Siegel Modular Forms
Lecture $\#11$

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Siegel's Main Theorem

The theta function $\vartheta_{E_8}^{(1)} \in M_4(\Gamma_1)$ associated to the unimodular lattice $L = E_8$ satisfies

$$E_4 = \vartheta_L^{(1)}$$

with E_4 the Eisenstein series. This fact allows a very wide generalization found by Siegel. A special case is the following.

Let $L_1, \ldots, L_n$ be the isometry classes of the even unimodular lattices of rank $r \equiv 0 \pmod 8$. For each L_i and fixed degree g we have the theta series

$$\vartheta_{L_i}^{(g)} = \sum_{l \in L_i^g} e^{\pi i \mathrm{Tr}(G_l \tau)}$$

with $G_l = (l_i, l_j)_{1\le i,j\le g}$ the Gram matrix of $l = (l_1, \ldots, l_g)$. The Fourier coefficient at n of $\vartheta_{L_i} = \sum a_{L_i}(n) q^n$ is

$$a_{L_i}(n) = \#\{l \in L_i^g : G_l = 2n\}$$

We put

$$w_i = \frac{\frac{1}{\#\mathrm{Aut}(L_i)}}{\sum_{j=1}^n \frac{1}{\#\mathrm{Aut}(L_j)}}$$

Theorem 1. *The modular form*

$$\sum_{j=1}^{n} w_j\, \vartheta_{L_j}^{(g)} \in M_{r/2}(\Gamma_g)$$

is an eigenform for the Hecke algebra.

In fact, Andrianov calculated in 1980 explicitly the action of the Hecke operators on theta series. His formulas show that the above form is an eigenform.

Corollary 1. *The Eisenstein series* $E_{r/2} \in M_{r/2}(\Gamma_g)$ *for* $r/2 > g+1$ *satisfies*

$$E_{r/2} = \sum_{j=1}^{n} w_j \, \vartheta_{L_j}^{(g)} \, .$$

Proof. By induction. The case $g = 1$ is clear. We claim: the Eisenstein series is the unique eigenform f of degree g with $\Phi^g(f) = 1$; indeed, by induction $\Phi(f)$ is the unique form f_1 of degree $g-1$ which is an eigenform and with $\Phi^{g-1}(f_1) = 1$. Then $E_{r/2} - f$ is a cusp form. But then the eigenvalues for the Hecke operators are too big (for a cusp form) unless $E_{r/2} - f = 0$.

To see this last point, recall that for $g = 1$ we have for an eigenform $f = \sum a(n)q^n \in S_k(\Gamma_1)$ that $a(n) = O(n^{k/2})$, while for the Eisenstein series of weight k we have $a(n) = O(n^{k-1})$. Similarly, for an eigenform $f \in S_k(\Gamma_g)$ the eigenvalue $\lambda(p)$ satisfies

$$|\lambda(p)| \leq p^{gk/2-g(g+1)/2} \prod_{j=1}^{g} (1 + p^j)$$

Indeed, the function $|f(\tau)| \det(y)^{k/2}$ is Γ_g-invariant and has a maximum in $\mathcal{H}_g$, say at τ_0. Applying now the definition of $T(p)$,

$$T(p)f = p^{gk-g(g+1)/2} \sum_{i=1}^{\deg(T(p))} f(\gamma_i(\tau)) \det(d)^{-k}$$

with the γ_i of the form $(a, b; 0, d)$ and with

$\det(\gamma_i) = p$ we get using

$$|f(\gamma_i(\tau))| \leq \left(\frac{\det(y_0)}{\det(\mathrm{Im}(\gamma_i(\tau_0)))}\right)^{k/2} |f(\tau_0)|$$

and

$$\frac{\det(y_0)}{\det(\mathrm{Im}(\gamma_i(\tau_0))} \leq \det(\frac{1}{p}d'd)$$

that

$$|\lambda(p)f| \leq p^{kg/2-g(g+1)/2}\deg(T(p))$$

that is the bound

$$p^{kg/2-g(g+1)/2}\prod_{j=1}^{g}(1+p^j)$$

But for the Eisenstein series we have

$$\lambda(p) = \prod_{j=1}^{g}(1+p^{k-j})\,.$$

The Slope

Let $f = \sum a(n)q^n \in M_\rho(\Gamma_g)$. We define $\mathrm{ord}_\infty(f)$ as

$$\inf\{m \in \mathbf{Z} : n \text{ represents } m \text{ and } a(n) \neq 0\ \}$$

Another way to define it is by the Fourier-Jacobi series

$$f = \sum_{r=0}^{\infty} \varphi_r(\tau_1, z) q_g^r, \qquad q_g = e^{2\pi i \tau_{gg}}$$

where $\varphi_r = 0$ for $r < \mathrm{ord}_\infty(f)$ and $\varphi_m \neq 0$ for $m = \mathrm{ord}_\infty(f)$. It also equals the order of vanishing of f along the divisor D of the rank 1 partial compactification: locally D can be defined by the vanishing of q_g.

