Siegel Modular Forms
Lecture $\#15$

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Summary

For $g = 1$ we have the formula (for $a > 0$ even)

$$e_c(\mathcal{A}_1, \mathbf{V}_a) = -S[a+2] - 1\,.$$

Here

$$e_c(\mathcal{A}_1, \mathbf{V}_a) = \sum_{i=0}^{2} (-1)^i H_c^i(\mathcal{A}, \mathbf{V}_a)$$

and $S[a+2]$ is the motive constructed by Scholl associated to $S_{a+2}(\Gamma_1)$. The motive $S[a+2]$ has a Hodge decomposition

$$S[a+2] = S_{a+2}(\Gamma_1) \oplus \overline{S_{a+2}(\Gamma_1)}$$

and has dimension $2 \dim S_{a+2}(\Gamma_2)$. We can also view it as a bookkeeping device such that

$$\mathrm{Tr}(F_p, S[a+2]) = \mathrm{Tr}(T(p), S_{a+2}(\Gamma_1)) \,.$$

We know by Deligne that for a normalized eigenform $f = \sum a(n)q^n \in S_{a+2}(\Gamma_1)$ we have

$$|a(p)| \leq 2p^{(a+1)/2} \,.$$

We have seen how to calculate traces by counting points on elliptic curves over $\mathbf{F}_p$. Having counted this for the (fixed) prime p we can calculate the trace of $T(p)$ on the spaces $S_{a+2}(\Gamma_1)$ for all a. Similarly for $T(p^2)$ etc.

Two more remarks. We have

$$e_c(\mathcal{A}_1, \mathbf{V}_a) = -S[a+2] - 1,$$
$$e(\mathcal{A}_1, \mathbf{V}_a) = -S[a+2] - L^{a+1}\,.$$

The Eisenstein series E_{a+2} has eigenvalue $1 + p^{a+1}$ at $T(p)$. Here $L := h^2(\mathbf{P}^1)$ is the Tate motive of weight 2.

The formulas remain valid for $a = 0$ with putting

$$S[2] = -L - 1$$

because

$$e_c(\mathcal{A}_1, \mathbf{V}_0) = L\,.$$

Genus Two

We have extended this to genus 2. For $g = 2$ we found a formula

$$e_c(\mathcal{A}_2, \mathbf{V}_{a,b}) = -S[a-b, b+3] + e_{2,extra}(a, b)$$

where $S[a-b, b+3]$ is a (hypothetical) motive or can be seen as a bookkeeping device such that

$$\mathrm{Tr}(F_p, S[a-b, b+3]) = \mathrm{Tr}(T(p), S_{a-b,b+3}(\Gamma_2))$$

Moreover,

$$\dim S[a-b, b+3] = 4 \dim S_{a-b,b+3}(\Gamma_2)$$

The ‘correction term’ $e_{2,\mathrm{extra}}(a, b)$ is expressible in $g = 1$-terms.

By listing isomorphism classes curves of genus 2 over $\mathbf{F}_p$ with their number of points and automorphisms we are able to calculate the trace of F_p on

$$e_c(\mathcal{A}_2 \otimes \overline{\mathbf{F}}_p, \mathbf{V}_{a,b}) \quad \text{and} \quad e_{2,extra}(a,b)$$

and deduce the trace of $T(p)$ on $S_{a-b,b+3}(\Gamma_2)$ for all (a,b).

For $(a,b)=(0,0)$ we define

$$S[0,3] = -(1+L)(1+L^2)$$

The Saito-Kurokawa lift of a Hecke eigenform $f \in S_{2a+4}(\Gamma_1)$ with a odd is a Hecke eigenform in $S_{0,a+3}(\Gamma_2)$ with spinor L-function

$$\zeta(s-a-2)L(f,s)\zeta(s-a-1)\,.$$

So the trace of $T(p)$ on the Maass subspace $S^*_{a+3}(\Gamma_2)$ equals

$$\mathrm{Tr}(T(p), S_{2a+4}(\Gamma_1)) + s_{2a+4}(p^{a+1}+p^{a+2})$$

with $s_k = \dim S_k(\Gamma_1)$.

