# Siegel Modular Forms
## Lecture #1

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# Elliptic Modular Forms

The group $\mathrm{SL}(2, \mathbf{R})$ acts on the upper half space

$$\mathcal{H} = \{\tau \in \mathbf{C} : \mathrm{Im}(\tau) > 0\}$$

via $\tau \mapsto (a\tau + b)(c\tau + d)^{-1}$ for

$$(a, b; c, d) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{R})$$

The elements $\pm 1_2$ act trivially and

$$\mathrm{PSL}(2, \mathbf{R}) = \mathrm{Aut}(\mathcal{H}),$$

the biholomorphic automorphism group.

$\mathrm{SL}(2, \mathbf{Z}) \subset \mathrm{SL}(2, \mathbf{R})$ is a discrete subgroup.

A holomorphic function $f : \mathcal{H} \to \mathbf{C}$ is a modular form of weight $k \in \mathbf{Z}$ on $\mathrm{SL}(2, \mathbf{Z})$ if

1)

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k f(\tau)$$

for all $(a, b; c, d) \in \mathrm{SL}(2, \mathbf{Z})$;

then $f(\tau + 1) = f(\tau)$, hence $f$ has a Fourier series

$$f = \sum_n a(n) q^n \qquad \text{with} \quad q = e^{2\pi i\tau}$$

2) $f = \sum_{n \geq 0} a(n) q^n$

" $f$ is holomorphic at $\infty$"

$M_k$: the $\mathbf{C}$-vector space of modular forms of weight $k$.

$$S_k = \{f \in M_k : f = \sum_{n \geq 0} a(n) q^n, a(0) = 0\}$$

the subspace of cusp forms.

$$f \in M_k,\ g \in M_l \Longrightarrow \quad f\,g \in M_{k+l}$$

$\bigoplus_k M_k$, the graded algebra of modular forms

Note: $-1 \in \mathrm{SL}(2, \mathbf{Z})$, so $M_k = (0)$ for $k$ odd.

Do such modular forms exist?

$$E_4 = \frac{1}{2} \sum_{m,n \in \mathbf{Z},(m,n)=1} (m\tau + n)^{-4} \quad \in M_4$$

$$E_6 = \frac{1}{2} \sum_{m,n \in \mathbf{Z},(m,n)=1} (m\tau + n)^{-6} \quad \in M_6$$

More generally, for even $k \geq 4$ we have the Eisenstein series

$$E_k = \frac{1}{2} \sum_{(m,n)=1} (m\tau + n)^{-k} \in M_k$$

with Fourier series

$$E_k = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n)\, q^n$$

with $\sigma_k(n) = \sum_{1 \leq d | n} d^k$ and with $B_i$ the $i$th Bernoulli number;

Recall the Bernoulli numbers are defined by

$$\sum B_k \frac{t^k}{k!} = \frac{t}{e^t - 1} = 1 - \frac{1}{2}t + \frac{1}{12}t^2 - \frac{1}{720}t^4 + \ldots$$