Definition 1. *For $f \in M_k(\Gamma_g)$ the slope of f is*

$$s(f) = \frac{k}{\mathrm{ord}_\infty(f)}\,.$$

The minimal slope for degree g is

$$\sigma(g) = \inf_{0 \neq f \in R_g} s(f)\,.$$

Example. $\sigma(1) = 12$. Every cusp form with $\mathrm{ord}_\infty(f) = m$ is divisible by Δ^m.

Example. $\sigma(2) = 10$. The surjective map

$$M_k^{\geq 2n}(\Gamma_2) \to \mathrm{Sym}^2(M_{k+2n}^{\geq n}(\Gamma_1))$$

and $\sigma(1) = 12$ implies $k + 2n \geq 12n$, hence $k \geq 10n$. Elements of $M_k^{\geq 2n}(\Gamma_2)$ vanish with multiplicity $\geq n$ along D. Hence $\sigma(2) \geq 10$.

But we have χ_{10} with slope 10. Using work of Eichler one can give bounds for the slope.

Proposition 1. *Let* $f \in S_k(\Gamma_g)$ *be non-zero. Assume that* $|f(\tau)|\,\det(y)^{k/2}$ *takes its maximum in* $\tau_0 = x_0 + iy_0$*. Then for every symmetric integral matrix* c *of size* g *we have*

$$\frac{k}{\mathrm{ord}_\infty(f)} \geq \frac{4\pi}{\mathrm{Tr}(c\, y_0^{-1})}$$

Proof. Fix τ. Look at $F(z) = f(\tau + zc)$ for $z \in \mathbf{C}$ with $\mathrm{Im}(z) > -\delta$ with appropriate δ. Then $F(z+1) = F(z)$, hence we can write

$$F(z) = \sum_{m=0}^{\infty} \left(\sum_{n:\mathrm{Tr}(n)=m} a(n) q^n \right) w^m$$

with $w = e^{2\pi i z}$ and $f = \sum a(n) q^n$.

We see that $w^{-m_0}F(z)$ with $m_0 = \mathrm{ord}_\infty(f)$ is holomorphic in a disc $|w| < e^{2\pi\delta}$. By the maximum principle it takes for $0 < \epsilon < \delta$ a maximum on the disc $|w| \le e^{2\pi\epsilon}$ on the boundary, say in z_ϵ with $\mathrm{Im}(z_\epsilon) = -\epsilon$, hence

$$|f(\tau)| \le e^{-2\pi m_0 \epsilon} |f(\tau + z_\epsilon c)|$$

On the other hand we have

$$|f(\tau_0 + z_\epsilon c)| \det(y_0 - \epsilon c)^{k/2} \le |f(\tau_0)| \det(y_0)^{k/2}$$

Comparing the red and blue inequalities for the case $\tau = \tau_0$ gives

$$\det(y_0)^{-k/2} \det(y_0 - \epsilon c)^{k/2} \le e^{-2\pi m_0 \epsilon}$$

that is

$$\det(1_g - \epsilon c y_0^{-1})^{k/2} \leq e^{-2\pi m_0 \epsilon}\,.$$

This holds for small ϵ; comparing linear terms we get

$$\mathrm{Tr}(c y_0^{-1}) \geq 4\pi m_0/k\,.$$

Corollary 2. *For* $0 \neq f \in S_k(\Gamma_g)$ *we have*

$$\mathrm{slope}(f) \geq \frac{2\pi\sqrt{3}}{\gamma_g}$$

with γ_g *the Hermite constant.*

Proof. Take $c = \xi'\xi$ with ξ such that $\xi' y_0^{-1}\xi$ assumes its minimum $\mu(y_0^{-1})$ for $\xi \in \mathbf{Z}^g \backslash (0)$. Geometry of numbers shows the existence of a constant γ_g, the Hermite constant, such that for all $y > 0$ we have

$$\mu(y) \leq \gamma_g \det(y)^{1/g}\,.$$

This gives $\mu(y_0)\mu(y_0^{-1}) \leq \gamma_g^2$. But since $\tau_0 \in F_g$ we know $\mu(y_0) \geq \sqrt{3}/2$, and we thus get

$$k/m_0 \geq \frac{4\pi}{\mathrm{Tr}(cy_0^{-1})} \geq \frac{2\pi\sqrt{3}}{\gamma_g^2}$$

