Siegel Modular Forms
Lecture $\#13$

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Summary

The moduli space $\mathcal{M}_2$ is an open dense part of $\mathcal{A}_2$ and has a description as a stack quotient $[\mathcal{Y}^0/\mathrm{GL}(V)]$ with $V = \langle x_1, x_2 \rangle$ and

$$\mathcal{Y}^0 \subset \mathrm{Sym}^6(V) \otimes \det(V)^{-2}$$

given by $\mathrm{disc}(f) \neq 0$. Using this over $\mathbf{C}$ we get a map

$$R_2 \to I \to R_2[1/\chi_{10}]$$

which extends to $M = \oplus_{j,k} M_{j,k}(\Gamma_2)$

$$M \to \mathcal{C} \to M[1/\chi_{10}]$$

with $\mathcal{C}$ the ring of covariants. A covariant is a polynomial in $a_0, \ldots, a_6$ and x_1, x_2 invariant under $\mathrm{SL}(V)$. Basic example: the universal binary sextic

$$f = \sum_{i=0}^{6} a_i \binom{6}{i} x_1^{6-i} x_2^i\,.$$

The map $\mathcal{C} \to M[1/\chi_{10}]$ sends a covariant c to a possibly meromorphic modular form of degree 2 by substituting for the a_i in c the coordinates α_i of $\chi_{6,-2} = \chi_{6,3}/\chi_5$ where

$$\chi_{6,-2} = \sum_{i=0}^{6} \alpha_i \binom{6}{i} X_1^{6-i} X_2^i$$

We have generators for the ring $\mathcal{C}$ due to Grace and Young. This gives an efficient recipe to construct (Fourier expansions of) Siegel modular forms of degree 2.

Degree 2 in positive characteristic

We consider scalar-valued modular forms of degree 2 in characteristic $p > 0$. These can be identified with elements of

$$H^0(\tilde{\mathcal{A}}_2 \otimes \mathbf{F}_p, \det(\mathbf{E})^k)$$

If X is a principally polarized abelian variety of dimension g in characteristic $p > 0$ we can factor multiplication by p

$$[p] : X \xrightarrow{F} X^{(p)} \xrightarrow{V} X$$

We call X is ordinary is V is separable. This can be detected by the determinant of the induced map

$$V^* : H^0(X, \Omega^1_X) \to H^0(X^{(p)}, \Omega^1_{X^{(p)}})$$

that is, for varying X, by a section of $\det(\mathbf{E})^{p-1}$. So the locus of non-ordinary abelian varieties is the zero divisor of a modular form h_g of weight $p-1$. It is called the Hasse invariant. We have

$$\Phi(h_g) = h_{g-1}\,.$$

If we restrict to degree 2 we also have a modular form of degree 10 due to the cycle relation

$$2[\mathcal{A}_{1,1}] + [D] \sim 10\lambda_1$$

with λ_1 the first Chern class of $\det(\mathbf{E})$. We call this form χ_{10}.

Also in positive characteristic we can use invariant theory to construct modular forms.

We know that for $p \geq 5$ the ring $R_2(\mathbf{F}_p)$ has the same stucture as in characteristic 0 (generators of weight $4, 6, 10, 12$ and a form of weight 35 whose square lies in $R^{\mathrm{ev}}(\mathbf{F}_p)$. Here $R_2^{\mathrm{ev}}(\mathbf{F}_p)$ is the subring of even weight.

We now do the case $p = 3$.

Theorem 1. *There exist modular forms ψ_2, χ_{10}, ψ_{12}, χ_{14} and χ_{36} in $R^{\mathrm{ev}}(\mathbf{F}_3)$ such that*

$$R_2^{\mathrm{ev}}(\mathbf{F}_3) = \mathbf{F}_3[\psi_2, \chi_{10}, \psi_{12}, \chi_{14}, \chi_{36}]/J$$

with J the ideal generated by

$$\psi_2^3 \chi_{36} - \chi_{10}^3 \psi_{12} - \psi_2^2 \chi_{10} \chi_{14}^2 + \chi_{14}^3$$

and $R_2(\mathbf{F}_3) = R_2^{\mathrm{ev}}(\mathbf{F}_3)[\chi_{35}]/(\chi_{35}^2 - P)$.