These numbers satisfy for $n \in \mathbf{Z}_{\geq 1}$

$$\zeta(1-2n) = -\frac{B_{2n}}{2n}$$

We thus find

$$E_4 = 1 + 240\,q + 2160\,q^2 + \cdots$$

$$E_6 = 1 - 504\,q + 10632\,q^2 - \cdots$$

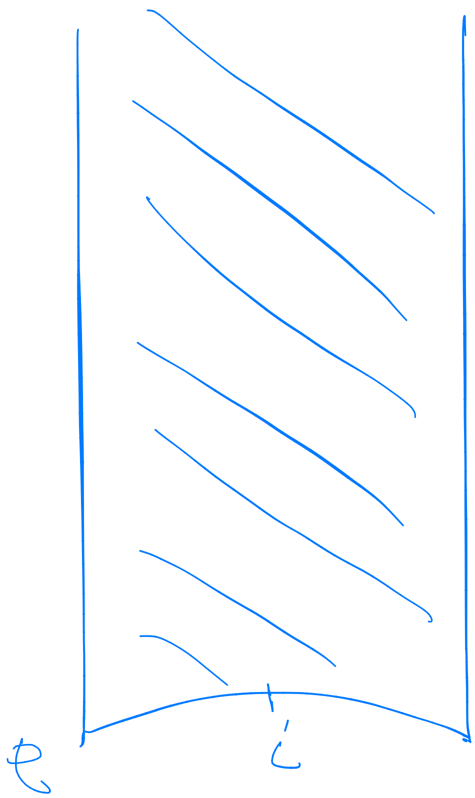
We define

$$\Delta = \frac{E_4^3 - E_6^2}{1728} = q - 24\,q^2 + 252\,q^3 - 1472 q^4 + \cdots$$

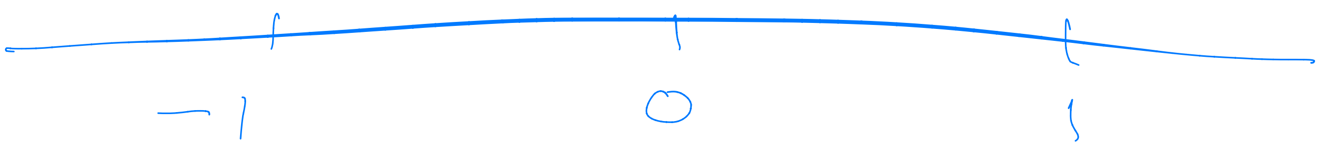
and find $\Delta \in S_{12}$.

A fundamental domain for the action of $\mathrm{SL}(2,\mathbf{Z})$ on $\mathcal{H}$ is

$$\mathcal{F} = \{\tau = x + iy \in \mathcal{H} : |x| \leq 1/2, |\tau| \geq 1\}$$





The group $\mathrm{SL}(2, \mathbf{Z})$ does not act effectively; $-1_2$ acts trivially. The group $\mathrm{PSL}(2, \mathbf{Z})$ does not act freely.

$\mathrm{PSL}(2, \mathbf{Z})$ has fixed points $P$. Representatives are:

$$P = \rho = e^{2\pi i/3}, \quad \#\mathrm{Stabilizer} = e_P = 3$$

and

$$P = i = \sqrt{-1} \quad \#\mathrm{Stabilizer} = e_P = 2$$

**Proposition 1.** *Let $f \in M_k$, $f \neq 0$. Then*

$$\mathrm{ord}_\infty(f) + \sum_{P \in \Gamma \backslash \mathcal{H}} \frac{1}{e_P} \mathrm{ord}_P(f) = \frac{k}{12}$$

*Proof.* Integrate $f'/f$ over the boundary of $\mathcal{F}$.

Example.

$$E_4: \qquad 0 + \frac{1}{3} + 0 = \frac{1}{3}$$

$$E_6: \qquad 0 + 0 + \frac{1}{2} = \frac{1}{2}$$

Similarly

$$\Delta: \qquad 1 + 0 + 0 = 1$$

So $\Delta$ has no zeros in $\mathcal{H}$.

We get an isomorphism (for $k \geq 12$)

$$M_{k-12} \overset{\sim}{\longleftrightarrow} S_k, \quad f \leftrightarrow \Delta f$$

Using this Proposition we see

$$\dim M_k = 0 \quad k < 0$$

and

$$M_0 = \mathbf{C}, \quad M_2 = (0)$$

and $\dim M_k = 1$ for $k = 4, 6, 8, 10$ and $\dim M_{12} = 2$. For example, for $k = 8$ we get $k/12 = 2/3$. This can only come from $e_P$ for $P = \rho$. Hence if $f \in M_8$ then $f/E_4^2$ constant.

Similarly, if $f \in M_{12}$ then $f - a(0)\, E_4^3 \in S_{12} \sim M_0$, hence $M_{12} = \langle E_4^3, \Delta \rangle$.

**Conclusion 1.** *We have*

$$\oplus_k M_k = \mathbf{C}[E_4, E_6]$$

$E_4$ and $E_6$ satisfy no non-trivial algebraic relation as looking at their zeros ($\tau = \rho$ and $\tau = i$ shows.