A general inequality (of Blichfeldt) gives

$$\gamma_g \leq \frac{2}{\pi}\Gamma(2+\frac{g}{2})^{2/g}$$

Corollary 3. *We have*

$$\sigma(g) \geq \frac{\sqrt{3}\pi^3}{2}\Gamma(2+g/2)^{-4/g}$$

But we have more precise results on γ_g for small g:

g	1	2	3	4	5	6	7	8
γ_g^g	1	$4/3$	2	4	8	$64/3$	64	2^8
$\sigma(g) \geq$	10.8	8.1	6.8	5.4	4.7	3.9	3.3	2.7

Now the existence of $\Delta \in S_{12}(\Gamma_1)$, $\chi_{10} \in S_{10}(\Gamma_2)$, $\chi_{18} \in S_{18}(\Gamma_3)$ and $\phi_8 \in S_8(\Gamma_4)$ show

g	1	2	3	4
$\sigma(g) \leq$	12	10	9	8

And indeed, these listed values are the correct values of $\sigma(g)$. We also know $\sigma(5) = 54/7$ by Farkas et al.

Differential Forms on $\mathcal{A}_g$

For $\Gamma \subset \Gamma_g$ torsion-free of finite index the cotangent bundle to $\mathcal{A}_\Gamma = \Gamma \backslash \mathcal{H}_g$ is $\mathrm{Sym}^2(\mathbf{E})$. If we have a 1-form on $\mathcal{H}_g$ its pull-back to $\mathcal{H}_g$ is

$$\omega = \sum_{1 \leq i \leq j \leq g} \omega_{ij} d\tau_{ij}$$

or $\omega = \mathrm{Tr}(f\, d\tau)$ with f a symmetric $g \times g$ matrix satisfying

$$f(\gamma(\tau)) = (c\tau + d)' f(\tau)(c\tau + d)$$

for $\gamma = (a, b; c, d) \in \Gamma_g$. Such a form can be extended to the rank 1 partial compactification $\tilde{\mathcal{A}}_\Gamma^{(1)}$ using (local) coordinates (τ_1, z, q) with $\tau_1 \in \mathcal{H}_{g-1}$, $z \in$

$\mathbf{C}^{g-1}$ and $q = e^{2\pi i \tau_{gg}}$. We have $dq/q = 2\pi i d\tau_{gg}$. We thus see

$$\Omega^1_{\tilde{\mathcal{A}}^{(1)}_\Gamma}(\log D) = \mathrm{Sym}^2(\mathbf{E})$$

If we consider a smooth compactification $\tilde{\mathcal{A}}_\Gamma$ then the complement of $\tilde{\mathcal{A}}^{(1)}_\Gamma$ is of codimension > 1, hence such differential forms extend. We thus find for torsion-free Γ

$$\Omega^1_{\tilde{\mathcal{A}}_\Gamma}(\log D) = \mathrm{Sym}^2(\mathbf{E})$$

and for $d = g(g+1)/2 = \dim \mathcal{A}_\Gamma$ that

$$\Omega^d_{\tilde{\mathcal{A}}_\Gamma}(\log D) = \det(\mathbf{E})^{g+1}\,.$$

From this we see for example

$$\Omega^{d-1}_{\tilde{\mathcal{A}}_\Gamma} = \mathrm{Sym}^2(\mathbf{E}) \otimes \det(\mathbf{E})^{g+1}$$

and thus $\Omega^{d-1}_{\tilde{\mathcal{A}}_\Gamma} = \mathbf{E}_\rho$ with $\rho = (g+1, g+1, \ldots, g+1, g-1)$.

If Γ, a group commensurable with Γ_g, does not acts freely, one can show that the locus of singular points of $\mathcal{A}_\Gamma$ for $g > 1$ is of codimension ≥ 2, hence holomorphic p-forms extend to all of a resolution of $\mathcal{A}_\Gamma$.

