Siegel Modular Forms
Lecture #10

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L-series

For $f = \sum a(n)q^n \in M_k(\Gamma_1)$, a normalized eigenform, we consider

$$L_f(s) = \sum_{n>0} a(n)\, n^{-s}$$

It has an Euler product

$$L_f(s) = \prod_p (1 - a(p)p^{-s} + p^{k-1-2s})^{-1}$$

with Euler factor

$$1 - a(p)X + p^{k-1}X^2 = (1 - \beta X)(1 - \overline{\beta} X)$$

This converges for cusp forms for $\mathrm{Re}(s) > k/2+1$. For $f \in S_k(\Gamma_1)$ the function $L_f(s)$ admits a holomorphic continuation to all of $\mathbf{C}$ and satisfies a functional equation

$$\Lambda(f,s) = (-1)^{k/2}\Lambda(f,k-s)$$

with $\Lambda(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s)$.

If $f \in M_\rho(\Gamma_g)$ is an eigenform of the Hecke algebra with eigenvalues $\lambda_f(T)$ for $T \in H(\Gamma, G)$, then we get via $T \mapsto \lambda_f(T)$ an element of

$$\mathrm{Hom}(H_p^0, \mathbf{C}) \cong (\mathbf{C}^*)^{g+1}/W_G$$

A representative $g+1$-tuple $(\alpha_0, \ldots, \alpha_g)$ is called the p-Satake parameters of f.

Remark 1. *The ring* $\mathbf{Z}[Y]^{W_G}$ *is the representation ring of the dual group* $\hat{G} = \mathrm{GSpin}_{2g+1}$ *of* $G = \mathrm{GSp}(2g)$ *given by the dual root datum.*

Last time we gave formulas expressing eigenvalues of the Hecke operators in terms of the p-Satake parameters:

$$\lambda(p) = \alpha_0(1 + \sigma_1 + \ldots + \sigma_g)$$

where $\sigma_j = \sigma_j(\alpha_1, \ldots, \alpha_g)$, the jth elementary symmetric function of $\alpha_1, \ldots, \alpha_g$ and

$$\lambda_i(p^2) = \sum_{j,k \geq 0, j+i \leq k} m_{k-j}(i) p^{-\binom{k-j+1}{2}} \alpha_0^2 \sigma_j \sigma_k$$

We can use the p-Satake parameters to construct several L-functions.

The Spinor L-Function

This L-function is defined as an Euler product

$$Z_f(s) = \prod_p Z_{f,p}(p^{-s})^{-1}$$

where the Euler factor $Z_{f,p}(t)$ is given by

$$Z_{f,p}(t) = \prod_I (1 - \alpha_0 \alpha_I \, t),$$

with I running over the 2^g subsets of $\{1, \ldots, g\}$ and α_I stands for

$$\alpha_I = \prod_{i \in I} \alpha_i$$

For $g = 1$ and $f \in S_k(\Gamma_1)$ upon writing

$$a(p) = \beta + \overline{\beta} \quad \text{with } \beta\overline{\beta} = p^{k-1}$$

we get

$$(1 - \alpha_0 t)(1 - \alpha_0\alpha_1 t) = (1 - \beta t)(1 - \overline{\beta} t)\,,$$

the usual L-factor.

The meromorphic continuation and functional equation are known for $g = 1, 2$ and $g = 3$, but this is open for $g \geq 4$.

There is a compatibility with the Siegel operator. If $f \in M_k(\Gamma_g)$ is an eigenform, then $\Phi(f) \in M_k(\Gamma_{g-1})$ is an eigenform and if $\Phi(f) \neq 0$ then

$$Z_f(s) = Z_{\Phi(f)}(s)\, Z_{\Phi(f)}(s + g - k)$$

Example. The Eisenstein series of weight k and degree g has p-Satake parameter

$$(1, p^{k-1}, \ldots, p^{k-g})$$

For $g = 2$ and even weight $k \geq 4$ we have

$$Z_{\psi_k}(s) = \zeta(s)\zeta(s-k+1)\zeta(s-k+2)\zeta(s-2k+3)$$

and $\Phi(\psi_k) = E_k$, the Eisenstein series of degree 1 with L-function $\zeta(s)\zeta(s-k+1)$, so

$$Z_{\psi_k}(s) = L(E_k, s)L(E_k, s-k+2)\,.$$

An example where spinor L-functions occur in nature is the L-function associated to the first cohomology of a simple abelian surface.

