Siegel Modular Forms

Lecture #14

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Traces of Hecke operators

In this lecture and the following one we will discuss a method (due to Faber-vdG and Bergström) to calculate traces of Hecke operators on the spaces $M_\rho(\Gamma_g)$ of Siegel modular forms of degree $g = 2$ and $g = 3$ for all ρ.

We will indicate the role that modular forms play in the cohomology of local systems on the moduli space of principally polarized abelian varieties.

Let us start with the simplest case: $g = 1$.

Elliptic Curves over Finite Fields

Modular forms are everywhere. Here is one example. Fix a finite field $\mathbf{F}_q$ of cardinality q. An elliptic curve E over $\mathbf{F}_q$ can be defined by

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with $a_i \in \mathbf{F}_q$. Then we know by Hasse

$$\#E(\mathbf{F}_q) = q + 1 - \alpha - \overline{\alpha}$$

with α an algebraic integer with $|\alpha| = \sqrt{q}$.

What is the average of $\#E(\mathbf{F}_q)$ as E varies?

We normalize and ask for

$$\sum_E \frac{q+1-\#E(\mathbf{F}_q)}{\#\mathrm{Aut}_{\mathbf{F}_q}(E)}$$

with the sum over all isomorphism classes defined over $\mathbf{F}_q$. More generally, put

$$h(k,E) = \alpha^k + \alpha^{k-1}\overline{\alpha} + \cdots + \overline{\alpha}^k \, .$$

Consider the sum

$$\sigma_k(q) = -\sum_E \frac{h(k,E)}{\#\mathrm{Aut}_{\mathbf{F}_q}(E)} \, ,$$

where E runs over all isomorphism classes defined over $\mathbf{F}_q$.

$\mathbf{F}_3$

f	$\#C(k)$	$1/\#\mathrm{Aut}_k(C)$	j
x^3+x^2+1	6	1/2	−1
x^3-x^2-1	2	1/2	−1
x^3+x^2-1	3	1/2	1
x^3-x^2+1	5	1/2	1
x^3+x	4	1/2	0
x^3-x	4	1/6	0
x^3-x+1	7	1/6	0
x^3-x-1	1	1/6	0

Example: σ_{10}

p	2	3	5	7	11
σ_{10}	-23	253	4831	-16743	534613

$$\Delta = q - 24\,q^2 + 252\,q^3 - 1472\,q^4 + 4830\,q^5 - 6048\,q^6 - 16744\,q^7 + \dots$$

and we see (experimentally)

$$\sigma_{10}(p) = \tau(p) + 1.$$

Cohomological Interpretation

Start with $g = 1$. We have the universal elliptic curve

$$\pi : \mathcal{X}_1 \to \mathcal{A}_1$$

Then $\mathbf{V} = R^1\pi_*\mathbf{Q}$ is a local system of rank 2 with fibre

$$H^1(X, \mathbf{Q})$$

over $[X]$. We can construct more local systems; for $a \geq 0$

$$\mathbf{V}_a = \mathrm{Sym}^a(\mathbf{V})$$

of rank $a+1$. We consider the cohomology

$$H^1(\mathcal{A}_1 \otimes \mathbf{C}, \mathbf{V}_a \otimes \mathbf{C}),$$

and since $\mathcal{A}_1$ is not complete (compact)

$$H^1_c(\mathcal{A}_1 \otimes \mathbf{C}, \mathbf{V}_a \otimes \mathbf{C}),$$

the compactly supported cohomology.

Cohomological interpretation of cusp forms:

Eichler-Shimura: for $k \geq 2$, even

$$H^1_c(\mathcal{A}_1, \mathbf{V}_k \otimes \mathbf{C}) \cong \mathbf{C} \oplus S_{k+2} \oplus \overline{S_{k+2}}.$$

with S_n the space of cusp forms of weight n on $\mathrm{SL}(2, \mathbf{Z})$. Indeed, we have an exact sequence on $\mathcal{A}_1$

$$0 \to \mathbf{E} \to H^1_{\mathrm{dR}} \to \mathbf{E}^\vee \to 0$$

and we have an induced map

$$\mathbf{E}^k \to \mathbf{V}_k \otimes_{\mathbf{Q}} \mathcal{O}$$

The de Rham resolution on $\mathcal{A}_1$

$$0 \to \mathbf{V}_k \otimes \mathbf{C} \to \mathbf{V}_k \otimes \mathcal{O} \to \mathbf{V}_k \otimes \Omega^1 \to 0$$

gives a long exact sequence and a connecting map

$$H^0(\mathbf{V}_k \otimes \Omega^1) \to H^1(\mathbf{V}_k \otimes \mathbf{C})\,.$$