Proof. The ring of invariants I for the action of $\mathrm{GL}(V)$ in characteristic 3 is generated by invariants A, B, C, D, E of degree $2, 4, 6, 10$ and 15. We also have the ring of covariants $\mathcal{C}$.

Again we have a description of $\mathcal{M}_2 \otimes \mathbf{F}_3$ as a stack quotient $[\mathcal{Y}^0/\mathrm{GL}(V)]$ with V the $\mathbf{F}_3$-vector space $\langle x_1, x_2 \rangle$ and $\mathcal{Y}^0 \subset \mathrm{Sym}^6(V) \otimes \det(V)^{-2}$ given by $\mathrm{disc}(f) \neq 0$. We thus have

$$[\mathcal{Y}^0/\mathrm{GL}(V)] \cong \mathcal{M}_2 \otimes \mathbf{F}_3 \hookrightarrow \mathcal{A}_2 \otimes \mathbf{F_3}$$

and the pull back of $\mathbf{E}$ to $\mathcal{Y}^0$ is the equivariant bundle V. This induces maps

$$R_2(\mathbf{F}_3) \to I \to R_2(\mathbf{F}_3)[1/\chi_{10}]$$

which extends to $M = \oplus_{j,k} M_{j,k}(\Gamma_2)$ with

$$M_\rho(\Gamma_3) = H^0(\tilde{\mathcal{A}}_2 \otimes \mathbf{F}_3, \mathbf{E}_\rho)$$

$$M \to \mathcal{C} \to M[1/\chi_{10}]\,.$$

The universal binary sextic f yields a meromorphic modular form $\chi_{6,-2}$ of weight $(6,-2)$. It becomes holomorphic after multiplication by χ_{10} (as one sees by semi-continuity).

The idea is to look at $\chi_{6,8} = \chi_{6,-2}\chi_{10}$. Restriction of $\chi_{6,8}$ to $\mathcal{A}_{1,1}$ is zero because we land in

$$\oplus_{r=0}^{6} S_{6-r+8}(\Gamma_1) \otimes S_{r+8}(\Gamma_1) = 0$$

The second term in the Taylor expansion along $\mathcal{A}_{1,1}$ lands in

$$\oplus_{r=0}^{6} S_{6-r+9}(\Gamma_1) \otimes S_{r+9}(\Gamma_1)$$

and only the middle term $S_{12}(\Gamma_1) \otimes S_{12}(\Gamma_1)$ is non-zero. We thus see that if we write

$$\chi_{6,-2} = \sum_{i=0}^{6} \alpha_i \, X_1^{6-i} X_2^i$$

then the order along $\mathcal{A}_{1,1}$ of the α_i is

$$\mathrm{ord}_{\mathcal{A}_{1,1}}(\alpha_0, \ldots, \alpha_6) = \\ (\geq 2, \geq 1, \geq 0, \geq -1, \geq 0, \geq 1, \geq 2)$$

In characteristic 3 the invariant A is

$$a_1 a_5 - a_2 a_4 \,,$$

in terms of the coefficients of $f = \sum a_i x_1^{6-i} x_2^i$. So substituting α_i for a_i it yields a non-zero regular form.

Restriction to $\mathcal{A}_{1,1}$ gives

$$0 \to M_{k-10}(\Gamma_2) \to M_k(\Gamma_2) \to \mathrm{Sym}^2(M_k(\Gamma_1))$$

Recall that $\oplus_k M_k(\Gamma_1) = \mathbf{F}_3[b_2, \Delta]$. This gives bounds on the dimensions:

$$\dim M_k(\Gamma_2) = \begin{cases} 1 & k = 2, 4, 6, 8 \\ 2 & k = 10 \end{cases}$$

Thus A must yield the Hasse invariant ψ_2 of weight 2 (up to a non-zero scalar factor).

From the dimensions we see that $B = (a_1 a_5 - a_2 a_4) a_3^2 + \ldots$ must have a pole (of order 2). We conclude that α_3 has order -1 along $\mathcal{A}_{1,1}$. If χ_B defines the meromorphic form obtained from B we see that

$$\chi_{10} \chi_B$$

defines a regular form of weight 14.