Two questions:

1) do these forms extend over a compactification?

2) which irreps ρ occur in the algebra $\wedge^* \mathrm{Sym}^2(\mathrm{St})$?

As to 1) consider the rank-1 compactification with coordinates τ_{ij}, $z_j = \tau_{1j}$ with $1 \leq i, j \leq g-1$ and $q = e^{2\pi i \tau_{gg}}$.

Suppose that ρ occurs in the dual of $\wedge^p \mathrm{Sym}^2(\mathrm{St})$ and $f \in M_\rho(\Gamma_g)$. If dq/q does not occur in the p-form defined by f then the form extends. Also if f is a cusp form then the form extends.

If $f = \sum a(n) q^n$ is not a cusp form and n is singular and $a(n) \neq 0$, then assuming

$$n = \begin{pmatrix} n_1 & 0 \\ 0 & 0 \end{pmatrix}$$

we know that $a(n) = \rho(u) a(n)$ for all $u \in G$ with

$$G = \{ \begin{pmatrix} 1_{g-1} & * \\ 0 & * \end{pmatrix} \in \mathrm{GL}(g, \mathbf{C}) \}$$

We write the standard representation as $\mathrm{St} = S \oplus S'$ with $\dim S = g - 1$ and $\dim S' = 1$.

Then $V = \mathrm{Sym}^2(\mathrm{St})$ decomposes

$$V = \mathrm{Sym}^2(S) \oplus (S \otimes S') \oplus \mathrm{Sym}^2(S')$$

Lemma 1. *For $p < g(g+1)/2$ the space of invariants of G for the action on $\wedge^p V$ is contained in $\wedge^p \left(\mathrm{Sym}^2(S) \oplus S \otimes S'\right)$.*

Corollary 4. *If $f \in M_\rho(\Gamma)$ defines a holomorphic Γ-invariant p-form on $\mathcal{H}_g$ with $p < g(g+1)/2$, then the form extends to a smooth compactification $\tilde{\mathcal{A}}_\Gamma$.*

For question 2 one uses a theorem of Kostant that tells which irreps ρ occur in the exterior algebra $\wedge^* \mathrm{Sym}^2(\mathrm{St})$. Namely if its dual is of the forms

$$w\delta - \delta$$

with $\delta = (g, g-1, \ldots, 1)$, the half sum of the positive roots. And w runs through the set W_0 of Kostant representatives. (These are certain representatives for W_G/W_M.) ($\#W_0 = 2^g$.) For example, for $g = 3$ we find the weights

$$(4,4,4), (4,4,2), (4,3,1), (4,1,1)$$

$$(3,3,0), (3,1,0), (2,0,0), (0,0,0)$$

Weissauer has a vanishing result on forms of singular weight.

Recall an irrep ρ of highest weight $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g)$ has

$$\mathrm{co}-\mathrm{rank}(\rho) = \#\{1 \leq i \leq g : \lambda_i = \lambda_g\}$$

Theorem 2. *Assume* ρ *has corank* $< g-\lambda_g$. *Then if*

$$\#\{1 \leq i \leq g : \lambda_i = \lambda_g + 1\} < 2(g - \lambda_g - \mathrm{corank}(\rho))$$

we have $M_\rho(\Gamma) = (0)$. *In particular,* $M_\rho(\Gamma) = (0)$ *if* $\lambda_g \leq g/2 - \mathrm{corank}(\rho)$.

Theorem 3. *Let* $\Gamma \subset \Gamma_g$ *of finite index and let* $\tilde{\mathcal{A}}_\Gamma$ *be a smooth compactification of* $\mathcal{A}_\Gamma$. *Then* $H^0(\tilde{\mathcal{A}}_\Gamma, \Omega^p_{\tilde{\mathcal{A}}_\Gamma}) = (0)$ *unless*

$$p = g(g+1)/2 - \alpha(\alpha+1)/2$$

for an integer $1 \leq \alpha \leq g$ *and then*

$$H^0(\tilde{\mathcal{A}}_\Gamma, \Omega^p_{\tilde{\mathcal{A}}_\Gamma}) \cong M_\rho(\Gamma)$$

with ρ of highest weight

$$(g+1, g+1, \ldots, g+1, \underbrace{g-\alpha, \ldots, g-\alpha}_{\alpha})$$

Top Differentials

For a torsion-free subgroup $\Gamma \subset \Gamma_g$ we have on a smooth compactification $\tilde{\mathcal{A}}_\Gamma$ of $\Gamma\backslash\mathcal{H}_g$ that