Modular forms show up everywhere in mathematics

Example 1. Take the lattice in $\mathbf{R}^8$

$$\mathbf{E}_8 = \{v \in \mathbf{Z}^8 \cup (\mathbf{Z} + 1/2)^8 : \sum_i v_i \quad \text{even.}\}$$

Put $a(n) = \#\{v \in \mathbf{E}_8 : \langle v, v \rangle = 2n\}$. Then $\sum_n a(n) q^n = E_4 \in M_4$; that is

$$a(n) = 240\, \sigma_3(n)$$

Example 2. Take a torus $E = \mathbf{C}/\mathbf{Z}+\mathbf{Z}\tau$ with $\tau \in \mathcal{H}$. Then $E$ is a compact Riemann surface and algebraic curve. We can construct meromorphic functions

$$y = 2\sum_{\lambda\in\Lambda} \frac{1}{(z+\lambda)^3}$$

with $\Lambda = \mathbf{Z}+\mathbf{Z}\tau$ and

$$x = \frac{1}{z^2} + \sum_{0\neq\lambda\in\Lambda} \frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2}$$

There exists $a$ and $b$ in $\mathbf{C}$ such that

$$y^2 = 4\,x^3 + ax + b$$

with $a = -60\,G_4(\tau)$, $b = -140\,G_6(\tau)$, where

$$G_k = \sum_{0\neq\lambda\in\Lambda} 1/\lambda^k = 2\zeta(k)\,E_k(\tau).$$

Example 3. Take an elliptic curve $E$ defined over $\mathbf{Q}$:

$$y^2 = 4x^3 + ax + b \quad \text{with} \quad \text{discr}(RHS) \neq 0.$$

Let $p$ be a prime $p$ such that $p$ does not divide $-16a^3 - 432b^2$. Define $a(p)$ by

$$p - a(p) = \#\{(x, y) \in \mathbf{F}_p^2 : y^2 = 4x^3 + ax + b\}$$

Then the $a(p)$ are Fourier coefficients of a modular form on $\Gamma_0(N)$ for $N$ the conductor of $E$.

The second example generalizes: Start with a compact Riemann surface $S$ of genus $g$.

Take a symplectic basis of the homology group $H_1(S,\mathbf{Z})$

$$\alpha_1,\ldots,\alpha_g,\beta_1,\ldots,\beta_g$$

with $\langle\alpha_i,\beta_j\rangle=\delta_{ij}$. The intersection matrix is

$$\begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$$

We know by Riemann: $\dim H^0(S,\Omega^1_S)=g$; basis $\omega_1,\ldots,\omega_g$. Can adapt the basis to satisfy

$$\int_{\alpha_j}\omega_i=\delta_{ij}$$

This yields a matrix

$$\tau = \left( \int_{\beta_j} \omega_i \right)$$

Riemann showed:

$$\text{i)}\, \tau^t = \tau, \quad \text{ii)}\, \mathrm{Im}(\tau) > 0\,.$$

So we look at

$$\mathcal{H}_g = \{\tau \in \mathrm{Mat}(g \times g, \mathbf{C}) : \tau^t = \tau, \mathrm{Im}(\tau) > 0\}$$

the Siegel upper half space.

But we can change the symplectic basis by a matrix $M \in \mathrm{GL}(2g, \mathbf{Z})$ with

$$M \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} M^t = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$$

We let $\Gamma_g = \mathrm{Sp}(2g, \mathbf{Z})$ be the group of all such $M$. This group acts on $\mathcal{H}_g$. Write

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, b, c, d$ blocks of size $g \times g$. The action is

$$\tau \mapsto (a\tau + b)(c\tau + d)^{-1}$$

We also have the notion of modular form.

A holomorphic function $f : \mathcal{H}_g \to \mathbf{C}$ for $g \geq 2$ satisfying

$$f((a\tau + b)(c\tau + d)^{-1}) = \det(c\tau + d)^k f(\tau)$$

for all $\gamma = (a, b; c, d) \in \Gamma_g$ is called a Siegel modular form.

