Siegel Modular Forms
Lecture #12

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Rings of Modular Forms

The structure of the ring

$$R_g = \oplus_k M_k(\Gamma_g)$$

is known only for $g = 1$ and $g = 2$. It is classical that $R_1 = \mathbf{C}[E_4, E_6]$ and we know by Igusa

$$R_2 = \mathbf{C}[\psi_4, \psi_6, \chi_{10}, \chi_{12}, \chi_{35}]/(\chi_{35}^2 = ...)$$

We know generators for $g = 3$ (34 by Tsuyumine, now reduced to 19 by Lercier and Ritzenthaler). Besides this not much is known.

The Case $g = 2$

We want to describe all vector-valued modular forms on Γ_2. We will use the Torelli map

$$t : \mathcal{M}_2 \to \mathcal{A}_2, \quad C \mapsto \mathrm{Jac}(C),$$

with $\mathcal{M}_2$ the moduli space of curves of genus 2. Both have dimension 3. This map embeds $\mathcal{M}_2$ into $\mathcal{A}_2$ as a dense open subset. The complement is the locus $\mathcal{A}_{1,1}$ of products of elliptic curves:

$$\mathcal{A}_1 \times \mathcal{A}_1 \to \mathcal{A}_{1,1} \subset \mathcal{A}_2$$

Over $\mathbf{C}$ this is induced by the map

$$\mathcal{H}_1 \times \mathcal{H}_1 \to \mathcal{H}_2$$

The moduli space $\mathcal{M}_2$ allows a compactification $\overline{\mathcal{M}}_2$ by allowing degenerations of smooth curves: stable curves. It turns out that the map $\mathcal{M}_2 \to \mathcal{A}_2$ can be extended to an identification

$$\overline{\mathcal{M}}_2 \xrightarrow{\sim} \tilde{\mathcal{A}}_2$$

with $\tilde{\mathcal{A}}_2$ a smooth toroidal compactification.

On $\mathcal{M}_2$ we have a Hodge bundle: a rank 2 vector bundle with as fibre over smooth C the space $H^0(C, \Omega^1_C)$. It extends to $\overline{\mathcal{M}}_2$ with fibre $H^0(C, \omega_C)$ over C.

The pull back of the Hodge bundle $\mathbf{E} = \mathbf{E}_2$ of $\mathcal{A}_2$ to $\mathcal{M}_2$ is the Hodge bundle of $\mathcal{M}_2$; it extends to $\overline{\mathcal{M}}_2$.

We can pull back modular forms via the map $\mathcal{A}_1 \times \mathcal{A}_1 \to \mathcal{A}_{1,1} \subset \mathcal{A}_2$. Note that the pullback of $\mathrm{Sym}^j(\mathbf{E}_2)$ is

$$\mathrm{Sym}^j(p_1^*\mathbf{E}_1 \oplus p_2^*\mathbf{E}_1)$$

that is,

$$\bigoplus_{r=0}^{j} p_1^*(\mathbf{E}_1)^{j-r} \otimes p_2^*(\mathbf{E}_1)^r \,.$$

This gives a map

$$M_{j,k}(\Gamma_2) \to \bigoplus_{r=0}^{j} M_{j-r+k}(\Gamma_1) \otimes M_{r+k}(\Gamma_1) \,.$$

Applying this to the case $k = 10$ we thus see that χ_{10} vanishes on $\mathcal{A}_{1,1}$ and with order 2. Indeed, $S_{10}(\Gamma_1) = 0$.

The Taylor development of χ_{10} along $\mathcal{A}_{1,1}$ is

$$\chi_{10} = \sum_{m=0}^{\infty} \xi_m \frac{z^m}{m!}$$

and this starts as (with $z = \tau_{12}$)

$$2\,\Delta \otimes \Delta \,\frac{z^2}{2} + 2\,\Delta e_2 \otimes \Delta e_2 \,\frac{z^4}{4!} + \ldots$$

with $e_2 = 1 - 24 \sum_n \sigma_1(n) q^n$, a quasi-modular form. The divisor of χ_{10} is

$$2\,\mathcal{A}_{1,1} + D\,.$$

Curves of Genus 2

A curve C of genus 2 is hyperelliptic: a choice ω_1, ω_2 of basis of $H^0(C, \Omega^1_C)$ defines a morphism of degree 2

$$C \to \mathbf{P}^1, \quad P \mapsto (\omega_1(P) : \omega_2(P)) .$$

Thus we find an equation $y^2 = f(x)$, with $\deg(f) = 6$, $\mathrm{discr}(f) \neq 0$. We can write f as homogeneous polynomial

$$f = \sum_{i=0}^{6} a_i \binom{6}{i} x_1^{6-i} x_2^i ,$$

that is, $f \in \mathrm{Sym}^6(V)$ with $V = \langle x_1, x_2 \rangle$.