In this way $S_{2a+4}(\Gamma_1)$ contributes

$$S[2a+4] + s_{2a+4}(L^{a+1}+L^{a+2})$$

to the hypothetical motive $S[a-b,b+3] = S[0,a+3]$.

But it contributes only $S[2a+4]$ to the interior cohomology $H^3_!(\mathcal{A}_2, \mathbf{V}_{a,a})$. This is the part of the cohomology where Frobenius F_p acts with eigenvalues of absolute value

$$p^{(a+b+3)/2} = p^{(2a+3)/2}\,.$$

Note that with $(j,k) = (a-b, b+3)$ we have Deligne weight

$$a+b+3 = j+2k-3\,.$$

Concerning the Maeda Conjecture:

In the cases where $\dim S_{j,k}(\Gamma_2)$ is 2-dimensional we found 4 cases where the space splits under $T(2)$ and $T(3)$:

$$(18,7),(10,13),(0,24),(0,26)$$

The last two cases were discovered by Skoruppa. The case $(10,3)$ contains Sym^3 of the space $S_{12}(\Gamma_1)$.

Level Two

We showed how one can use counting points on curves of genus 2 over finite fields to obtain the traces of Hecke operators on Siegel modular forms on Γ_2.

In joint work with Bergström and Faber we extended all this to the case of degree 2 and level 2. Here the group Γ_2 is replaced by the congruence subgroup

$$\Gamma_2[2] = \ker(\mathrm{Sp}(4, \mathbf{Z}) \to \mathrm{Sp}(4, \mathbf{Z}/2\mathbf{Z}))$$

Since $\mathrm{Sp}(4, \mathbf{Z}/2\mathbf{Z}) \cong \mathfrak{S}_6$, the symmetric group on six letters, everything comes with an action of this group. We formulated the conjectural analogue of the above theorem.

The cohomology and spaces of modular forms appear as representation spaces of $\mathfrak{S}_6$.

A curve C of genus 2 admits a unique morphism of degree 2 to $\mathbf{P}^1$ with six ramification points. Let $\mathcal{M}_2(w)$ be the moduli space of curves of genus 2 together with six ordered Weierstrass points $r_1, \ldots, r_6$. The non-zero points of $\mathrm{Jac}(C)[2]$ correspond $1-1$ to the 15 pairs (r_i, r_j). The map

$$(\mathbf{Z}/2\mathbf{Z})^6 \to \mathrm{Jac}(C)[2], \quad a \mapsto \sum a_i(r_i - r_1)$$

induces an identification $\mathrm{Jac}(C)[2] \cong K$ with $K = \ker(\Sigma)/(\mathbf{Z}/2\mathbf{Z})$, that is, the kernel of

$$\Sigma : (\mathbf{Z}/2\mathbf{Z})^6 \to \mathbf{Z}/2\mathbf{Z}, \quad a \mapsto \sum a_i\,,$$

modulo the diagonally embedded $\mathbf{Z}/2\mathbf{Z}$. The symplectic form on $\mathrm{Jac}(C)[2]$ can be identified with

$$(a,b) \mapsto \sum a_i b_i \,.$$

Therefore $\mathfrak{S}_6$ embeds into $\mathrm{Sp}(4,\mathbf{Z}/2\mathbf{Z})$ and this fixes an isomorphism.

By associating to a curve its Jacobian we get an embedding

$$\mathcal{M}_2(w) \hookrightarrow \mathcal{A}_2[2]$$

On this moduli space $\mathcal{A}_2[2]$ we have the local systems $\mathbf{V}_{a,b}$ and we can study the motivic Euler characteristic

$$e_c(\mathcal{A}_2[2],\mathbf{V}_{a,b}) = \sum_{i=0}^{6}(-1)^i[H_c^i(\mathcal{A}_2[2],\mathbf{V}_{a,b})] \,.$$

We look again at the interior cohomology, that is, the image $H^*_!(\mathcal{A}_2[2], \mathbf{V}_{a,b})$ of

$$H^*_c(\mathcal{A}_2[2], \mathbf{V}_{a,b}) \to H^*(\mathcal{A}_2[2], \mathbf{V}_{a,b})\,.$$