We also have the complex conjugate of this map

$$\begin{array}{c} H^0(\mathcal{A}_1, \Omega^1 \otimes \mathbf{E}^k) \oplus \overline{H^0(\mathcal{A}_1, \Omega^1 \otimes \mathbf{E}^k)} \\ \downarrow \\ H^1(\mathcal{A}_1, \mathbf{V}_k \otimes \mathbf{C}) \end{array}$$

The image of

$$H^0(\tilde{\mathcal{A}}_1, \Omega^1 \otimes \mathbf{E}^k) \oplus \overline{H^0(\tilde{\mathcal{A}}_1, \Omega^1 \otimes \mathbf{E}^k)}$$

lands in the image of the compactly supported cohomology. We thus get

$$H^0(\tilde{\mathcal{A}}_1, \Omega^1 \otimes \mathbf{E}^k) \cong S_{k+2}$$

sitting in the (image of the) compactly supported cohomology. These forms can be given as

$$f(\tau)\, d\tau\, dz^{\otimes k}$$

Now $\mathcal{A}_1$ is defined over $\mathbf{Z}$, so we have the spaces $\mathcal{A}_1 \otimes \mathbf{F}_p$ for every prime p.

In characteristic $p > 0$ information on cohomology can be obtained by counting points over finite fields. We can use the ℓ-adic variants of the $\mathbf{V}_k$ and etale cohomology and compute the trace of Frobenius.

The cohomology groups $H^i_c(\mathcal{A}_1 \otimes \overline{K}, \mathbf{V}_k)$ come with an action of $\mathrm{Gal}(\overline{K}/K)$.

There is a natural surjection

$$\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$$

For $p \neq \ell$ there is an isomorphism

$$H^i_c(\mathcal{A}_1 \otimes \overline{\mathbf{F}}_p, \mathbf{V}_k) \xrightarrow{\cong} H^i_c(\mathcal{A}_1 \otimes \overline{\mathbf{Q}}_p, \mathbf{V}_k)$$

as $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-representations.

In characteristic p the Hecke correspondence $T(p)$ decomposes: if E and E' are elliptic curves with j-invariants j and j' and $E \to E'$ is an isogeny of degree p then $j = j'^p$ or $j^p = j'$. This gives

$$T_p = F_p + F_p^t \,.$$

We thus get (Deligne (1968))

$$\mathrm{Tr}(F_p, H^1_c(\mathcal{A}_1 \otimes \overline{\mathbf{F}}_p, \mathbf{V}_k)) =$$

$$1 + \mathrm{Tr}(T(p), S_{k+2}(\Gamma_1)) \,.$$

This explains the relation

$$\sigma_{10}(p) = \tau(p) + 1 \,,$$

where $\Delta = \sum \tau(n) q^n$.

Notationally

$$H^1_c(\mathcal{A}_1, \mathbf{V}_{10}) = S[12] + 1$$

with for k even $S[k]$ the motive of cusp forms of weight k on $SL(2,\mathbf{Z})$. This is a motive of weight $k-1$, rank $= 2\dim S_k(\Gamma_1)$ and Hodge type

$$S[k] = S_k(\Gamma_1) \oplus \bar{S}_k(\Gamma_1).$$

The formula

$$H^1_c(\mathcal{A}_1, \mathbf{V}_k) = S[k+2] + 1$$

tells us:

$$\sigma_k(p) = \text{Trace of } T(p) \text{ on } S_{k+2} + 1$$

There is a natural map

$$H^*_c(\mathcal{A}_1, \mathbf{V}_k) \to H^*(\mathcal{A}_1, \mathbf{V}_k)$$

and the image is called interior cohomology $H_!^*$. We have (for $k > 0$ even)

$$H_c^1(\mathcal{A}_1, \mathbf{V}_k) = S[k+2] + 1$$

and

$$H_!^1(\mathcal{A}_1, \mathbf{V}_k) = S[k+2].$$

The difference comes from an Eisenstein series

$$E_{k+2} := \frac{1}{2} \sum_{(c,d)=1} (cz+d)^{-k-2}$$

with Hecke eigenvalue at $T(p)$

$$1 + p^{k+1}.$$

Conclusion: We can calculate $\mathrm{Tr}(T(p))$ on $S_{k+2}(\Gamma_1)$ for all even $k > 0$ by

1) making a list of all elliptic curves $/\mathbf{F}_p$ up to $\cong_{\mathbf{F}_p}$;

2) for each elliptic curve E determine $\#E(\mathbf{F}_p)$ and $\#\mathrm{Aut}_{\mathbf{F}_p}(E)$.