Similarly, $C = 2a_3^6 + Aa_3^4 + \ldots$ has a pole of order 6. Thus $\chi_C \chi_{10}^3$ gives a regular form of weight 36.

In degree 10 there is a new invariant

$$D = (a_1 a_5)^3 a_3^4 + \ldots$$

and one checks that the order of χ_D along $\mathcal{A}_{1,1}$ is ≥ 2. Hence up to a scalar factor it is χ_{10}.

In degree 12 we find that $B^3 + A^3C - A^2B^2$ gives a regular form ψ_{12}. This describes the ring $R^{\mathrm{ev}}(\mathbf{F}_3)$ as one can see by looking at the bounds on the dimensions.

Finally, the invariant $E = (a_1 a_4^2 - a_2^2 a_5) a_3^6 + \cdots$ of degree 15 has order -3 along

$\mathcal{A}_{1,1} \otimes \mathbf{F}_3$. Thus

$$\chi_{35} = \chi_E \chi_{10}^2$$

is a form of weight 35.

Degree 2 Characteristic 2

Theorem 2. *The ring $R_2(\mathbf{F}_2)$ is generated by modular forms of weights $1, 10, 12, 13$ and 48 with one relation of weight 52:*

$$R_2(\mathbf{F}_2) = \mathbf{F}_2[\psi_1, \chi_{10}, \psi_{12}, \chi_{13}, \chi_{48}]/(R)$$

with $R = \chi_{13}^4 + \psi_1^3\chi_{10}\chi_{13}^3 + \psi_1^4\chi_{48} + \chi_{10}^4\psi_{12}$. The ideal of cusp forms is generated by χ_{10}, χ_{13} and χ_{48}.

Though curves of genus 2 in characteristic 2 can no longer be given by binary sextics one still can use the invariant theory of binary sextics to construct modular forms.

A curve of genus 2 is still a double cover of $\mathbf{P}^1$, but given by an equation

$$y^2 + a\,y = b$$

with $a, b \in k[x]$ of degree 3 and 6. We now have an action of $\mathrm{GL}(V) \ltimes \mathrm{Sym}^3(V)$ with $v \in \mathrm{Sym}^3(V)$ acting by

$$(a, b) \mapsto (a, b + v^2 + va)\,.$$

And we must twist. Still we can look at invariants and covariants. An example is the square root of the discriminant of a:

$$a_0 a_3 + a_1 a_2\,.$$

This defines the Hasse invariant of weight 1. In general, one lifts the curve to the Witt ring with coefficients $(\tilde{a}, \tilde{b})$ and considers invariants (or covariants) of the binary sextic defined by $\tilde{a}^2 + 4\tilde{b}$, divides these by an appropriate power of 2 and reduces modulo 2. This defines invariants.

The Case $g = 3$

The graded ring $R_3(\Gamma_3)$ was described by Tsuyumine who gave 34 generators and the generating function

$$\sum \dim M_k(\Gamma_3)\, t^k \,.$$

Igusa had observed the exact sequence

$$0 \to \chi_{18} R_3(\Gamma_3) \to R_3(\Gamma_3) \xrightarrow{r} I(2,8)$$

with $I(2,8)$ the ring of invariants of binary octics. The map r is induced by restriction to the hyperelliptic locus H_3. Hyperelliptic curves of genus 3 can be described by an equation $y^2 = f$ with f of degree 8. The zero locus of χ_{18} is the closure H_3 of the locus of Jacobians of hyperelliptic curves of genus 3. Shioda determined the ring of invariants of binary octics. We know the generating function for the dimensions of $M_k(\Gamma_3)$.

In a recent paper Lercier and Ritzenthaler reduced the number of generators to 19.

The Torelli morphism

$$t : \mathcal{M}_3 \longrightarrow \mathcal{A}_3, \quad C \mapsto \mathrm{Jac}(C)$$

is of degree 2, ramified along the hyperelliptic locus H_3. Indeed, every abelian variety has an automorphism -1, but a generic curve does not have non-trivial automorphisms. But hyperelliptic curves have always an automorphism of order 2.