This is analogue to the L-functions of elliptic curves defined over $\mathbf{Q}$. Here we have the modularity of elliptic curves (proved by Wiles, Taylor-Wiles, Breuil and others (1995-2001); if E is an elliptic curve over $\mathbf{Q}$ with conductor N then there is a normalized eigenform $f \in S_2(\Gamma_0(N))$ such that $L(E, s) = L(f, s)$.

There is a conjecture due to Brumer and Kramer making a conjecture of Yoshida more precise (2014):

Conjecture 1. *Let X be an abelian surface defined over $\mathbf{Q}$ of conductor N with $\mathrm{End}(X) = \mathbf{Z}$. Then there exists a Siegel (para-)modular form $f \in S_2(\Gamma_2(N))$ that is an eigenform with rational eigenvalues such that*

$$L(X, s) = Z_f(s)$$

For a good prime p the p-factor of $L(X, s)$ is given by the

$$\det(1 - tF_p) \quad \text{acting on } T_\ell(X)$$

with F_p the geometric Frobenius automorphism and T_ℓ the Tate module for $\ell \neq p$.

If $X = \mathrm{Jac}(C)$ then $L_p(X, t)$ can be given by zeta function of $C \otimes \mathbf{F}_p$:

$$\exp\left(\sum_{n=1}^{\infty} \#C(\mathbf{F}_{p^n})\frac{t^n}{n}\right) = \frac{L_p(X,t)}{(1-t)(1-pt)}$$

The paramodular group $\Gamma_2(N)$ is defined by the congruence condition

$$\mathrm{Sp}(4,\mathbf{Q}) \cap \begin{pmatrix} \mathbf{Z} & N\mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & N^{-1}\mathbf{Z} \\ \mathbf{Z} & N\mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ N\mathbf{Z} & N\mathbf{Z} & N\mathbf{Z} & \mathbf{Z} \end{pmatrix}$$

Here is an example (due to Brumer et al). Let C be the curve of genus 2 defined by

$$y^2 + (x^3 + x^2 + x + 1)y = -x^2 - x$$

It has conductor 277. There is a unique non-lifted eigenform $f \in S_2(\Gamma_2(277))$ with

$L(\mathrm{Jac}(C), s) = Z_f(s)$. There is also a unique isogeny class of abelian surfaces of conductor 277.

For prime $N < 277$ the conjecture holds: there are no such abelian surfaces and no such eigenforms as shown by Poor and Yuen.

Similar results are obtained for $N = 353$ and $N = 587$.

Standard Zeta Function

We define for $f \in S_\rho(\Gamma_g)$ (or $M_\rho(\Gamma_g)$)

$$D_f(s) = \prod_p \frac{1}{D_{f,p}(p^{-s})}$$

where

$$D_{f,p}(t) = (1-t)\prod_{i=1}^{g}(1-\alpha_i t)(1-\alpha_i^{-1} t)$$

For $g = 1$ and a normalized eigenform $f = \sum_n a(n)q^n \in S_k(\Gamma_1)$ this zeta function is related to the Rankin zeta function:

$$\frac{D_f(s-k+1)}{\prod_p (1+p^{-s+k+1})} = \sum_n a(n^2)n^{-s}$$

This converges absolutely for $\mathrm{Re}(s) > k+g+1$ and for cusp forms for $\mathrm{Re}(s) > g+1$.

We know by Böcherer that $D_f(s)$ can be extended meromorphically to the whole complex plane with finitely many poles and when provided with the right Gamma factors it satisfies a functional equation relating the values at s and $1-s$. In fact, put

$$\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2), \quad \Gamma_{\mathbf{C}}(s) = 2\,(2\pi)^{-s}\Gamma(s)$$

and set

$$\Lambda_f(s) = \Gamma_{\mathbf{R}}(s+\epsilon)\prod_{j=1}^{g}\Gamma_{\mathbf{C}}(s+k-j)D_f(s)$$

Then $\Lambda_f(1-s) = \Lambda_f(s)$.