$$K_{\tilde{\mathcal{A}}_\Gamma} = L^{g+1} \otimes O_{\tilde{\mathcal{A}}_\Gamma}(-D)$$

with $L = \det(\mathbf{E})$ and D the divisor at ∞. An element of $S_{g+1}(\Gamma)$ extends to a section of $K_{\tilde{\mathcal{A}}_\Gamma}$. If Γ is not torsion-free the expression $f(\tau) \wedge_{i \leq j} d\tau_{ij}$ defines a top differential on the smooth locus $\mathcal{A}_g^0$. For $g > 1$ the fixed point locus is of codimension ≥ 2, also in the rank 1 partial compactification, hence it extends to all of $\tilde{\mathcal{A}}_\Gamma^{(1)}$, and then to all of a smooth compactification. We thus have

$$S_{g+1}(\Gamma) \cong H^0(\tilde{\mathcal{A}}_\Gamma, K_{\tilde{\mathcal{A}}_\Gamma})$$

The Kodaira Dimension of $\mathcal{A}_g$

For a torsion-free finite index subgroup $\Gamma \subset \Gamma_g$ an element $f \in M_{r(g+1)}(\Gamma)$ defines a Γ-invariant pluri-canonical form

$$f(\tau)\,(\wedge_{i \leq j} d\tau_{ij})^{\otimes r}$$

It will extend over the divisor D if f vanishes with multiplicity $\geq r$ along D; that is, if its slope $\leq g+1$.

If Γ does not act freely, we have to analyze the singularities, that is, the fixed points of Γ. Let us say that γ acts on the tangent space of a fixed point with eigenvalues $e^{2\pi\alpha_j}$ woth $0 \leq \alpha_j < 1$. Then there is a criterion of Reid-Tai that says: if for every γ and every fixed point we have

$$\sum_{j=1}^{g(g+1)/2} \alpha_j \geq 1$$

then the pluricanonical form extends over the resolution of the singularity. Tai checked that this works for $g \geq 5$:

Proposition 2. *If $g \geq 5$ then for each fixed point $\tau \in \mathcal{H}_g$ and each $\gamma \in \Gamma_g$ every Γ_g-invariant pluri-canonical form extends over the resolution of the singularity.*

For a smooth algebraic variety X the Kodaira dimension $\kappa(X)$ is defined as the largest dimension of the image of X under the (rational) map $X \to \mathbf{P}^N$ defined by the sections of $K_X^{\otimes n}$ (where n runs). It is

$-\infty$ if $H^0(X, K^n) = (0)$ for all n, otherwise $0 \leq \kappa(X) \leq \dim(X)$. If $\kappa(X) = \dim(X)$ the variety is called 'of general type'. The invariant $\kappa(X)$ is a birational invariant.

Let $Y_g \subset \tilde{\mathcal{A}}_g^{(1)}$ be the open subset of $\tilde{\mathcal{A}}_g^{(1)}$ parametrizing rank 1 degenerations with $\#\mathrm{Aut} = \{\pm 1\}$. Then we saw that $\tilde{\mathcal{A}}_g^{(1)} - Y_g$ has codimension ≥ 2 and so pluri-canonical forms extend.

Moreover, the Picard group

$$\mathrm{Pic}(Y_g) \otimes \mathbf{Q} = \mathbf{Q}[L] + \mathbf{Q}[D] \,.$$

with $L = \det(\mathbf{E})$, a result due to Borel, and we saw

$$K_{Y_g} = (g+1)[L] - [D]$$

If we find an effective divisor F with $F \sim a[L] - b[D]$ with $a/b \leq g+1$ we can write

$$K_{Y_g} \sim (g+1-a/b)L + F$$

hence K_{Y_g} is ample.

It is known that $\kappa(\mathcal{A}_g) = -\infty$ for $g = 1, \ldots, 5$. In fact, our bound on the slope with $\sigma(g) \geq g+1$ for $g \leq 4$ shows that there are no non-zero pluri-canonical forms for $g \leq 4$. It is known that $\mathcal{A}_g$ is unirational for $g \leq 5$.