The vector space $M_k(\Gamma_g)$ of such functions turns out be finite-dimensional.

Siegel modular forms also appear everywhere, e.g. as generating functions, in the study of moduli spaces of curves and abelian varieties, in relation to zeta functions of varieties, in mathematical physics, etc.

Here we used only $\det(c\tau + d)$. But $c\tau + d$ is $g \times g$-matrix. This naturally leads to vector-valued Siegel modular forms.

# The Siegel Modular Group

Let $L = \mathbf{Z}^{2g}$ with basis $e_1, \ldots, e_g, f_1, \ldots, f_g$. Let $\langle\,,\,\rangle$ be a non-degenerate alternating pairing with

$$\langle e_i, e_j \rangle = 0 = \langle f_i, f_j \rangle, \quad \langle e_i, f_j \rangle = \delta_{ij}$$

Define

$$\Gamma_g = \mathrm{Sp}(2g, \mathbf{Z}) = \mathrm{Aut}(L, \langle\,,\,\rangle)$$

We can write any element $\gamma \in \Gamma_g$ as

$$\gamma = (a, b; c, d) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d$ are $g \times g$ blocks.

These $\gamma = (a, b; c, d)$ satisfy

$$\gamma J \gamma' = J \qquad \text{with} \quad J = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$$

where $x'$ denotes the transpose; equivalently $\gamma' J \gamma = J$; that is,

$$ab' \quad \text{and} \quad cd' \quad \text{symmetric}, \quad ad' - bc' = 1_g.$$

Then $\mathrm{Sp}(2g, \mathbf{Z})$ is an algebraic group and we have $\mathrm{Sp}(2g, \mathbf{Q})$ and $\mathrm{Sp}(2g, \mathbf{R})$, again defined by $\gamma J \gamma' = J$.

Note: $\mathrm{Sp}(2g, \mathbf{R}) \subseteq \mathrm{SL}(2g, \mathbf{R})$. Indeed,

$$\mathrm{Pf}(J) = \mathrm{Pf}(\gamma J \gamma') = \det(\gamma)\mathrm{Pf}(J)$$

**Proposition 2.** *$\Gamma_g$ is generated by $J$ and the matrices*

$$\begin{pmatrix} 1_g & b \\ 0 & 1_g \end{pmatrix} \qquad b \quad \text{symmetric, integral}$$

Recall

$$\mathcal{H}_g = \{\tau \in \mathrm{Mat}(g\times g, \mathbf{C}) : \tau' = \tau, \mathrm{Im}(\tau) > 0\},$$

a complex manifold of dimension $g(g+1)/2$. We write

$$\tau = x + i\,y, \quad x, y \in \mathrm{Mat}(g \times g, \mathbf{R}),\ y > 0$$

Observe: $\tau \in \mathcal{H}_g \Leftrightarrow$

$$* \qquad \begin{cases} (\,\tau' \quad 1_g\,)\, J \begin{pmatrix} \tau \\ 1_g \end{pmatrix} = 0 \\ \frac{1}{2i}(\,\tau' \quad 1_g\,)\, J \begin{pmatrix} \bar{\tau} \\ 1_g \end{pmatrix} > 0 \end{cases}$$

indeed, $\tau - \tau' = 0$ and $(1/2i)(\tau' - \bar{\tau}) > 0$.

Now $\gamma \in \Gamma_g$ means $\gamma J \gamma' = J$. Substitute this in *; we get

$$(a\tau + b)'(c\tau + d) - (c\tau + d)'(a\tau + b) = 0$$

and

$$\frac{1}{2i}((a\tau + b)'(c\bar{\tau} + d) - (c\tau + d)'(a\bar{\tau} + b)) > 0$$

In fact rewrite the blue equation as

$$(c\tau + d)'(\gamma(\tau) - \overline{\gamma(\tau)})(c\bar{\tau} + d)$$

$$= (1/2i)(\tau - \bar{\tau}) = y\,.$$

**Claim 1.** $(c\tau + d)$ *is invertible.*

*Proof.* Let $\xi \in \mathbf{C}^g$ such that $(c\tau + d)\xi = 0$. By the last eqn we have $\xi' y \bar{\xi} = 0$, hence $\xi = 0$ because $y > 0$.