We made a choice of basis of $H^0(C, \Omega^1_C)$. A different choice corresponds to the action of an element of $\mathrm{GL}(V)$. The group $\mathrm{GL}(V) = \mathrm{GL}(2)$ acts on $\mathrm{Sym}^2(V)$ on the right:

$$f(x_1, x_2) \mapsto f(ax_1 + bx_2, cx_1 + dx_2)$$

The action on $y^2 = f(x)$ is induced by

$$y \mapsto \frac{y}{(cx+d)^3},$$

and the equation becomes

$$y^2 = (cx+d)^6 f((ax+b)/(cx+d)).$$

This makes that we have to divide by the action of $\mathrm{GL}(2)$ on $\mathrm{Sym}^6(V)$.

Note also that the generic curve of genus 2 has an automorphism group of order 2. Therefore, we want that the generic point has a stabilizer of order 2.

Therefore we twist the action by $\det^{-2}$:

$$f \mapsto (ad - bc)^{-2} f(ax_1 + bx_2, cx_1 + dx_2)$$

We twisted by $\det(V)^{-2}$ so that $-\mathrm{id}_V$ acts by $(x, y) \mapsto (x, -y)$ and the generic curve has stabilizer of order 2. Then $\mathcal{M}_2$ is a stack quotient

$$[\mathcal{Y}^0/\mathrm{GL}(V)]$$

with

$$\mathcal{Y}^0 \subset \mathcal{Y} = \mathrm{Sym}^6(V) \otimes \det(V)^{-2},$$

with $\mathcal{Y}^0$ referring to $\mathrm{disc}(f) \neq 0$.

A curve $y^2 = f$ comes with differentials

$$dx/y,\ xdx/y$$

and $\mathrm{GL}(V)$ acts by the standard representation:

$$\begin{pmatrix} xdx/y \\ dx/y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} xdx/y \\ dx/y \end{pmatrix}$$

We get

$$[\mathcal{Y}^0/\mathrm{GL}(V)] \cong \mathcal{M}_2 \hookrightarrow \mathcal{A}_2$$

The pull back of $\mathbf{E}$ to $\mathcal{Y}^0$ is the equivariant bundle V.

And pull back of $\det(\mathbf{E})$ to $\mathrm{Sym}^6(V)$ is $\det(V)^3$ because we twisted.

Invariant Theory

An invariant for the action of $\mathrm{GL}(V) = \mathrm{GL}(2, \mathbf{C})$ on $\mathrm{Sym}^6(V)$ is a polynomial in the coefficients $a_0, \ldots, a_6$ of f invariant under $\mathrm{SL}(2, \mathbf{C})$. E.g.: the discriminant $\mathrm{discr}(f)$.

Invariants form a ring (Bolza, Clebsch,..)

$$I = \mathbf{C}[A, B, C, D, E]/(E^2 = \ldots)$$

Here the degrees of the invariants are

	A	B	C	D	E
deg	2	4	6	10	15

For example, with the above normalization of f we have

$$A = a_0 a_6 - 6a_1 a_5 + 15a_2 a_4 - 10a_3^2$$

The invariant D is the discriminant.

Since the pull back of $\det \mathbf{E}$ is $\det(V)^{\otimes 3}$, the embedding $\mathcal{M}_2 \hookrightarrow \mathcal{A}_2$ gives us a map

$$R_2 \longrightarrow I$$

with (up to a normalization)

$$\begin{array}{ccccc} \psi_4 & \psi_6 & \chi_{10} & \chi_{12} & \chi_{35} \\ & & \downarrow & & \\ B & C - AB & D & AD & ED^2 \end{array}$$

Igusa (1962) made such a map using theta functions and Thomae's formulas. Here Thomae's formulas are formulas expressing theta constants in terms of cross ratios of points on $\mathbf{P}^1$.