By Faltings we know that $H^3_!(\mathcal{A}_2[2], \mathbf{V}_{a,b})$ carries a Hodge structure with Hodge filtration

$$(0) \subseteq F^{a+b+3} \subseteq F^{a+2} \subseteq F^{b+1} \subseteq F^0$$

and

$$F^{a+b+3} \cong S_{a-b,b+3}(\Gamma_2[2])\,.$$

We also know that for regular (a, b) we have

$$H^i_c(\mathcal{A}_2, \mathbf{V}_{a,b}) \neq (0) \Rightarrow i = 3\,.$$

In fact, Faltings gives an interpretation of the Hodge filtration on $H_c^\bullet(\mathcal{A}_2, \mathbf{V}_{a,b})$ in terms of coherent cohomology of bundles made from the Hodge bundle:

$$F^0/F^{b+1} \cong H_c^\bullet(\tilde{\mathcal{A}}_2, \mathbf{E}_{a-b,-b}(-D))$$

$$F^{b+1}/F^{a+2} \cong H_c^{\bullet-1}(\tilde{\mathcal{A}}_2, \mathbf{E}_{a+b+2,-a}(-D))$$

$$F^{a+2}/F^{a+b+3} \cong H_c^{\bullet-2}(\tilde{\mathcal{A}}_2, \mathbf{E}_{a+b+2,1-b}(-D))$$

$$F^{a+b+3} \cong H_c^{\bullet-3}(\tilde{\mathcal{A}}_2, \mathbf{E}_{a-b,b+3}(-D))$$

with $\mathbf{E}_{m,n} = \mathrm{Sym}^m(\mathbf{E}) \otimes \det(\mathbf{E})^n$ and D the divisor at infinity.

We counted again points on curves together with Weierstrass points in order to calculate the trace of Frobenius on the Euler characteristic

$$e_c(\mathcal{A}_2[2], \mathbf{V}_{a,b}) = \sum_i (-1)^i H_c^i(\mathcal{A}_2[2] \otimes \overline{\mathbf{F}}_q, \mathbf{V}_{a,b})$$

where H_c^i stands for compactly supported etale cohomology. The action of $\mathfrak{S}_6$ induces a decomposition into pieces corresponding to the irreducible representations of $\mathfrak{S}_6$. We then count the isotypical part $e_{c,\mu}(\mathcal{A}_2[2], \mathbf{V}_{a,b})$ for an irrep μ of $\mathfrak{S}_6$.

We have to determine the Eisenstein cohomology $e_{\rm Eis}(\mathcal{A}_2[2], \mathbf{V}_{a,b})$ defined as the kernel of of the map $H_c^* \to H^*$.

Theorem 1. *For regular pairs (a, b) the Eisenstein cohomology for $\mathbf{V}_{a,b}$ on $\mathcal{A}_2[2]$ is given by*

$$15\, s_{a-b+2}(\Gamma_1[2]) - 15\, s_{a+b+4}(\Gamma_1[2])\, L^{b+1} +$$

$$+15 \begin{cases} S[\Gamma_1[2], b+2] + 3 & \text{if } b \text{ even} \\ -S[\Gamma_1[2], a+3] & \text{if } b \text{ odd.} \end{cases}$$

with $s_k(\Gamma_1[2]) = \dim S_k(\Gamma_1[2])$.

Here $\Gamma_1[2] = \ker(\mathrm{SL}(2, \mathbf{Z}) \to \mathrm{SL}(2, \mathbf{Z}/2\mathbf{Z}))$. The 15 corresponds to the fact that $\mathcal{A}_2[2]^*$ has 15 boundary components of dimension 1 and also of dimension 0.

In level 1 we encountered the Saito-Kurokawa lifting to scalar-valued forms. Here we also have liftings to vector-valued forms. These are called Yoshida lifts.

Theorem 1. *For Hecke eigenforms* $f \in S_{a+b+4}(\Gamma_0(2))^{\mathrm{new}}$ *and* $g \in S_{a-b+2}(\Gamma_0(2))^{\mathrm{new}}$ *there is a Hecke eigenform* $F \in S_{a-b,b+3}(\Gamma_2[2])$ *with spinor* L*-function*

$$L(F, s) = L(f, s)L(g, s - b - 1).$$

The form F *will appear with multiplicity* 5 *in* $S_{a-b,b+3}(\Gamma_2[2])$ *if* f *and* g *have the same eigenvalue* $\pm$ *under* w_2 *and with multiplicity* 1 *if they have opposite eigenvalues under* w_2*.*

Here w_2 is the Atkin-Lehner involution.