The red equation says that

$$\gamma(\tau) = (a\tau + b)(c\tau + d)^{-1} \quad \text{is symmetric}$$

and the blue one that $\mathrm{Im}(\gamma(\tau)) > 0$. Hence $\mathrm{Sp}(2g, \mathbf{R})$ acts on $\mathcal{H}_g$. We note the formula

$$\gamma(\tau) - \overline{\gamma(\tau)} = ((c\tau + d)')^{-1}(\tau - \overline{\tau})(c\tau + d)^{-1}$$

so

$$\det \mathrm{Im}(\gamma(\tau)) = |\det(c\tau + d)|^{-2} \det(y)$$

If $\gamma \in \mathrm{Sp}(2g, \mathbf{R})$ acts trivially then

$$\tau(c\tau + d) - a\tau - b = 0$$

hence $\tau$ is not general unless $\gamma = \pm 1_{2g}$. So $\mathrm{Sp}(2g, \mathbf{R})/\langle \pm 1 \rangle$ acts effectively.

The stabilizer of a point, say $\tau = i\,1_g \in \mathcal{H}_g$:

$$\mathrm{Stab}_{i_g} = \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{Sp}(2g, \mathbf{R}), aa' + bb' = 1\}$$

and this is isomorphic to (via $\mapsto a + bi$)

$$U_g = \{u \in \mathrm{Mat}(g \times g, \mathbf{C}) : u'\,\bar{u} = 1_g\}$$

the unitary group. The action of $\mathrm{Sp}(2g, \mathbf{R})$ on $\mathcal{H}_g$ is transitive: given $\tau = x + iy$ write $y = a'a$ and put $b' = xa^{-1}$. Then

$$\begin{pmatrix} a' & b \\ 0 & a^{-1} \end{pmatrix} : i_g \mapsto (a'i_g + b')a = x + iy = \tau$$

So

$$\mathrm{Sp}(2g, \mathbf{R})/U_g \longleftrightarrow \mathcal{H}_g, \quad \gamma U_g \mapsto \gamma(i_g)$$

The subgroup $K = U_g$ is a so-called maximal compact subgroup. Let $B$ be the Borel subgroup of $\mathrm{Sp}(2g, \mathbf{R})$ of matrices $(a, b; 0, d)$ with $a$ upper triangular, $d$ lower. Let $N$ be its unipotent radical

$$N = \{\gamma \in B : \mathrm{diag}(\gamma = (1, \ldots, 1))\}$$

and set $A$ the algebraic torus of diagonal matrices. Then the map

$$\pi : K \times A \times N \to \mathrm{Sp}(2g, \mathbf{R}), \quad (k, a, n) \mapsto k \cdot a \cdot n$$

is a diffeomorphism (Iwasawa decomposition).

Moreover, the map $\mathrm{Sp}(2g, \mathbf{R}) \to \mathcal{H}_g$ given by $\gamma \mapsto \gamma(i_g)$ is diffeomorphic to the projection $K \times A \times N \to A \times N$.

**Lemma 1.** *If* $\Gamma \subset \mathrm{Sp}(2g, \mathbf{R})$ *is a discrete subgroup then* $\Gamma$ *acts properly discontinuously.*

This means: if $C_1$, $C_2$ are compact in $\mathcal{H}_g$ then

$$\{\gamma \in \Gamma : \gamma(C_1) \cap C_2 \neq \emptyset\}$$

is finite.

*Proof.* If $\gamma(C_1) \cap C_2 \neq \emptyset$ then there exist $\gamma_i \in \pi^{-1}(C_i)$ such that $\gamma = \gamma_1 \gamma_2^{-1}$. But the map

$$\pi : K \times A \times N \to A \times N$$

is a proper map, hence the image of $\pi^{-1}(C_1) \times \pi^{-1}(C_2)$ under

$$(\gamma_1, \gamma_2) \mapsto \gamma_1 \gamma_2^{-1}$$

is compact, hence intersection with $\Gamma$ finite.