An invariant defines a modular form on $\mathcal{M}_2 \subset \mathcal{A}_2$. Not every invariant gives a modular form because $\mathcal{M}_2 \neq \mathcal{A}_2$; complement is $\mathcal{A}_{1,1}$, zero locus of χ_{10}, e.g. $A \longmapsto \chi_{12}/\chi_{10}$. So we get

$$R_2 \longrightarrow I \longrightarrow R_2[1/\chi_{10}]$$

The composition is the identity. Thus we can understand the structure of R_2 as a subring of the ring I.

We now want to extend this to vector-valued modular forms.

For this we consider covariants, that is, the invariants for the action of $\mathrm{SL}(2, \mathbf{C})$ on

$$V \oplus \mathrm{Sym}^6(V)$$

These are expressions in $a_0, \ldots, a_6, x_1, x_2$ invariant under $\mathrm{SL}(2, \mathbf{C})$. For example $f = \sum a_i \binom{6}{i} x_1^{6-i} x_2^i$.

An alternative description is as follows. If

$$U \hookrightarrow \mathrm{Sym}^d(\mathrm{Sym}^6(V))$$

is an GL-equivariant embedding or, equivalently, if we have a map of $\mathrm{GL}(V)$-representations

$$\mathbf{C} \xrightarrow{\phi} \mathrm{Sym}^d(\mathrm{Sym}^6(V)) \otimes U^\vee$$

then $\Phi = \phi(1)$ is a covariant. If U has highest weight (λ_1, λ_2) then Φ is a form of degree d in $a_0, \ldots, a_6$ and degree $\lambda_1 - \lambda_2$ in x_1, x_2.

Example: $d = 1$: $\mathrm{Sym}^6(V) \cong \mathrm{Sym}^6(V)$. Then $\Phi = f = \sum a_i \binom{6}{i} x_1^{6-i} x_2^i$, our "universal sextic".

Example: $d = 2$. In this case we can write $\mathrm{Sym}^2(\mathrm{Sym}^6(V))$ as

$$U[12,0] + U[10,2] + U[8,4] + U[6,6]$$

with $U[a,b]$ the irrep $\mathrm{Sym}^{a-b}(St) \otimes \det(St)^b$ of $\mathrm{GL}(V)$. We thus get four covariants

$$f^2, \mathrm{Hessian}, \ldots, A\,.$$

Here the Hessian is given by $f_{x_1x_1}f_{x_2x_2} -$

$f^2_{x_1x_2}$, or explicitly

$$(a_0a_2 - a_1^2)x_1^8 + 4(a_0a_3 - a_1a_2)x_1^7x_2 + \cdots$$
$$\cdots + (a_4a_6 - a_5^2)x_2^8$$

Covariants form a ring $\mathcal{C}$, with 26 generators (Cayley, Grace and Young, 19/20th century). These covariants are made in a way analoguous to that of the Hessian:

The kth transvectant $(\phi, \psi)_k$ of two forms $\phi \in \mathrm{Sym}^m(V)$, $\psi \in \mathrm{Sym}^n(V)$ is defined as

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{\partial^k\phi}{\partial^{k-j}x_1\partial^j x_2}\frac{\partial^k\psi}{\partial^j x_1\partial^{k-j}x_2}$$

If ϕ is of bidegree (a, m) and ψ of bidegree (b, n) then (ϕ, ψ_k) is of bidegree $(a + b, m + n - 2k)$.