Freitag proved that $\mathcal{A}_g$ is of general type if $g > 0$ and $24|g$. Tai proved that $\mathcal{A}_g$ is of general type if $g \geq 9$ using estimates on the dimension of spaces of cusp forms vanishing sufficiently often at infinity. Mumford refined this to $g \geq 7$ by another method.

Mumford looked at N_0, the locus in $\mathcal{A}_g$ of principally polarized abelian varieties (X, Θ) such that Θ is singular and its closure $\overline{N}_0$ in $\tilde{\mathcal{A}}_g^{(1)}$.

This has a part that we know: the zeros of

$$\prod_{\epsilon \text{ even}} \vartheta[\epsilon]$$

is a modular form of weight $2^{g-1}(2^g+1)\cdot\frac{1}{2}$. By explicitly analyzing at the boundary one finds it vanishes with multiplicity

$$2^{2g-5}$$

at the divisor D. This gives us an effective divisor ϑ_{null} in $\tilde{\mathcal{A}}_g^{(1)}$ corresponding to singular points of Θ at points of order 2 of X. It appears in $\overline{N}_0$:

$$\overline{N}_0 = \vartheta_{\rm null} + 2\overline{N}_0^*$$

and the rest $\overline{N}_0^*$ appears with multiplicity 2 because if $x \in \Theta \subset X$ is singular then $-x$ too.

Mumford calculates the divisor class of $\overline{N}_0$

$$[\overline{N}_0] = \left(\frac{g+1)!}{2} + g!\right)[L] - \frac{(g+1)!}{12}[D]$$

and we saw

$$[\vartheta_{\rm null}] = 2^{g-2}(2^g+1)[L] - 2^{2g-5}[D]$$

This difference gives us an effective class $\overline{N}_0^*$. It is zero for $g = 2$ and 3 and equal to $8[L] - [D]$ for $g = 4$. Thus we find an effective class $a[L]-b[D]$ in $\mathrm{Pic}\otimes\mathbf{Q}$ with

$$a = \big(g!(\tfrac{g+3}{4}) - 2^{g-3}(2^g+1)\big)$$
$$b = \left(\frac{(g+1)!}{24} - 2^{2g-6}\right)$$

If $a/b < g+1$ then $\mathcal{A}_g$ is of general type. Indeed, we then can write

$$K = \alpha L + R$$

with R effective, $\alpha > 0$ and L ample. The inequality holds for $g \geq 7$.

For $g = 6$ the Kodaira dimension $\kappa(\mathcal{A}_6)$ is unknown. But recently it was shown (by Dittmann et al) that there is a cusp form $f \in S_{14}(\Gamma_6)$ vanishing with multiplicity 2 at infinity, hence with slope $7 = 6+1$. So there is an effective pluricanonical divisor, hence $\kappa(\mathcal{A}_6) \geq 0$. They use harmonic theta functions of unimodular lattices of rank 24 to construct a basis of $S_{14}(\Gamma_6)$. Note that $\dim S_{14}(\Gamma_6) = 9$. Then they show that there is an element vanishing with order 2 along infinity.

Dimension Formulas

We want formulas for the dimension of $M_\rho(\Gamma_g)$ and $S_\rho(\Gamma_g)$. For $g=1$ and level 1 we have the generating series

$$\sum_k \dim M_k(\Gamma_1) t^k = \frac{1}{(1-t^4)(1-t^6)}$$

and for congruence subgroups $\Gamma \subset \mathrm{SL}(2,\mathbf{Z})$ one can apply Riemann-Roch and find formulas for $\dim S_k(\Gamma)$ like

$$(k-1)(G-1)+(k/2-1)c+\sum [k(r_i-1)/2r_i]$$

where G is the genus of the Riemann surface $\Gamma\backslash\mathcal{H}_1^*$, c is the number of cusps and the fixed points have order r_i (in the image of Γ in

$\mathrm{PSL}(2, \mathbf{Z})$). An alternative is to apply the so-called trace formula.

For $g = 2$ we know the generating series for $R_2 = \oplus M_k(\Gamma_2)$. For principal congruence subgroups Yamazaki gave a formula for $\dim S_k(\Gamma_2[n])$ using Hirzebruch-Riemann-Roch for the smooth compactification. Morita used Selberg's trace formula for this. Tsushima extended Yamazaki's result to $g = 3$. Tsushima gave formulas for $\dim S_{j,k}(\Gamma_2)$ for $k \geq 4$.

Taïbi has used the trace formula for calculating dimensions of spaces of (vector-valued) Siegel modular forms.

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