicative group of multiples of the diagonal.

Conclusion 8.1 The pull back of the Hodge bundle $\mathbb{E}$ on $\mathcal{M}_3^{\mathrm{nh}}$ under

$$V_{4,0,-1}^0 \to [V_{4,0,-1}^0]/\mathrm{GL}_2] \xrightarrow{\sim} \mathcal{M}_3^{\mathrm{nh}}$$

is the equivariant bundle V.

Therefore we now look at the invariant theory of GL_3 acting on ternary quartics $\mathrm{Sym}^4(V)$ with $V = \langle x, y, z \rangle$ the standard representation of $\mathrm{GL}_3(V)$. We write the universal ternary quartic f as

$$f = a_0 x^4 + a_1 x^3 y + \cdots + a_{14} z^4$$

in a lexicographic way. We fix coordinates for $\wedge^2 V$

$$\hat{x} = y \wedge z,\ \hat{y} = z \wedge x,\ \hat{z} = x \wedge y\,.$$

Recall that an irreducible representation ρ of GL_3 is determined by its highest weight $(\rho_1 \geq \rho_2 \geq \rho_3)$. This representation appears in

$$\mathrm{Sym}^{\rho_1-\rho_2}(V) \otimes \mathrm{Sym}^{\rho_2-\rho_3}(\wedge^2 V) \otimes \det(V)^3$$

An invariant for the action of GL_3 on $\mathrm{Sym}^4(V)$ is a polynomial in $a_0, \ldots, a_{14}$ invariant under SL_3. Instead of the notion of covariant we consider here the notion of a concomitant. A concomitant is a polynomial in $a_0, \ldots, a_{14}$ and in x, y, z and $\hat{x}, \hat{y}, \hat{z}$ that is invariant under the action of SL_3. The most basic example is the universal ternary quartic f.

Concomitants can be obtained as follows. One takes an equivariant map of GL_3-representations

$$U \hookrightarrow \mathrm{Sym}^d(\mathrm{Sym}^4(V))$$

or equivalently the equivariant embedding

$$\varphi : \mathbb{C} \longrightarrow \mathrm{Sym}^d(\mathrm{Sym}^4(V)) \otimes U^\vee$$

Then $\Phi = \varphi(1)$ is a concomitant. If U is an irreducible representation of highest weight $\rho_1 \geq \rho_2 \geq \rho_3$ then Φ is of degree d in $a_0, \ldots, a_{14}$, of degree $\rho_1 - \rho_2$ in x, y, z and degree $\rho_2 - \rho_3$ in $\hat{x}, \hat{y}, \hat{z}$.

The invariants form a ring $I(3,4)$ and the concomitants $\mathcal{C}(3,4)$ form a module over $I(3,4)$. For more on the ring $I(3,4)$ see [11].

9 Concomitants of Ternary Quartics and Modular Forms of Degree 3

The starting point for the construction of modular forms of degree 3 is the Torelli morphism

$$t : \mathcal{M}_3 \to \mathcal{A}_3$$

defined by associating to a curve of genus 3 its Jacobian. This is a morphism of Deligne-Mumford stacks of degree 2 ramified along the hyperelliptic locus $\mathcal{H}_3$. Indeed, every abelian variety has an automorphism of order 2, but a generic curve of genus 3 does not have non-trivial automorphisms. Hyperelliptic curves have an automorphism of order 2 that induces -1_{Jac} on the Jacobian.

There is a Siegel modular form $\chi_{18} \in S_{18}(\Gamma_3)$ constructed by Igusa [21]. It is defined as the product of the 36 even theta constants of order 2. The divisor of χ_{18} in the standard compactification (defined by the second Voronoi fan) $\tilde{\mathcal{A}}_3$ is

$$\mathcal{H}_3 + 2D$$

with D the divisor at infinity.