1	$C_{1,6} = f$			
2	$C_{2,0} = (f, f)_6$	$C_{2,4} = (f, f)_4$	$C_{2,8} = (f, f)_2$	
3	$C_{3,2} = (f, C_{2,4})_4$	$C_{3,6} = (f, C_{2,4})_2$	$C_{3,8} = (f, C_{2,4})$	$C_{3,12} = (f, C_{2,8})$
4	$C_{4,0} = (C_{2,4}, C_{2,4})_4$	$C_{4,4} = (f, C_{3,2})_2$	$C_{4,6} = (f, C_{3,2})$	$C_{4,10} = (C_{2,8}, C_{2,4})$
5	$C_{5,2} = (C_{2,4}, C_{3,2})_2$	$C_{5,4} = (C_{2,4}, C_{3,2})$	$C_{5,8} = (C_{2,8}, C_{3,2})$	
6	$C_{6,0} = (C_{3,2}, C_{3,2})_2$	$C_{6,6}^{(1)} = (C_{3,6}, C_{3,2})$	$C_{6,6}^{(2)} = (C_{3,8}, C_{3,2})_2$	
7	$C_{7,2} = (f, C_{3,2}^2)_4$	$C_{7,4} = (f, C_{3,2}^2)_3$		
8	$C_{8,2} = (C_{2,4}, C_{3,2}^2)_3$			
9	$C_{9,4} = (C_{3,8}, C_{3,2}^2)_4$			
10	$C_{10,0} = (f, C_{3,2}^3)_6$	$C_{10,2} = (f, C_{3,2}^3)_5$		
12	$C_{12,2} = (C_{3,8}, C_{3,2}^3)_6$			
15	$C_{15,0} = (C_{3,8}, C_{3,2}^4)_8$			

As we saw, for $g = 2$ the R_2-module

$$M = \oplus_{j,k} M_{j,k}(\Gamma_2)$$

can be made into a ring. Now using that under

$$\mathcal{Y}^0 \to \mathcal{M}_2 \hookrightarrow \mathcal{A}_2$$

the pull back of $\mathbf{E}_\rho$ is the equivariant bundle V_ρ, we see that we get maps

$$M \longrightarrow \mathcal{C} \longrightarrow M \otimes R_2[1/\chi_{10}]$$

The composition is the identity. Apply this to the universal sextic $f \in \mathcal{C}$. What do we get?

$$f \mapsto \text{merom. section of} \quad \mathrm{Sym}^6(\mathbf{E}) \otimes \det(\mathbf{E})^{-2}$$

In other words, a meromorphic modular form of weight $(6, -2)$.

This form can be identified: we have six odd thetas $\vartheta_\epsilon(\tau, z)$, living on $\mathcal{H}_2 \times \mathbf{C}^2$, hence six gradients

$$g_\epsilon(\tau, z) = (\partial\vartheta_\epsilon/\partial z_1, \partial\vartheta_\epsilon/\partial z_2)\,|_{(\tau,0)}$$

The product of these is a modular form $\chi_{6,3}$ with character on Γ_2. We then have

$$f \longmapsto \chi_{6,3}/\chi_5 \qquad \text{with } \chi_5^2 = \chi_{10}$$

The form $\chi_{6,3}$ can be written as

$$\chi_{6,3} = \sum_{i=0}^{6} \alpha_i \binom{6}{i} X_1^{6-i} X_2^i$$

with X_1, X_2 dummy variables and α_i holomorphic on $\mathcal{H}_2$.

Recall that a covariant of bidegree (d, e) is a form in $a_0, \ldots, a_6, x_1, x_2$ and it is of degree d in the a_i and degree e in x_1, x_2. The map

$$s : \mathcal{C} \to M \otimes R_2[1/\chi_{10}]$$

is then given by substituting α_i for a_i in a covariant and then dividing by χ_5^d. This is an extremely effective method. This was done in joint work with Fabien Cléry and Carel Faber.

Let us do an example. We can decompose $\mathrm{Sym}^2(\mathrm{Sym}^6(V))$ as

$$U[12,0] + U[10,2] + U[8,4] + U[6,6]$$

The covariant defined by $U[10,2]$ is the

Hessian and gives rise to a form $\chi_{8,8} \in S_{8,8}(\Gamma_2)$. Its fourier expansion starts

$$\begin{pmatrix} 0 \\ 0 \\ (r-2+1/r \\ 3(r-1/r) \\ 4r+10+4/r \\ 3(r-1/r) \\ r-2+1/r \\ 0 \\ 0 \end{pmatrix} q_1 q_2 + \cdots$$

Restriction to $\mathcal{H}_1 \times \mathcal{H}_1$ shows that it starts as the transpose of $(0, \ldots, 0, \Delta \otimes \Delta, 0, \ldots, 0)$, hence it is not divisible by χ_5.