Similarly, for eigenforms $f \in S_{a+b+4}(\Gamma_0(4))^{\mathrm{new}}$ and $g \in S_{a-b+2}(\Gamma_0(4))^{\mathrm{new}}$ there is a form $F \in S_{a-b,b+3}(\Gamma_2[2])$ with spinor L-function

$$L(F,s) = L(f,s)L(g,s-b-1)$$

and it will appear with multiplicity 5 in $S_{a-b,b+3}(\Gamma_2[2])$.

To the Hecke eigenforms f and g we find a contribution

$$\xi \otimes M_f + \eta \otimes L^{b+1} M_g$$

in $e_c(\mathcal{A}_2[2], \mathbf{V}_{a,b})$. Here M_f and M_g are the motives (of dimension 2) associated to f and g and ξ and η are zero or different representations of $\mathfrak{S}_6$. The weights are $a+b+3$ and $a+2$.

In the interior cohomology we only see M_f. To find a 4-dimensional piece we have to add the $L^{b+1}M_g$. Note that the absolute values of eigenvalues of F_p on M_f are $p^{(a+b+3)/2}$.

To give an idea a table follows here. Here we will write $\Phi_{N,k} := S[\Gamma_0(N), k]$; here a motive associated to a single newform. $S[n,m]$ stands for the motive associated to $S_{n,m}(\Gamma_2[2])$.

(a,b)	$e_c(\mathcal{A}_2[2], \mathbf{V}_{a,b})$
$(0,0)$	$L^3 + L^2 - 14L + 16$
$(2,0)$	$-30L + 30$
$(1,1)$	$5L^3 - 10L^2$
$(4,0)$	$-45L + 45 - 10L\Phi_{4,6}$
$(3,1)$	$-30L^2 - 15\Phi_{4,6}$
$(2,2)$	$9L^4 - 21L^3 - \Phi_{2,8}$
$(6,0)$	$-60L + 60 - 31L\Phi_{2,8} - \Phi_{2,10}$
$(5,1)$	$-45L^2 + 15 - 30\Phi_{2,8} - 20L\Phi_{4,6} - 5\Phi_{4,10}$
$(4,2)$	$-45L^3 + 45 - S[2,5]$
$(3,3)$	$10L^5 - 35L^4 - 15\Phi_{4,6} - 5\Phi_{2,10}$
$(8,0)$	$-75L + 75 - 25L\Phi_{4,10} - 40L\Phi_{2,10} - 5\Phi_{4,12}$
$(7,1)$	$-60L^2 + 30 - 15\Phi_{4,10} - 30\Phi_{2,10} - 40L^2\Phi_{2,8} - S[6,4]$
$(6,2)$	$-60L^3 + 60 - 20L^3\Phi_{4,6} - S[4,5]$
$(5,3)$	$-60L^4 - 30\Phi_{2,8} - S[2,6]$
$(4,4)$	$15L^6 - 45L^5 + 30 - 15\Phi_{4,6} - 5\Phi_{4,12}$

For example, take the case $(a, b) = (5, 1)$:

$$-45L^2 + 15 - 30\Phi_{2,8} - 20L\Phi_{4,6} - 5\Phi_{4,10}.$$

This can be split in the isotypical parts for $\mathfrak{S}_6$: First there is the Eisenstein cohomology,

$$\begin{gathered}-S[\Gamma_0(2), 8]^{\text{new}}(s[2^3] + s[3,2,1] + s[4,2]) \\ -L^2(s[3,2,1] + s[3^2] + s[4,1^2] + s[4,2] + s[5,1]) \\ +(s[3^2] + s[4,1^2]),\end{gathered}$$

then the endoscopic cohomology,

$$-L^2\, S[\Gamma_0(4), 6]^{\text{new}}(s[3,1^3] + s[4,1^2]),$$

and finally there is a lift of Yoshida type, contributing

$$-S[\Gamma_0(4), 10]^{\text{new}}\, s[2,1^4].$$

We formulated a completely explicit conjecture on the shape of $e_c(\mathcal{A}_2[2], \mathbf{V}_{a,b})$ and this allowed us to calculate the traces of Hecke operators on spaces of Siegel modular forms.