We have for an eigenform $f \in S_k(\Gamma_g)$ with the property that $\Phi(f) \neq 0$ that

$$D_f(s) = \zeta(s+k-g)\zeta(s-k+g)D_{\Phi(f)}(s)$$

proven by Zharkovskaya.

Example. Take the Leech lattice L and build the harmonic theta series

$$f = \sum_{l \in L^{24}} \det(G_l) e^{\pi i \mathrm{Tr}(G_l \tau)}.$$

Freitag constructed this form $f \in S_{13}(\Gamma_{24})$ in 1982. It is an eigenform and Chenevier showed

$$D_f = \prod_{i=0}^{11} L_\Delta(s+11-i) \prod_{j=0}^{24} \zeta(s+12-j)$$

with $\Delta \in S_{12}(\Gamma_1)$.

Liftings

If $f = \sum a(n)q^n \in S_k(\Gamma_1)$ is a normalized eigenform we have for every prime:

$$|a(p)| \leq 2\, p^{(k-1)/2}$$

This was shown for $f \in S_2(\Gamma_0(N))$ by Eichler and in general by Deligne by reducing it to the Weil conjectures in 1968 and then by proving the Weil conjectures in 1974. This means that the roots of the Euler factor

$$1 - a(p)X + p^{k-1}X^2$$

have absolute value $1/p^{(k-1)/2}$.

For $g = 2$ the analogous polynomial is the spinor Euler factor $Z_{f,p}(X)$

$$1 - \lambda(p)X + (\lambda(p)^2 - \lambda(p^2) - p^{2k-4})X^2 \\ -\lambda(p)p^{2k-3}X^3 + p^{4k-6}X^4$$

It was tacitly assumed in the 1960-70s that the the roots of this polynomial would have absolute values

$$1/p^{(2k-3)/2}$$

But then Saito and Kurokawa found that for $\chi_{10} \in S_{10}(\Gamma_2)$ the first eigenvalues $\lambda(p)$ satisfy

$$\lambda(p) = p^8 + a(p) + p^9$$

with $a(p)$ given by

$$f_{18} = \sum a(n)q^n \in S_{18}(\Gamma_1)\,.$$

Here

$$a(p) = \beta + \overline{\beta} \quad \text{with } |\beta| = p^{17/2}.$$

In other words:

$$L(\chi_{10}, s) = \zeta(s-9)L(f_{18}, s)\zeta(s-8)$$

with $L(\chi_{10}, s)$ the spinor L-function.

On the basis of this Kurokawa conjectured a lift

$$S_{2k-2}(\Gamma_1) \to S_k(\Gamma_2)\,.$$

The Maass Subspace: Spezialschar

Maass identified a subspace (k even)

$$M_k^*(\Gamma_2) \subset M_k(\Gamma_2)$$

$$M_k^*(\Gamma_2) = \{F = \sum a(N) q^N \in M_k(\Gamma_2) : \\ a(N) \text{ depends only on } d(N) \text{ and } e(N)\}$$

where for

$$N = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix},$$

also written as $[n, r, m]$ or $nx^2 + rxy + my^2$, we have the discriminant and content

$$d(N) = 4mn - r^2, \quad e(N) = \text{g.c.d.}(n, r, m)$$

Equivalently: $F \in M_k^*(\Gamma_2) \iff$

$$a([n,r,m]) = \sum_{0<d|(n,r,m)} d^{k-1} a([1, \frac{r}{d}, \frac{mn}{d^2}])$$

The Maass space is invariant under the Hecke algebra.

Theorem 1. *There is a* $1{-}1$ *correspondence between Hecke eigenspaces in* $S_{2k-2}(\Gamma_1)$ *and Hecke eigenspaces in* $S_k^*(\Gamma_2)$ *given by*

$$f \in S_{2k-2}(\Gamma_1) \text{ corresponds to } F \in S_k(\Gamma_2)$$

$$\iff L(F,s) = \zeta(s-k+1)L(f,s)\zeta(s-k+2)$$

This theorem is due to Maass and Andrianov with simplifications by Zagier.