In practice, for a prime p we list how often a trace $t = \alpha + \overline{\alpha}$ occurs in our list with the frequency taken to be

$$w(t) = \sum_{E:\#E(\mathbf{F}_p)=p+1-t} \frac{1}{\#\mathrm{Aut}_{\mathbf{F}_p}(E)}$$

For example for $p = 17$ we get the list

t	± 8	± 7	± 6	± 5	± 4	± 3	± 2	± 1	0
w	$1/4$	$1/2$	$3/2$	$1/2$	1	$3/2$	$7/4$	$1/2$	2

and have

$$\mathrm{Tr}(T(17), S_{k+2}(\Gamma_1)) + 1 =$$

$$-\sum_t w(t)\ \left(\alpha_t^k + \alpha_t^{k-1}\bar{\alpha}_t + \cdots + \bar{\alpha}_t^k\right)$$

for all k. Here α_t and $\bar{\alpha}_t$ are the solutions of $X^2 - t\,X + 17 = 0$.

Reformulation

Put

$$e_c(\mathcal{A}_1, \mathbf{V}_k) = \sum_{i=0}^{2} (-1)^i H_c^i(\mathcal{A}_1, \mathbf{V}_k)$$

in a Grothendieck group of Hodge structures or Galois representations. Then we have for even $k > 0$

$$e_c(\mathcal{A}_1, \mathbf{V}_k) = -S[k+2] - 1$$

with $S[k+2]$ a motive (due to Scholl) or a bookkeeping device with

$$\mathrm{Tr}(F_p, S[k+2]) = \mathrm{Tr}(T(p), S_{k+2}(\Gamma_1))$$

Question Can we use genus 2 curves in a similar way to get information on Siegel modular forms?

We have seen

$$\mathcal{A}_2 = t(\mathcal{M}_2) \sqcup \mathcal{A}_{1,1}\,,$$

that is, we can describe it using curves of genus 2 and degenerate curves (unions of two elliptic curves).

The corresponding moduli space $\mathcal{A}_2$ is defined over $\mathbf{Z}$ and has dimension 3.

It has natural local system of rank 4:

$$\mathbf{V} := \mathrm{Sp}(4,\mathbf{Z})\backslash \mathcal{H}_2 \times \mathbf{Q}^4$$

More intrinsically, if $\pi : \mathcal{X} \to \mathcal{A}_2$ is the universal abelian surface over $\mathcal{A}_2$ then

$$\mathbf{V} := R^1\pi_*\mathbf{Q}\,.$$

There is a symplectic pairing

$$\mathbf{V} \otimes \mathbf{V} \to \mathbf{Q}(-1).$$

coming from the non-degenerate alternating pairing on the fibres $H^1(X, \mathbf{Q})$.

For each pair (a, b) with $a \geq b \geq 0$ we have an irreducible representation of $\mathrm{Sp}(4, \mathbf{Q})$: $V_{a,b}$. It occurs for the 'first time' in

$$\mathrm{Sym}^{a-b}(U) \otimes \mathrm{Sym}^b(\wedge^2 U)$$

with $U = V_{1,0}$ the standard representation of $\mathrm{Sp}(4, \mathbf{Q})$.

There is a corresponding local system $\mathbf{V}_{a,b}$.

$$\begin{aligned} \mathbf{V} &= \mathbf{V}_{1,0} \\ \mathbf{V}_{1,1} &= \text{primitive part of } \wedge^2\mathbf{V} \end{aligned}$$

If $a > b > 0$ we call (a, b) regular.

We have to use the ℓ-adic versions (for ℓ-adic etale cohomology) of these in characteristic p with $\ell \neq p$.

We are interested in

$$e_c(\mathcal{A}_2, \mathbf{V}_{a,b}) = \sum (-1)^i [H_c^i(\mathcal{A}_2, \mathbf{V}_{a,b})].$$

View this in a Grothendieck group of Hodge structures or Galois representations.