The covariant corresponding to $U[8,4]$ gives a form $\chi_{4,10} \in S_{4,10}(\Gamma_2)$. Finally, the summand $U[6,6]$ gives the form

$$\chi_{12} = \alpha_0\alpha_6 - 6\alpha_1\alpha_5 + 15\alpha_2\alpha_4 - 10\alpha_3^2$$

which after normalization starts as

$$\chi_{12}(\tau) = (1 + 10 + 1/r)q_1q_2 + \dots$$

Remark 1. *We have the maps*

$$M \to \mathcal{C} \to M[1/\chi_{10}]\,.$$

Recall also that M is not finitely generated, but $\mathcal{C}$ is finitely generated.

The website smf.compositio.nl gives the Fourier series for all cases where $\dim S_{j,k}(\Gamma_2) = 1$.

We determined the R_2-module structure for $\oplus_k M_{j,k}(\Gamma_2,\epsilon)$ for $j=2,4,6,8$ and 10. Here ϵ is the unique quadratic character of Γ_2 given by

$$\Gamma_2/\Gamma_2[2] \cong \mathfrak{S}_6 \to \{\pm 1\}.$$

The structure of the R_2^{ev}-modules

$$\oplus_k M_{j,2k}(\Gamma_2), \quad \oplus_k M_{j,2k+1}(\Gamma_2)$$

has been determined for $j=2,4,6,8,10$ by Ibukiyama, Kiyuna, van Dorp and Takemori using various methods. Using covariants one can uniformly treat these cases and the cases

$$\oplus_k M_{j,2k}(\Gamma_2,\epsilon), \qquad \oplus_k M_{j,2k+1}(\Gamma_2,\epsilon)$$

For example

Proposition 1. *The R_2^{ev}-module*

$$\oplus_k M_{2,2k+1}(\Gamma_2, \epsilon)$$

is free with generators of weight $(2, 9)$, $(2, 11)$ *and* $(2, 17)$.

We know here that the generating series is

$$\frac{t^9 + t^{11} + t^{17}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$$

We consider covariants

$$\xi_1 = 4\, C_{2,0}C_{3,2} - 15\, C_{5,2}$$
$$\xi_2 = 32 C_{2,0}^2 C_{3,2} + 135\, C_{2,0}C_{5,2} - 300\, C_{3,2}C_{4,0} - 15750\, C_{7,2}$$
$$\xi_3 = C_{3,2}$$

One can check that the orders of these ξ_i along $\mathcal{H}_1 \times \mathcal{H}_1$ are $-1, -1, -3$. Thus we get holomorphic modular forms by

$$f_{2,9} = \nu(\xi_1)\chi_5,\ f_{2,11} = \nu(\xi_2)\chi_5,\ f_{2,17} = \nu(\xi_3)\chi_5^3$$

and one can write down Fourier expansions. Using these series one sees that

$$f_{2,9} \wedge f_{2,11} \wedge f_{2,17} \neq 0 \, .$$

This proves the result.

In a similar manner we treated the cases $j = 2, 4, 6, 8, 10$. For example, the module $\mathcal{M}_{10}^{\mathrm{ev}}(\Gamma_2, \epsilon)$ is free with 10 generators.

Degree 2, Small k

It is known (by Skoruppa) that

$$\dim S_{j,1}(\Gamma_2) = 0$$

He proved this by looking at Fourier-Jacobi forms.

We conjecture

$$\dim S_{j,2}(\Gamma_2) = 0$$

We showed $S_{j,2}(\Gamma_2) = (0)$ for $j \leq 52$.

Compare: the smallest j such that $S_{j,3}(\Gamma_2) \neq (0)$ is 36. We can construct the generator of $S_{36,3}(\Gamma_2)$ easily using covariants.

In order to put our evidence for the vanishing of $S_{j,2}(\Gamma_2)$ in perspective we show a small table that gives for each value of k the smallest j_0 such that $\dim S_{j_0,k}(\Gamma_2) \neq 0$.

k	3	4	5	6	7	8
j_0	36	24	18	12	12	6

Degree 2 over $\mathbf{Z}$

Igusa determined the ring $R_2(\mathbf{Z})$ of scalar-valued modular forms of degree 2 over $\mathbf{Z}$. By this he meant modular forms whose Fourier series have integral Fourier coefficients.

The ring of modular forms of degree 1 over $\mathbf{Z}$ is classically known:

$$R_{\mathbf{Z}}(\Gamma_1) = \mathbf{Z}[e_4, e_6, \Delta]$$

with $1728\,\Delta = e_4^3 - e_6^2$.