Siegel proved:

**Proposition 3.** *The group* $\mathrm{Sp}(2g, \mathbf{R})/\langle \pm 1_{2g} \rangle$ *is the biholomorphic automorphism group of the domain* $\mathcal{H}_g$*.*

In the case $g = 1$ we have

$$\mathcal{H}_1 \cong \{z \in \mathbf{C} : |z| < 1\}$$

This generalizes. Define

$$D_g = \{z \in \mathrm{Mat}(g \times g, \mathbf{C}) : z \cdot \bar{z} < 1_g\}$$

Here $z \cdot \bar{z} < 1_g$ means $1_g - z\bar{z}$ defines a positive definite hermitian metric on $\mathbf{C}^g$. Then

$$\mathcal{H}_g \xrightarrow{\sim} D_g$$

via the Cayley transform

$$\tau \mapsto (\tau - i1_g)(\tau + i1_g)^{-1}$$

Here $D_g$ is a so-called bounded symmetric domain. It has a symmetry $z \leftrightarrow -z$

corresponding to $\tau \leftrightarrow -\tau^{-1}$. Using the transitive action every point has a symmetry.

# A fundamental domain

We first look at 0-dim. cusps: for $g = 1$ we look at the orbits of $\mathrm{SL}(2, \mathbf{Z})$ on $\mathbf{P}^1(\mathbf{Q})$ with $\mathbf{P}^1(\mathbf{Q})$ lying in the boundary of $\mathcal{H}_1$. Such points are given by co-prime pairs $(b, d)$, that is, pairs $(b, d) \in \mathbf{Z}^2$ such that there exist $a, c$ such that $(a, b; c, d) \in \mathrm{SL}(2, \mathbf{Z})$. The pair $(1, 0)$ represents the cusp at $\infty$.

We generalize. We look at pairs $(b, d)$ of integral matrices of size $g$ such that there exists a matrix $(a, b; c, d) \in \Gamma_g$:

$$\Sigma = \{(u, v) : \exists \gamma = \begin{pmatrix} * & u \\ * & v \end{pmatrix} \in \Gamma_g\}$$

the set of 0-dimensional cusps. The group $\Gamma_g$

acts transitively on $\Sigma$ via

$$(u, v) \mapsto (au + bv, cu + dv)$$

We will now define for $\sigma = (u, v) \in \Sigma$, $\tau \in \mathcal{H}_g$ a ‘distance function’. Put