Proof. Write $F \in M_k(\Gamma_2)$ as a Fourier-Jacobi series

$$F = \sum_m \varphi_m(\tau_1, z) q_2^m$$

with $\varphi_m \in J_{k,m}$, the space of Jacobi forms of weight k and index m. On the space $J_{k,m}$ we have a sort of Hecke operators

$$V_l : J_{k,m} \to J_{k,ml}$$

The action on $\varphi_m = \sum_{n,r} c(n,r) q^n \zeta^r$ with $q = e^{2\pi i \tau_1}$ and $\zeta = e^{2\pi i z}$ is

$$\varphi_{m|_{k,m}V_l} = \sum_{n,r} \sum_{d|(n,r,l)} d^{k-1} c(\frac{nl}{d^2}, \frac{r}{d}) q^n \zeta^r$$

One checks using generators of Γ_2 that for

$\varphi \in J_{k,1}$ the expression

$$v(\varphi) = \sum_{m \geq 0} \varphi_{|V_m}(\tau, z) q_2^m$$

is a Siegel modular form in $M_k(\Gamma_2)$. We thus get maps

$$J_{k,1} \xrightarrow{v} M_k(\Gamma_2) \to \oplus_m J_{k,m} \xrightarrow{\mathrm{pr}} J_{k,1}$$

The composition is the identity. So v is injective and the image consists of the modular forms $F = \sum_m \varphi_m q_2^m$ with

$$\varphi_m = \varphi_{1|V_m}$$

This implies a relation for the Fourier

coefficients:

$$a([n,r,m]) = \sum_{d|(n,r,m)} d^{k-1} c(\frac{4mn-r^2}{d^2})$$

with $c(N)$ given by

$$c(N) = \begin{cases} a([n,0,1]) & N = 4n \\ a([n,1,1]) & N = 4n-1 \end{cases}$$

In this way we see that the image is the Maass subspace because

$$a([n,r,m]) = \sum_{d|(n,r,m)} d^{k-1} a([nm/d^2, r/d, 1])$$

But it is known that we have isomorphisms

$$J_{k,1} \cong M^{+}_{k-1/2} \cong M_{2k-2}(\Gamma_1)$$

Here $M^+_{k-1/2}$ is the so-called Kohnen plus space in $M_{k-1/2}(\Gamma_0(4))$. It consists of modular forms on $\Gamma_0(4)$ which have the same transformation behavior as $\vartheta(\tau)^{2k-1}$ with

$$\vartheta = \sum_{n\in\mathbf{Z}} q^{n^2}.$$

The plus refers to:

$h = \sum_n c(n)q^n \in M_{k-1/2}(\Gamma_0(4))$ lies in $M^+_{k-1/2}$ if $c(n) = 0$ for $n \equiv 1$ or $\equiv 2 (\bmod 4)$.

The first isomorphism $M^+_{k-1/2} \xrightarrow{\sim} J_{k,1}$ is given by

$$f = \sum c(n)q^n \mapsto \sum_{n\equiv -r^2(\bmod 4)} c(n)\, q^{(n+r^2)/4}\zeta^r$$

So we have

$$M_k^*(\Gamma_2) \cong J_{k,1} \cong M_{k-1/2}^+$$

The identification

$$M_{k-1/2}^+ \cong M_{2k-2}(\Gamma_1)$$

is not canonical at all, but depends on a choice of a discriminant.

For a Hecke eigenform $f \in S_k(\Gamma_2)$ the fact that it lies in the Maass subspace can be read off from the spinor zeta function: whether it has a (simple) pole for $s = k$ (as observed by Evdokimov).

One final remark: for a Saito-Kurokawa lift F that is an eigenform and lift of $f \in$

$S_{2k-2}(\Gamma_1)$ the spinor zeta function is

$$\zeta(s-k+1)\zeta(s-k+2)L(f,s)$$

and the standard zeta function is

$$\zeta(s)L(f,s+k-1)L(f,s+k-2)\,.$$

Ikeda generalized the Saito-Kurokawa lift: if $f = \sum a(n)q^n \in S_{2k}(\Gamma_1)$ is an eigenform and if $g \equiv k (\mathrm{mod}\, 2)$ then f lifts to an eigenform $F \in S_{k+g}(\Gamma_{2g})$ such that the standard zeta function of F is given by

$$\zeta(s) \prod_{j=1}^{2g} L(f, s+k+g-j)$$

If $a(p) = \beta + \overline{\beta}$ with $\beta\overline{\beta} = p^{2k-1}$, the Satake parameters of F are $(\alpha_0, \alpha_1, \ldots, \alpha_{2g})$ with