Igusa showed that the ring $R_{\mathbf{Z}}(\Gamma_2)$ is generated by fifteen forms of degree

$$4, 6, 10, 12, 12, 16, 18, 24, 28, \\ 30, 35, 36, 40, 42, 48 \quad .$$

Positive Characteristic

The quotient $\Gamma_g \backslash \mathcal{H}_g$ is the complex fibre of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g. This moduli space is defined over $\mathbf{Z}$. We thus can consider

$$\mathcal{A}_g \otimes \mathbf{F}_p \, .$$

It is the moduli space of principally polarized abelian varieties of dimension g in characteristic $p > 0$. It carries a universal principally polarized abelian variety

$$\pi : \mathcal{X}_g \to \mathcal{A}_g$$

We thus find also a vector bundle $\mathbb{E}$ of rank

g over $\mathcal{A}_g$:

$$\pi_* \Omega^1_{\mathcal{X}_g/\mathcal{A}_g}$$

The fibre over a point $[X]$ is $H^0(X, \Omega^1_X)$.

We thus can for each representation ρ of $\mathrm{GL}(g)$ construct a vector bundle $\mathbf{E}_\rho$. In particular we have the line bundle $\det(\mathbf{E})$.

We define for $g \geq 2$ a modular form of weight ρ to be a section of $\mathbf{E}_\rho$:

$$M_\rho = H^0(\mathcal{A}_g \otimes \mathbf{F}_p, \mathbf{E}_\rho)$$

It is known that there exist appropriate (toroidal) compactifications $\tilde{\mathcal{A}}_g \otimes \mathbf{F}_p$ over which $\mathbf{E}$ and $\mathbf{E}_\rho$ extend. These were constructed by Faltings-Chai extending earlier work of Mumford et al.

Let D be the divisor added to compactify

$\mathcal{A}_g \otimes \mathbf{F}_p$. Then we put for all g

$$M_\rho = H^0(\tilde{\mathcal{A}}_g \otimes \mathbf{F}_p, \mathbf{E}_\rho)$$

and

$$S_\rho = H^0(\tilde{\mathcal{A}}_g \otimes \mathbf{F}_p, \mathbf{E}_\rho \otimes O(-D))$$

Let us look at the case $g = 1$. If X is an elliptic curve we can factor multiplication by p as

$$[p] : X \xrightarrow{F} X^{(p)} \xrightarrow{V} X$$

with F the Frobenius map and V, Verschiebung, its dual. Now F is always inseparable, but V not always. It is inseparable if the induced map

$$H^0(X, \Omega^1_X) \to H^0(X^{(p)}, \Omega^1)$$

is zero. This happens exactly for supersingular elliptic curves.

For varying X this is a map $\mathbf{E} \to \mathbf{E}^p$, or equivalently a section of $\mathbf{E}^p \otimes \mathbf{E}^{-1} = \mathbf{E}^{p-1}$. So the locus of X with X supersingular is the zero locus of the section of $\mathbf{E}^{p-1}$. That is, it is the zero locus of a modular form of weight $p-1$.

We thus see that there is a modular form of weight $p-1$ in characteristic $p > 0$. In particular, for $p = 2$ and $p = 3$ we have modular forms of weights that do not occur in characteristic 0.

Deligne showed that the ring $R_1(\mathbf{F}_p)$ of modular forms of degree 1 in characteristic

$p > 0$ is given by

$$R_1(\mathbf{F}_2) = \mathbf{F}_2[a_1, \Delta]$$

and

$$R_1(\mathbf{F}_3) = \mathbf{F}_3[b_2, \Delta]$$

and for $p \geq 5$

$$R_1(\mathbf{F}_p) = \mathbf{F}_p[c_4, c_6]$$

For degree 2 we can approach the question of the nature of $R_2(\mathbf{F}_p)$ by invariant theory.

For finite fields $\mathbf{F}_p$ with $p \geq 5$ the ring $R_2(\mathbf{F}_p)$ is generated by forms of degree

$$4, 6, 10, 12 \quad \text{and } 35$$

as in characteristic 0. In fact, the reduction map

$$R_2(\mathbf{Z}_p) \to R_2(\mathbf{F}_p)$$

is surjective as Ichikawa and Nagaoka showed.

But it is different for $p = 2$ and $p = 3$.

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