Our conjectures were later proven by M. Rösner. For many examples the values can be found on the website http://smf.compositio.nl .

p	$S_{2,5}(\Gamma_2[2])^{[2^2,1^2]}$	$S_{6,4}(\Gamma_2[2])^{[2^2,1^2]}$
3	$-2^3 \cdot 5$	$-2^3 \cdot 5 \cdot 7$
5	$-2^2 \cdot 5^2 \cdot 13$	$-2^2 \cdot 5 \cdot 149$
7	$2^4 \cdot 3 \cdot 5 \cdot 13$	$-2^4 \cdot 3 \cdot 5 \cdot 401$
11	$2^3 \cdot 11 \cdot 13 \cdot 31$	$2^3 \cdot 36383$
13	$-2^2 \cdot 5 \cdot 3469$	$2^2 \cdot 5 \cdot 37 \cdot 251$
15	$-2^2 \cdot 5 \cdot 11 \cdot 13 \cdot 197$	$2^2 \cdot 5 \cdot 19 \cdot 6983$
17	$2^3 \cdot 5^2 \cdot 11 \cdot 13 \cdot 17$	$-2^3 \cdot 5 \cdot 29 \cdot 6287$
19	$-2^4 \cdot 5 \cdot 13 \cdot 311$	$-2^4 \cdot 5 \cdot 43 \cdot 2267$

This approach – by counting points over finite fields– can also be applied to other cases. In the next lecture we will look at the case $g = 3$. Other examples regard ball quotients. For example, one can look at curves of genus 3 that are triple cycle covers of the projective line:

$$y^3 = f(x)$$

with f of degree 4. The Jacobians of such curves have multiplication by $F = \mathbf{Q}(\sqrt{-3})$ and their moduli are parametrized by an (open part of) a ball quotient $\Gamma\backslash B_2$ with Γ a discrete subgroup of the group of similitudes of a 3-dimensional vector space F^3 with hermitian form

$$z_1\bar{z}_2 + z_2\bar{z}_1 + z_3\bar{z}_3$$

Jonas Bergström and I counted and obtained similar results on Picard modular forms.

Shimura gave in 1964 a list of rational ball quotients, that is, birationally equivalent to projective space. These ball quotients turn out to be moduli spaces of curves that are covers of the projective line of given degree and therefore are amenable to our approach.

The Igusa Quartic

By work of Igusa we know that the ring

$$R^{\mathrm{ev}}(\Gamma_2[2]) = \oplus_{k\,\mathrm{even}} M_k(\Gamma_2[2])$$

is generated by a 5-dimensional vector space of modular forms of weight 2 that form a representation of type $[2,1,1,1,1]$ for the group $\mathfrak{S}_6$ satisfying a quartic relation.

If we write the fourth powers $\vartheta[\epsilon](\tau,0)^4$ of the ten even theta constants as x_i with $i = 1,\ldots,10$ (in an appropriate order) then the

x_i satisfy linear relations defining a $\mathbf{P}^4 \subset \mathbf{P}^9$

$$
\begin{gathered}
x_6 = x_1 - x_2 + x_3 - x_4 - x_5, \\
x_7 = x_2 - x_3 + x_5 \\
x_8 = x_1 - x_4 - x_5, \\
x_9 = -x_3 + x_4 + x_5, \\
x_{10} = x_1 - x_2 - x_5
\end{gathered}
$$

and one quartic equation

$$\left(\sum_{i=1}^{10} x_i^2\right)^2 - 4\sum_{i=1}^{10} x_i^4 = 0$$

the Igusa quartic. This defines the Satake compactification $\mathcal{A}_2[2]^*$. The ring of modular forms of all weights is a quadratic extension generated by χ_5.