The modular form $\chi_{18} = \prod_\epsilon \vartheta[\epsilon](\tau, 0)$ in

$S_{18}(\Gamma_3)$ has divisor

$$H_3 + 2D \qquad \text{in } \widetilde{\mathcal{A}}_3$$

We pull back the Hodge bundle $\mathbf{E}$ and similarly the $\mathbf{E}_\rho$ and extend to $\overline{\mathcal{M}}_3$. We define

$$T_\rho = H^0(\overline{\mathcal{M}}_3, \mathbf{E}_\rho),$$

the space of Teichmüller modular forms of weight ρ. (Here ρ is an irrep of $\mathrm{GL}(3, \mathbf{C})$.)

Example: $\chi_9 = \sqrt{\chi_{18}} \in T_9$ (with $\rho = \det^9$). Due to Ichikawa.

There is an involution ι (coming from $\mathcal{M}_3 \xrightarrow{2:1} \mathcal{A}_3$) and we have eigenspaces

$$T_\rho = T_\rho^+ \oplus T_\rho^-.$$

We can identify

$$T_\rho^+ \cong S_\rho(\Gamma_3),$$

while T_ρ^- is the space of *genuine* Teichmüller forms; we have

$$\chi_9 \, T_\rho^- \subset S_{\rho'}(\Gamma_3) \quad \text{with } \rho' = \rho \otimes \det^9.$$

We have another description of the non-hyperelliptic part $\mathcal{M}_3^{\mathrm{nh}}$ of $\mathcal{M}_3$. A non-hyperelliptic curve embeds as a smooth quartic curve in $\mathbf{P}^2$. Choose a basis $\omega_1, \omega_2, \omega_3$ of $H^0(C, \Omega_C^1)$ and use it to map C to $\mathbf{P}^2$

$$C \to \mathbf{P}^2, \quad P \mapsto (\omega_1(P) : \omega_2(P) : \omega_3(P)) \,.$$

The image is given by a quartic equation $f = 0$ with $f \in \mathrm{Sym}^4(V)$, where $V = \langle x, y, z \rangle$.

Then $\mathrm{GL}(V)$ acts. We look at the space of smooth ternary quartics $\mathcal{Y}^0$ inside all ternary quartics $\mathcal{Y} = \mathrm{Sym}^4(V)$, but we twist.

The moduli of non-hyperelliptic curves is a stack quotient

$$\mathcal{M}_3^{\mathrm{nh}} \cong [\mathcal{Y}^0/\mathrm{GL}(V)]$$

$$\mathcal{Y}^0 \subset \mathrm{Sym}^4(V) \otimes \det^{-1}(V).$$

(We twisted such that $c \cdot \mathrm{id}_V$ acts as $c \cdot \mathrm{id}_{\mathcal{Y}}$.) The pull back of the Hodge bundle on $\mathcal{M}_3^{\mathrm{nh}}$ to $\mathcal{Y}^0$ is the equivariant bundle V.

Thus we look at invariants for the action of $\mathrm{GL}(V)$ on $\mathrm{Sym}^4(V)$. If we write the 'universal' ternary quartic $f \in \mathrm{Sym}^4(V)$ as

$$f = a_0 x^4 + a_1 x^3 y + \cdots + a_{14} z^4$$

then an invariant is a polynomial in the coefficients $a_0, \ldots, a_{14}$ of f invariant under $\mathrm{SL}(3, \mathbf{C})$. The structure of the ring of invariants is known (Salmon, Shioda, Dixmier). The generating series of this ring is

$$\frac{P(t)}{(1-t^3)(1-t^6)(1-t^9)(1-t^{12})(1-t^{15})(1-t^{18})(1-t^{27})}$$

with $P(t)$ a palindromic polynomial

$$t^{75} + t^{66} + t^{60} + 2t^{57} + \cdots + t^{12} + t^9 + t + 1$$

If we also want to treat vector-valued forms we need to look at concomitants.