When the weight $a+b$ is odd, $e_c(\mathbf{V}_{a,b}) = 0$ because of the hyperelliptic involution.

For a curve C of genus 2 over $\mathbf{F}_q$ there exist algebraic integers α_1, α_2 of absolute value $\sqrt{q}$ such that (Weil)

$$\#C(\mathbf{F}_q) = q + 1 - \alpha_1 - \bar{\alpha}_1 - \alpha_2 - \bar{\alpha}_2.$$

The trace of Frobenius on $\mathbf{V}_{a,b}$ is calculated by summing certain symmetric expressions in these α's over all C (up to $\cong_{\mathbf{F}_q}$).

What we did was counting curves of genus 2 over finite fields; for given field $\mathbf{F}_q$ with $q \leq 37$ (later $q < 200$) we compiled a list of possible α's and their frequencies as the curve ran through all curves of genus 2 up to $\cong_{\mathbf{F}_q}$ (with factor $1/\#\mathrm{Aut}_{\mathbf{F}_q}(C)$) and added the contribution from the degenerate curves (that is, union of two elliptic curves intersecting in one point).

What we want is an analogue for $g = 2$ of

$$e_c(\mathcal{A}_1, \mathbf{V}_k) = -S[k+2] - 1 \,.$$

We have a natural map

$$H_c^*(\ldots) \to H^*(\ldots)$$

with image the interior cohomology $H_!^*(\ldots)$.

We define the Eisenstein cohomology:

$$e_{\mathrm{Eis}}(\mathcal{A}_2, \mathbf{V}_{a,b}) := e_c(\mathcal{A}_2, \mathbf{V}_{a,b}) - e_!(\mathcal{A}_2, \mathbf{V}_{a,b}).$$

For $g = 1$ we have

$$e_{\mathrm{Eis}}(\mathcal{A}_1, \mathbf{V}^k) = -1 \quad \text{for } k \geq 2 \text{ even,}$$

Eisenstein cohomology was studied by Harder, Schwermer and Pink.

Theorem 1. *For (a, b) regular the Euler characteristic $e_{\mathrm{Eis}}(\mathcal{A}_2, \mathbf{V}_{a,b})$ of the Eisenstein cohomology is*

$$-S[a+3] - s_{a+b+4}L^{b+1} + S[b+2] + s_{a-b+2} \cdot 1 + \begin{cases} 1 & a \text{ even,} \\ 0 & l \text{ odd,} \end{cases}$$

with $s_n = \dim S_n(\Gamma_1)$.

Here L is the Lefschetz motive. The ℓ-adic realization of L is $\mathbf{Q}_\ell(-1)$ with $\mathrm{Tr}(F_q, \mathbf{Q}_\ell(-1)) = q$.

Faltings proved that $H^*(\mathcal{A}_2, \mathbf{V}_{a,b})$ and $H^*_c(\mathcal{A}_2, \mathbf{V}_{a,b})$ possess a Hodge filtration; $H^3_!(\mathcal{A}_2, \mathbf{V}_{l,m})$ has a pure Hodge structure:

$$0 \subset F^{a+b+3} \subset F^{a+2} \subset F^{b+1} \subset F^0$$

with $F^0 = H^3_!(\mathcal{A}_2, \mathbf{V}_{a,b})$. Moreover, if the weight (a, b) is regular,

$$H^i_!(\mathcal{A}_2, \mathbf{V}_{a,b}) = (0) \quad \text{for} \quad i \neq 3.$$

Now F^{a+b+3} has an interpretation in terms of Siegel modular forms.

$$F^{a+b+3} \cong S_{a-b,b+3}(\Gamma_2).$$

Still there is another contribution from the case $g = 1$: endoscopic lifting from $N = \mathrm{GL}(2) \times \mathrm{GL}(2)/\mathbf{G}_m$

Such endoscopic lift was studied among others by KUDLA, RALLIS, WEISSAUER.

We (F-vdG) conjectured:

For (a, b) regular the endoscopic contribution of N to $e_!(\mathcal{A}_2, \mathbf{V}_{a,b})$ has zero intersection with F^{a+b+3} and is equal to

$$e_{\mathrm{endo}}(\mathcal{A}_2, \mathbf{V}_{a,b}) = -s_{a+b+4} S[a - b + 2] L^{b+1}.$$

Rest should come from genus 2 Siegel modular forms in a “true sense”: trace of Frobenius on

$$S[a-b,b+3]$$

defined as

$$H_!^3((\mathcal{A}_2,\mathbf{V}_{a,b})-H^3_{\mathrm{endo}}(\mathcal{A}_2,\mathbf{V}_{a,b})$$

should be the trace of the Hecke operator $T(p)$ on $S_{a-b,b+3}(\Gamma_2)$. Our results enabled us to calculate (assuming the conjecture) the traces of Hecke operators on $S_{j,k}$ for all $p \leq 200$ and *all* j,k.