$$\alpha_0 = p^{gk-g(g+1)/2},$$

and for $i = 1, \ldots, g$

$$\alpha_i = \beta p^{i-1/2}, \quad \alpha_{g+i} = \beta^{-1} p^{i-1/2}\,.$$

Kohnen has given a description in terms of Fourier coefficients. He interprets as a linear map

$$S^+_{k+1/2} \to S_{k+g}(\Gamma_{2g})$$

sending $f = \sum c(n)q^n$ (sum over n with $(-1)^k n \equiv 0, 1 \pmod 4$) to $\sum a(N)q^N$ with $a(N)$ an explicit expression in the $c(n)$ and N with

$$a(N) = \sum_{d|f_N} d^{k-1}\phi(d, N)c(\frac{|D_N|}{d^2})$$

with $D_N = (-1)^g \det(2N)$ the discriminant of N and $D_N = D_{N,0}f_N^2$ with $D_{N,0}$ the corresponding fundamental discriminant and ϕ an explicit function depending on the quadratic space defined by reductions of N $\mathrm{mod}\ p$.

Example. Take $k = 6$ and $g = 2$. The Ikeda lift

$$S_{12}(\Gamma_1) \to S_8(\Gamma_4)$$

takes Δ to the Schottky form ϕ_8 vanishing on the Jacobian locus.

Example. Take $k = 6$ and $g = 6$. The Ikeda lift

$$S_{12}(\Gamma_1) \to S_{12}(\Gamma_{12})$$

sends Δ to a form that can also be obtained by theta functions for unimodular lattices, namely associated to the Leech lattice.

There are many more conjectures about liftings. Here is one.

Suppose $r, n \in \mathbf{Z}$ with $n + r \equiv k (\mathrm{mod} 2)$ and $f \in S_{2k}(\Gamma_1)$ a normalized eigenform.

The Ikeda lift F of f lives in $S_{k+n+r}(\Gamma_{2n+2r})$. For $h \in S_{k+n+r}(\Gamma_r)$ define $\mathcal{F}_{f,h}$ by

$$\int_{\Gamma_r \backslash \mathcal{H}_r} F \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \overline{h(-\overline{\tau_1})} \det \mathrm{Im}(\tau_2)^{k+n-1} d\tau_2$$

Here is the Miyawaki-Ikeda conjecture.

Conjecture 2. *If $\mathcal{F}_{f,h} \neq 0$ then $\mathcal{F}_{f,h}$ is an eigenform in $S_{k+n+r}(\Gamma_{2n+r})$ such that*

$$D_{\mathcal{F}_{f,h}}(s) = D_h(s) \prod_{j=1}^{2n} L(f, s+k+n-j)$$

Example. Take $k = 10$ and $n = r = 1$. For $0 \neq f \in S_{12}(\Gamma_3)$ we have

$$D_f(s) = L(f_{20}, s+9) D_\Delta(s) L(f_{20}, s+10)$$

with $f_{20} \in S_{20}(\Gamma_1)$ the normalized eigenform. We have

$$\dim S_{12}(\Gamma_3) = 1 = \dim S_{20}(\Gamma_1)$$

Maeda Conjecture

For $g = 1$ there is the Maeda conjecture. It says that $S_k(\Gamma_1)$ is an irreducible Hecke module; in other words, if $f \in S_k(\Gamma_1)$ is a normalized eigenform and $K_f = \mathbf{Q}(a(n), n = 1, 2, \ldots)$ the field of eigenvalues then

$$[K_f : \mathbf{Q}] = \dim S_k(\Gamma_1)$$

This conjecture is open, but has been checked up to very large k.

For $g = 2$ there is no direct analogue since we saw a rational subspace of $S_k(\Gamma_2)$, the Maass subspace. But we could consider the orthogonal complement $S'_k(\Gamma_2)$ of the Maass subspace. Skoruppa discovered that there are cases where this space splits: for $k = 24$ and $k = 26$. In both cases $\dim S_k(\Gamma_2) - \dim S'_k(\Gamma_2) = 2$.

But apart from these counter examples the Hecke module $S'_k(\Gamma_2)$ seems to be irreducible, at least up to $k = 150$ (Raum).

Literature

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