In the case of $\mathrm{GL}(3)$ the irreducible representations occur in

$$\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(\wedge^2 V) \otimes \det(V)^{\otimes c}$$

The irreducible representation ρ of highest weight $(\rho_1 \geq \rho_2 \geq \rho_3)$ occurs for the 'first time' in

$$\mathrm{Sym}^{\rho_1-\rho_2}(V) \otimes \mathrm{Sym}^{\rho_2-\rho_3}(\wedge^2 V) \otimes \det(V)^{\rho_3}\,.$$

We will write

$$M_\rho(\Gamma_3) = M_{i,j,k}(\Gamma_3)$$

with $i = \rho_1 - \rho_2$, $j = \rho_2 - \rho_3$ and $k = \rho_3$.

We fix coordinates on $\wedge^2 V$:

$$\hat{x} = y \wedge z, \quad \hat{y} = z \wedge x, \quad \hat{z} = x \wedge y\,.$$

Take an equivariant map of $\mathrm{GL}(V)$-reps

$$U \hookrightarrow \mathrm{Sym}^d(\mathrm{Sym}^4(V))\,,$$

or equivalently

$$\varphi : \mathbf{C} \longrightarrow \mathrm{Sym}^d(\mathrm{Sym}^4(V)) \otimes U^\vee$$

Then $\Phi = \varphi(1)$ is a concomitant. That is, it is an expression in $a_0, \ldots, a_{14}, x, y, z$ and $\hat{x}, \hat{y}, \hat{z}$ invariant under the action of $\mathrm{SL}(3, \mathbf{C})$. Most basic example: the universal ternary quartic f.

The concomitants form a module $\mathcal{C}$ over the ring I of invariants for $\mathrm{GL}(3, \mathbf{C})$.

For the modular forms we have a module

$$\Sigma = \oplus_\rho T_\rho$$

of vector-valued Teichmüller modular forms over the ring T of scalar-valued Teichmüller modular forms.

The pull back of $\mathbf{E}$ under $\mathcal{Y}^0 \to \mathcal{M}_3^{\mathrm{nh}}$ is $\mathcal{Y}^0 \times V$. This gives us

$$\begin{array}{ccccc} T & \longrightarrow & I & \longrightarrow & T[1/\chi_9] \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma & \longrightarrow & \mathcal{C} & \stackrel{s}{\longrightarrow} & \Sigma[1/\chi_9] \end{array}$$

because the zero locus of χ_9 is the closure of the hyperelliptic locus. Modular forms give concomitants, concomitants give meromorphic modular forms.

Where does the concomitant $f \in \mathcal{C}$, the universal quartic, go under s?

$$f \longmapsto \chi_{4,0,-1},$$

a meromorphic section of

$$\mathrm{Sym}^4(\mathbf{E}) \otimes \det(\mathbf{E})^{-1}$$

on $\overline{\mathcal{M}}_3$ with

$$\chi_{4,0,-1} \cdot \chi_9 = \chi_{4,0,8} \in S_{4,0,8}(\Gamma_3),$$

a holomorphic Siegel modular form.

How to get it? Note $\dim S_{4,0,8}(\Gamma_3) = 1$.

We take the Schottky form for $g = 4$ (of weight 8 vanishing on the Torelli locus) and develop it along

$$\mathcal{H}_3 \times \mathcal{H}_1 \hookrightarrow \mathcal{H}_4$$

The first non-zero term in the Taylor expansion is

$$\chi_{4,0,8} \otimes \Delta \in S_{4,0,8}(\Gamma_3) \otimes S_{12}(\Gamma_1)$$

Its Fourier expansion starts as follows: write
$q_i = e^{2\pi i \tau_{ii}}$ $(i = 1, 2, 3)$ and
$u = e^{2\pi i \tau_{12}}, v = e^{2\pi i \tau_{13}}, w = e^{2\pi i \tau_{23}})$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ (v-1)^2(w-1)^2/vw \\ (u-1)(v-1)(w-1)(-1+1/vw+1/uw-1/uv) \\ (u-1)^2(w-1)^2/uw \\ 0 \\ (u-1)(v-1)(w-1)(-1+1/vw-1/uw+1/uv) \\ (u-1)(v-1)(w-1)(-1-1/vw+1/uw+1/uv) \\ 0 \\ 0 \\ 0 \\ (u-1)^2(v-1)^2/uv \\ 0 \\ 0 \end{pmatrix} q_1 q_2 q_3 + \cdots ,$$

We describe the map $s : \mathcal{C} \to \Sigma[1/\chi_9]$.