The Igusa quartic admits another model in $\mathbf{P}^4 \subset \mathbf{P}^5$ defined by

$$\sigma_1 = 0, \quad \sigma_2^2 - 4\sigma_4 = 0$$

with σ_i the ith elementary symmetric function of $y_1, \ldots, y_6$.

Mukai discovered that if one considers the subgroup $\Gamma' \subset \Gamma_2$ defined by

$$\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2 : c \equiv\ 0 \mod 2\}$$

then the Satake compactification $\mathcal{A}^*_{\Gamma'}$ is also given by the Igusa quartic. Since we have a Galois covering

$$\mathcal{A}^*_2[2] \to \mathcal{A}^*_{\Gamma'}[2]$$

with group $(\mathbf{Z}/2\mathbf{Z})^3$, this means that the Igusa quartic admits a self-map of degree 8. Mukai used geometry. One may rephrase this in terms of modular forms as follows.

The ring of modular forms on $\Gamma_2[2]$ is generated by x_1, x_2, x_3, x_4 and $\xi = x_5 - x_6$ of weight 2. These satisfy the relation

$$(s_1^2 - 4s_2 - \xi^2)^2 = 64\, s_4$$

with s_i the ith elementary symmetric function of $x_1, \ldots, x_4$.

Consider now the forms

$$\begin{aligned} X_1 &= (x_1 + x_2 + x_3 + x_4)^2, \\ X_2 &= (x_1 - x_2 + x_3 - x_4)^2, \\ X_3 &= (x_1 + x_2 - x_3 - x_4)^2, \\ X_4 &= (x_1 - x_2 - x_3 + x_4)^2, \end{aligned}$$

and

$$\eta = 2(s_1^2 - 4s_2 - \xi^2)$$

One shows that these forms generate $M_4(\Gamma')$. One checks that these satisfy

$$(\gamma_1 - 4\gamma_2 - \eta^2)^2 = 64\,\gamma_4$$

with γ_i the ith elementary symmetric function of the X_i.

We thus observe

$$\dim M_k(\Gamma_2[2]) = \dim M_{2k}(\Gamma')$$

but also

$$\dim M_{2,k}(\Gamma_2[2]) = \dim M_{2,2k}(\Gamma')$$

and

$$\dim M_{2,k+1}(\Gamma_2[2]) = \dim M_{2,2k+1}(\Gamma')$$

The hyperplane bundle of the Igusa quartic (restricted to $\mathcal{A}_2[2]$) is the anti-canonical bundle. If a group Γ acts freely we have with $L = \det(\mathbf{E})$ in the Picard group $K_{\mathcal{A}_\Gamma} = 3[L]$ since the cotangent bundle is $\mathrm{Sym}^2(\mathbf{E})$. But for non-freely acting group we have to correct this; the map $\mathcal{H}_2 \to \mathcal{A}_2[2]$ is ramified along $\mathcal{A}_{1,1}$. The corrected formula is

$$K_{\mathcal{A}_{\Gamma_2[2]}} = 3[L] - 5[L] = -2[L]\,.$$

For Γ' we observe that

$$\mathcal{A}_{\Gamma_2[2]} \to \mathcal{A}_{\Gamma'}$$

is ramified along the surface given by the vanishing of ξ. This gives

$$K_{\mathcal{A}_{\Gamma'}} = 3[L] - (5+2)[L] = -4[L]\,.$$

By the Koecher Principle this suffices.

The cohomological calculations also provide information on modules of vector-valued forms. The direct sums

$$\oplus_k M_{j,k}(\Gamma_2[2])$$

for fixed value of j are modules over $R(\Gamma_2[2])$. We can calculate the dimensions of all $\mathfrak{S}_6$-isotypical parts.

We determined the structure of such modules for small values of j.