$$\mu(\sigma, \tau) = \frac{\det(y)}{|\det(-v\tau + u)|^2}$$

Example: For $\sigma = (1_g, 0_g)$ we have

$$\mu(\sigma, \tau) = \det(y)\,.$$

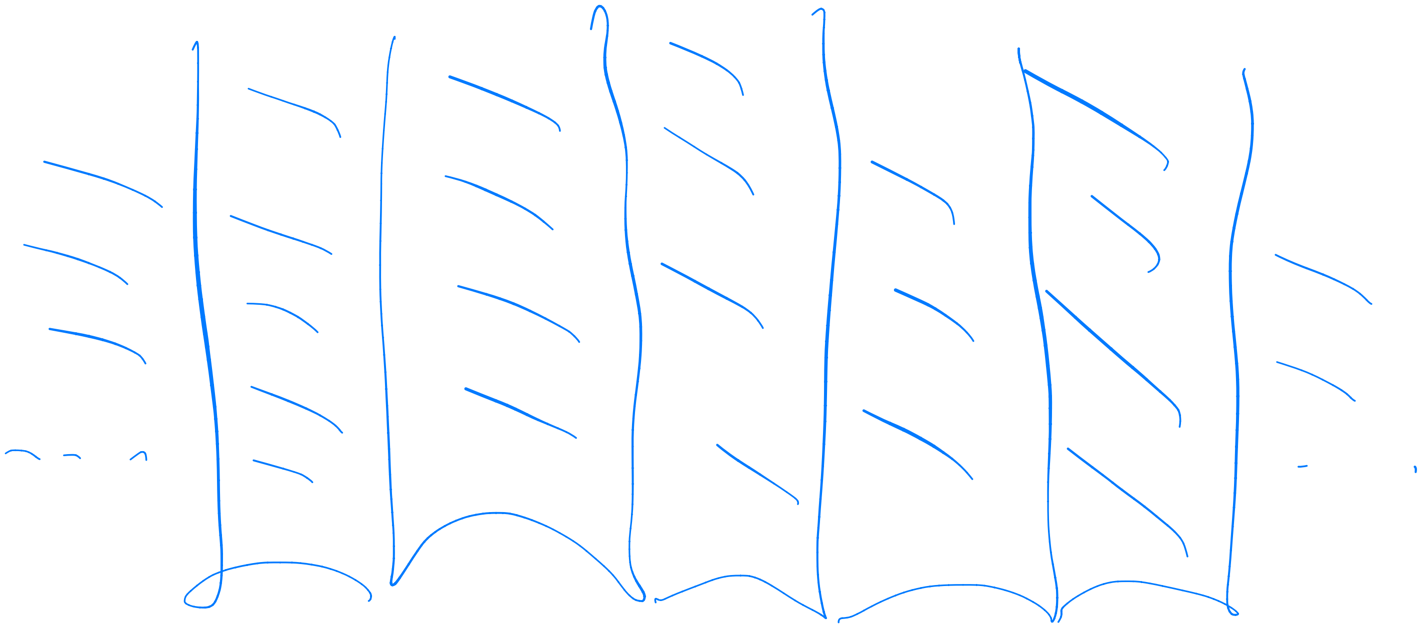
**Lemma 2.** *We have $\mu(\gamma\sigma, \gamma\tau) = \mu(\sigma, \tau)$.*

$1/\mu(\sigma, \tau)$ measures the distance to the $\sigma$.

Define the sphere of influence of the cusp at infinity as

$$V_\infty = \{\tau \in \mathcal{H}_g : \mu(\infty, \tau) \geq \mu(\sigma, \tau), \forall \sigma \in \Sigma\}$$

For $g = 1$ this set $V_\infty$ consists of the usual fundamental domain and its translates under $\tau \mapsto \tau + n$ with $n \in \mathbf{Z}$.



The condition $\mu(\infty, \tau) \geq \mu(\sigma, \tau)$ translates as

$$|\det(c\tau + d)| \geq 1 \quad \forall \gamma \in \Gamma_g$$

Since $\Gamma_g$ acts transitively on $\Sigma$ we get a fundamental domain by taking the quotient

$$V_\infty / P_\infty,$$

$$P_\infty = \{\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma_g\}$$

the isotropy group of $\infty$, a semi-direct product of $\mathrm{GL}(g, \mathbf{Z})$ with $S$, the addive group of symmetric integral matrices of size $g$.

The action of $\mathrm{GL}(g, \mathbf{Z})$ on $y = \mathrm{Im}(\tau)$ is

$$y \mapsto aya'$$

that is: reduction of quadratic forms.

**Proposition 4.** *A fundamental domain $\mathcal{F}_g$ is given by*

$$\mathcal{F}_g = \{x + iy \in V_\infty : |x_{ij}| \le 1/2, y \textit{ reduced}\}$$

*reduced* means Minkowski reduced:

- $u'yu \ge y_{kk}$ for $k = 1, \ldots, g$ for all $u \in \mathbf{Z}^g$ with $(u_k, \ldots, u_g)$ primitive.

- $y_{k,k+1} \ge 0$ for $k = 1, \ldots, g-1$

For $g = 2$ Minkowski reduced means

$$y_{22} \ge y_{11} \ge 2y_{12} \ge 0$$

We check that $\mathcal{F}_g$ is connected and that $\Gamma_g$-equivalent points lie on the boundary.

# Some literature

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