Write the universal quartic f as $\sum a_I x^I$. Then write $\chi_{4,0,8}$ as

$$\chi_{4,0,8} = \sum_I \alpha_I X^I$$

as a ternary quartic with dummy variables X_1, X_2, X_3. The α_I are holomorphic on $\mathcal{H}_4$, given by their Fourier expansion.

A concomitant is a polynomial in the a_I, x_1, x_2, x_3 and u_1, u_2, u_3 with $u_1 = x_2 \wedge x_3$, $u_2 = x_1 \wedge x_3$ and $u_3 = x_2 \wedge x_3$.

The map s is given by substituting α_I for a_I and dividing by χ_9^d.

Is the result holomorphic? In other words, which concomitants give holomorphic modular forms?

Define the order of a concomitant c along the locus of double conics as the order w.r.t. t of the evaluation of c on the quartic $tf + h^2$ where f is a general quartic and h a general conic.

Theorem 3. *Let c be a concomitant of degree d. Let $\nu(c)$ be its order of vanishing along the locus of double conics. If d is odd then $s(c)\chi_9$ is a Siegel modular form vanishing with order $\nu(c) - (d-1)/2$ along the hyperelliptic locus. If d is even, then the order of $s(c)$ is $\nu(c) - d/2$.*

Example: Let c be the discriminant, an invariant of degree $d = 27$. Now we know that

$$s(c)\chi_9 = \chi_{18}$$

vanishes with order 1 along $\mathcal{H}_3$, the hyperelliptic locus. So c vanishes with order 14 along the locus of double conics (confirms a result of Aluffi-Cuckierman).

Let $m \geq 0$ be an integer. We denote by $C_{d,\rho}(-mDC)$ the vector space of concomitants of type (d,ρ) that have order $\geq m$ along the locus DC of double conics. (Type (d,ρ) means belonging to an irrep ρ occurring in $\mathrm{Sym}^d(\mathrm{Sym}^4(V))$.)

By $M_{i,j,k}^{(m)}(\Gamma_3)$ we mean the vector space of modular forms of weight (i,j,k) with $\mathrm{ord}_\infty(f) \geq m$.

Theorem 4. *There exists an isomorphism*

$$C_{d,\rho}(-mDC) \xrightarrow{\sim} M^{(d-2m)}_{\rho_1-\rho_2,\rho_2-\rho_3,\rho_3+9(d-2m)}$$

given by

$$c \mapsto s(c)\chi_9^{d-2m}.$$

Corollary 1. *The space $M^{(2k)}_{18k}$ is generated by χ_{18}^k.*

In principle we can describe all modular forms of genus 3. Though we do not know generators of the module of concomitants.

Example. For $d = 2$ we find the decomposition of $\mathrm{Sym}^2(\mathrm{Sym}^4(V))$ as

$$V[8,0,0] + V[6,2,0] + V[4,4,0] \,.$$

The component $V[4,4,0]$ defines a concomitant. It is a form of degree 2 in the a_I and degree 4 in u_1, u_2, u_3, the generators of $\wedge^2 V$. It gives a meromorphic Siegel modular form of weight $(0,4,-2)$. It becomes holomorphic after multiplication by χ_9^2, hence a form in $S_{0,4,16}$ vanishing twice at ∞.

We find isomorphisms

$$C_{2,(8,0,0)} \cong M^{(2)}_{8,0,16} \quad C_{2,(6,2,0)} \cong M^{(2)}_{4,2,16}$$

and

$$C_{2,(4,4,0)} \cong M^{(2)}_{0,4,16}$$

via $c \mapsto s(c)\chi_{18}$.

Example. The catalecticant is an invariant of degree $d = 6$ associated to

$$V[8,8,8] \subset \mathrm{Sym}^6(\mathrm{Sym}^4(V)).$$

It defines a Siegel modular form of weight 56 vanishing with order 6 at ∞ and order 16 along $\mathcal{A}_{2,1}$. It confirms a result of Ottaviani-Sernesi who view the catalecticant as a section $\mathcal{O}(56\lambda - 6\delta_0 - 16\delta_1)$ on $\overline{\mathcal{M}}_3$.

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