One example. Consider the $R^{\mathrm{ev}}(\Gamma_2[2])$-module

$$\Sigma_2 = \oplus_{k:\mathrm{odd}} S_{2,k}(\Gamma_2[2])\,.$$

$S_{2,k} \backslash P$	$[6]$	$[5,1]$	$[4,2]$	$[4,1^2]$	$[3^2]$	$[3,2,1]$	$[3,1^3]$	$[2^3]$	$[2^2,1^2]$	$[2,1^4]$	$[1^6]$
$S_{2,5}$	0	0	0	0	0	0	0	0	1	0	0
$S_{2,7}$	0	0	0	1	1	1	0	0	1	0	0
$S_{2,9}$	0	1	0	2	1	2	1	0	3	1	1
$S_{2,11}$	0	2	1	4	3	5	2	0	4	1	1
$S_{2,13}$	0	2	2	6	5	9	4	1	8	2	1

The representation type of $M_{0,k}(\Gamma[2])$ for even k with $2 \leq k \leq 12$.

$k \backslash P$	$[6]$	$[5,1]$	$[4,2]$	$[4,1^2]$	$[3^2]$	$[3,2,1]$	$[3,1^3]$	$[2^3]$	$[2^2,1^2]$	$[2,1^4]$	$[1^6]$
2	0	0	0	0	0	0	0	1	0	0	0
4	1	0	1	0	0	0	0	1	0	0	0
6	1	0	1	0	0	0	1	2	0	1	0
8	1	0	3	0	0	1	1	3	0	0	0
10	2	0	3	0	0	2	3	4	0	2	0
12	3	1	6	1	0	3	4	5	0	2	0

We introduce the modular forms

$$\Phi_i = [\vartheta_i^4, \chi_5]/\vartheta_i^4 \in S_{2,5}(\Gamma_2[2])\,.$$

with ϑ_i $(i = 1, \ldots, 10)$ the ten even theta characteristics. Since we know the decomposition of $S_{2,k}$ as a $\mathfrak{S}_6$-representation we can read off the relations.

Theorem 2. *The R_2^{ev}-module Σ_2 is generated by the ten modular forms Φ_i.*

Then ten modular forms $\Phi_i \in S_{2,5}$ satisfy the relation

$$\sum_{i=1}^{10} \Phi_i = 0$$

and they span a 9-dimensional $\mathfrak{S}_6$-representation of type $[2,2,1,1]$. In order to prove the result one uses the fact that the vector bundle $\mathrm{Sym}^2(\mathbf{E})$ is 3-regular with respect to $\det(\mathbf{E})$. This gives a bound on the weights of the generators and relations.

First we show that the Φ_i generate $S_{2,7}(\Gamma_2[2])$ and $S_{2,9}(\Gamma_2[2])$. For example, a linear relation

$$x_1 - x_4 - x_6 - x_7 = 0$$

gives by taking the bracket $[\chi_5, --]$ a linear relation

$$x_1\Phi_1 - x_4\Phi_4 - x_6\Phi_6 - x_7\Phi_7 = 0$$

We thus get an irrep $s[2,1^4] \otimes s[1^6] = s[5,1]$ of relations. We then consider

$$M_2(\Gamma_2[2]) \otimes S_{2,5}(\Gamma_2[2]) = s[2^3] \otimes s[2^2,1^2]$$

and comparing $s[2^3] \otimes s[2^2, 1^1]$, that is,

$$= s[5,1]+s[4,1^2]+s[3^2]+s[3,2,1]+s[2^2,1^2]$$

with

$$S_{2,7}(\Gamma_2[2]) = s[4,1^2]+s[3^2]+s[3,2,1]+s[2^2,1^2]$$

we see that we need exactly the $s[5,1]$ of relations that we got.

Literature

J. Bergström, C. Faber, G. van der Geer: *Siegel modular forms of genus 2 and level 2: Cohomological computations and conjectures.* IMRN 2008, rnn 100, 20 pp.

F. Cléry, G. van der Geer, S. Grushevsky: *Modular forms of genus* 2 *and level* 2 . Int. J. of Math. Vol. 26, (2015).

S. Mukai: *Igusa quartic and Steiner surfaces.* In: Compact Moduli Spaces and Vector Bundles, Cont. Math., 564 (2012).

M. Rösner: *Parahoric restriction for* $GSp(4)$ *and the inner cohomology of Siegel modular threefolds.* Ph.D. thesis, University of Heidelberg, 2016.

G. Shimura: *On purely transcendental fields of automorphic functions of several variables.* Osaka Math. Journal 1 (1964), 1-14.

websites:

smf.compositio.nl

van-der-geer.nl