It turned out to be better to take the Eisenstein cohomology and endoscopic contribution together. A formula for the Eisenstein cohomology was proved for regular (a, b).

Based on our numerical evidence we conjectured:

Theorem 2. *The trace of $T(p)$ on $S_{a-b,b+3}(\Gamma_2)$ equals*

$$-\mathrm{Tr}(F_p, e_c(\mathcal{A}_2 \otimes \mathbf{F}_p, \mathbf{V}_{a,b})) + \mathrm{Tr}(F_p, e_{2,\mathrm{extra}}(a, b))$$

with $e_{2,\mathrm{extra}}(a, b)$ a correction term given by

$$s_{a-b+2} + s_{a+b+4}(S[a-b+2]+1)L^{b+1}$$

$$+\begin{cases} S[b+2]+1 & a \text{ even,} \\ -S[a+3] & a \text{ odd .} \end{cases}$$

The conjecture later was shown to be true by work of Weissauer and Petersen.

Reformulation

$$e_c(\mathcal{A}_2, \mathbf{V}_{a,b}) = -S[a-b, b+3] + e_{2,\mathrm{extra}}(a, b)$$

with

$$S[a-b, b+3]$$

a hypothetical motive of dimension $4 \dim S_{a-b,b+3}(\Gamma_2)$, or a bookkeeping device with

$$\mathrm{Tr}(T(p), S_{a-b,b+3}(\Gamma_2)) = \mathrm{Tr}(F_p, S[a-b, b+3])$$

For completeness we define

$$S[0,3] = -L^3 - L^2 - L - 1\,.$$

The first case one encounters is $S_{6,8}(\Gamma_2)$. We have $\dim S_{6,8}(\Gamma_2) = 1$.

At our request Ibukiyama constructed a form in 2001. He constructed it via harmonic theta series. Now we have more easily $\chi_{6,8} = \chi_{6,3}\chi_5$. For $g = 2$ it is difficult to calculate Hecke eigen values from Fourier coefficients.

The characteristic polynomial at the prime $p = 2$ is

$$X^4 - 204800X^2 + 274877906944$$

and at $p = 7$

$$X^4 + 107822000\,X^3 + 5939548912704750\,X^2 \\ + 122905167667730302454600\,X \\ + 129934811447123020117172145698449.$$

The Euler factor of the spinor L-function is:

$$1-\lambda(p)X+(\lambda(p)^2-\lambda(p^2)-p^{a+b+2})X^2+ \\ -\lambda(p)p^{a+b+3}X^3+p^{2a+2b+6}X^4.$$

p	$\lambda(p)$	$\lambda(p^2)$	slopes
2	0	466944	$13/2, 25/2$
3	-27000	143765361	$3, 7, 12, 16$
5	2843100	-7734928874375	$2, 7, 12, 17$
7	-107822000	7314448369205699	$0, 6, 13, 19$

Let us look at the form χ_{35} constructed by Igusa. It occurs in the cohomology of the local system $\mathbf{V}_{32,32}$ where $e_{2,\mathrm{extra}}$ has the form

$$e_{2,\mathrm{extra}}(32,32) = 5\,L^{34} + S[34]$$

and one finds as eigenvalue $\lambda(p)$ of $T(p)$ for $p \leq 37$:

Observe that the eigenvalues grow like $p^{67/2}$ with $67 = a + b + 3 = j + 2k - 3$.

p	$\lambda(p)$ on $S_{0,35}$
2	−25073418240
3	−1182455157157840
5	9470081642319930937500
7	−10370198954152041951342796400
11	−8015071689632034858364818146947656
13	−20232136256107650938383898249808243380
17	118646313906984767985086867381297558266980
19	2995917272706383250746754589685425572441160
23	−1911372622140780013372223127008015060349898320
29	−2129327273873011547769345916418120573221438085460
31	−157348598498218445521620827876569519644874180822976
37	−47788585641545948035267859493926208327050656971703460

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