The pullback under the Torelli morphism of the Hodge bundle $\mathbb{E}$ on $\mathcal{A}_3$ is the Hodge bundle of $\mathcal{M}_3$. The Hodge bundle on $\mathcal{M}_3$ extends to the Hodge bundle over $\overline{\mathcal{M}}_3$, denoted again by $\mathbb{E}$. For each irreducible representation ρ of GL_3 have a bundle $\mathbb{E}_\rho$ on $\overline{\mathcal{M}}_3$ constructed by applying a Schur functor. We thus can consider

$$T_\rho = H^0(\overline{\mathcal{M}}_3, \mathbb{E}_\rho)$$

and elements of it are called Teichmüller modular forms of weight ρ and genus (or degree) 3. There is an involution ι acting on the stack $\mathcal{M}_3$ associated to the double cover $\mathcal{M}_3 \to \mathcal{A}_3$. If the characteristic is not 2 we can thus split T_ρ into ± 1-eigenspaces under ι

$$T_\rho = T_\rho^+ \oplus T_\rho^- \,.$$

We can identify the invariants under ι with Siegel modular forms

$$T_\rho^+ = M_\rho(\Gamma_3) \tag{6}$$

while the space T_ρ^- consists of the genuine Teichmüller modular forms.

The pullback of χ_{18} to $\mathcal{M}_3$ is a square χ_9^2 with χ_9 a Teichmüller modular form of weight 9 constructed by Ichikawa [17, 18].

Using the identification (6) we have

$$\chi_9 \, T_\rho^- \subset S_{\rho'}(\Gamma_3) \quad \text{with } \rho' = \rho \otimes \det{}^9.$$

We will now use the invariant theory of ternary quartics Conclusion 8.1 implies that the pullback of a scalar-valued Teichmüller modular form of weight k is an invariant of weight $3k$ in $I(3,4)$. An invariant of degree $3d$ defines a meromorphic Teichmüller modular form of weight d on $\overline{\mathcal{M}}_3$ that becomes holomorphic after multiplication by an appropriate power of χ_9. Indeed, an invariant of degree $3d$ is defined by an equivariant embedding $\det(V)^{4d} \hookrightarrow \mathrm{Sym}^{3d}(\mathrm{Sym}^4(V))$ or taking care of the necessary twisting by

$$\det(V)^d \hookrightarrow \mathrm{Sym}^{3d}(\mathrm{Sym}^4(V)) \otimes \det(V)^{-3d} \,.$$

We thus get

$$T \longrightarrow I(3,4) \longrightarrow T[1/\chi_9] \,,$$

where the composition of the arrows is the identity. In particular, the Teichmüller modular form χ_9 maps to an invariant of degree 27 and since it is a cusp form one can check that it must be divisible by the discriminant, hence is a multiple of the discriminant.

We can extend this to vector-valued Teichmüller modular forms

$$\Sigma \longrightarrow \mathcal{C}(3,4) \xrightarrow{\nu} \Sigma[1/\chi_9]$$

with the T-module Σ defined as

$$\Sigma = \oplus_\rho T_\rho$$

with ρ running through the irreducible representations of GL_3.

We can ask what the image $\nu(f)$ of the universal ternary quartic is. By construction it is a meromorphic modular form of weight $(4, 0, -1)$. Here the weight refers to the irreducible representation $\mathrm{Sym}^4(V) \otimes \det(V)^{-1}$ of GL_3.

We know that there exists a holomorphic modular cusp form $\chi_{4,0,8}$ of weight $(4, 0, 8)$, see [4] and below.

Proposition 9.1 *Over $\mathbb{C}$ the Siegel modular modular form $\chi_9\,\nu(f)$ is a generator of $S_{4,0,8}(\Gamma_3)(\mathbb{C})$.*

Proof The cusp form $\chi_{4,0,8}$ maps to a concomitant of degree 28 that is divisible by the discriminant. Therefore, $\chi_{4,0,-1} = \chi_{4,0,8}/\chi_9$ corresponds to a concomitant of degree 1. This must be a non-zero multiple of f. □

If we write the universal ternary quartic lexicographically as

$$f = a_0x^4 + a_1x^3y + \cdots + a_{14}z^4$$

and we write the meromorphic Teichmüller form $\chi_{4,0,-1}$ similarly lexicographically as

$$\chi_{4,0,-1} = \alpha_0X^4 + \alpha_1X^3Y + \cdots + \alpha_{14}Z^4$$

with dummy variables X, Y, Z to indicate the coordinates of $\chi_{4,0,-1}$, we arrive at the analog for degree 3:

Proposition 9.2 *The map $\nu : \mathcal{C}(3,4) \to T[1/\chi_9]$ is given by substituting α_i for a_i (and X, Y, Z for x, y, z and $\hat{X}, \hat{Y}, \hat{Z}$ for $\hat{x}, \hat{y}, \hat{z}$).*

In the following, we restrict to $F = \mathbb{C}$. One way to construct a generator of $S_{4,0,8}(\Gamma_3)(\mathbb{C})$ is to take the Schottky form of degree 4 and weight 8 that vanishes on the Torelli locus. We can develop it along $\mathcal{A}_{3,1}$, the locus in $\mathcal{A}_4$ of products of abelian threefolds and elliptic curves. It restriction to $\mathcal{A}_{3,1}$ is a form in $S_8(\Gamma_3) \otimes S_8(\Gamma_1)$ and thus vanishes. The first non-zero term in the Taylor expansion along $\mathcal{A}_{3,1}$ is

$$\chi_{4,0,8} \otimes \Delta \in S_{4,0,8}(\Gamma_3) \otimes S_{12}(\Gamma_1)$$

Since the Schottky form can be constructed explicitly with theta functions we can easily obtain the beginning of the Fourier expansion. We refer to [4] for the details.

In [9] we formulated a criterion that tells us which elements of $\mathcal{C}(3,4)$ will give holomorphic modular forms. We can associate to a concomitant its order along the locus of double conics by looking at its order in t when we evaluate it on the ternary quartic $tf + q^2$ where q is a sufficiently general quadratic form in x, y, z. Then the result is the following, see [9].

Theorem 9.3 *Let c be a concomitant of degree d and $v(c)$ its order along the locus of double conics. If d is odd then $\nu(c)\chi_9$ is a Siegel modular form with order $v(c) - (d-1)/2$ along the hyperelliptic locus. If d is even, then the order of $\nu(c)$ is $v(c) - d/2$.*

We formulate a corollary. Let $M_{i,j,k}(\Gamma_3)^{(m)}$ be the space of Siegel modular forms of weight (i, j, k) vanishing with multiplicity $\geq m$ at infinity. (The weight (i, j, k) corresponds to the irreducible representation of GL_3 of highest weight $(i+j+k, j+k, k)$.) Moreover, let $\mathcal{C}_{d,\rho}(-m\, DC)$ be the vector space of concomitants of type (d, ρ) that have order $\geq m$ along the locus of double conics. (Type (d, ρ) means belonging to an irreducible representation U of highest weight ρ occurring in $\mathrm{Sym}^d(\mathrm{Sym}^4(V))$.)

Corollary 9.4 *The exists an isomorphism*

$$\mathcal{C}_{d,\rho}(-m\, DC) \xrightarrow{\sim} M^{(d-2m)}_{\rho_1-\rho_2,\rho_2-\rho_3,\rho_3+9(d-2m)}$$

given by $c \mapsto \nu(c)\chi_9^{d-2m}$.

This allows now the construction of Siegel modular forms and Teichmüller modular forms of degree 3. In fact, in principle, all of them. As a simple example, we decompose

$$\mathrm{Sym}^2(\mathrm{Sym}^4(V)) = V[8,0,0] + V[6,2,0] + V[4,4,0]\,.$$

The concomitant corresponding to $U = V[8,0,0]$ yields via ν the symmetric square of $\nu(f)$. The concomitant corresponding to $V[6,2,0]$ yields a form in that after multiplication by χ_{18} becomes a holomorphic form in $S_{4,2,16}$ vanishing with multiplicity 2 at infinity. Similarly, the concomitant c corresponding to $U = V[4,4,0]$ yields a cusp form $\nu(c)\chi_{18} \in S_{0,4,16}$ vanishing with multiplicity 2 at infinity. We refer for more examples to [9].

The method also allows to treat the positive characteristic case. We hope to come back to it on another occasion.

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