Siegel Modular Forms
Lecture $\#16$

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The Case $g = 3$

For $g = 1$ we have the formula

$$e_c(\mathcal{A}_1, \mathbf{V}_a) = -S[a+2] - 1$$

and for $g = 2$ we found a formula

$$e_c(\mathcal{A}_2, \mathbf{V}_{a,b}) = -S[a-b, b+3] + e_{2,\mathrm{extra}}(a,b)$$

where $S[a-b, b+3]$ is a motive or bookkeeping device such that the trace of $T(p)$ on $S_{a-b,b+3}(\Gamma_2)$ equals

$$\mathrm{Tr}(F_p, S[a-b, b+3])$$

Moreover,

$$\mathrm{rk}\, S[a+2] = 2 \dim S_{a+2}(\Gamma_1)$$

and

$$\mathrm{rk}\, S[a-b, b+3] = 4 \dim S_{a-b,b+3}(\Gamma_2)\,.$$

We calculate the trace of F_p on $e_c(\mathcal{A}_2, \mathbf{V}_{a,b})$ and $e_{2,\mathrm{extra}}(a,b)$ and then deduce the trace of $T(p)$ on $S_{a-b,b+3}(\Gamma_2)$.

Now $g = 3$. We have the universal abelian variety

$$\pi : \mathcal{X}_3 \to \mathcal{A}_3$$

and the corresponding local system

$$\mathbf{V} = R^1\pi_*\mathbf{Q}$$

of rank 6. For each triple $\lambda = (a, b, c)$ with $a \geq b \geq c \geq 0$ we have a local system $\mathbf{V}_\lambda$.

We want an analogue of this formula to calculate the trace of $T(p)$ on

$$S_{a-b,b-c,c+4}(\Gamma_3)\,.$$

We put

$$e_c(\mathcal{A}_3,\mathbf{V}_\lambda)=\sum_i(-1)^i H_c^i(\mathcal{A}_3,\mathbf{V}_\lambda)$$

We call this the motivic Euler characteristic of the local system $\mathbf{V}_\lambda$. We sometimes use the numerical Euler characteristic

$$n_c(\mathcal{A}_3,\mathbf{V}_\lambda)=\sum_i(-1)^i \dim H_c^i(\mathcal{A}_3,\mathbf{V}_\lambda)$$

These numbers are known (by Bini-vdG).

Based on numerical evidence we (B-F-vdG) conjectured the following.

Conjecture 1. *The motivic Euler characteristic* $e_c(\mathcal{A}_3 \otimes \overline{\mathbf{F}}_p, \mathbf{V}_\lambda)$ *is equal to*

$$S[a-b, b-c, c+4] + e_{3,\mathrm{extra}}(a,b,c)$$

with $e_{3,\mathrm{extra}}(a,b,c)$ *defined by*

$$\begin{aligned} -e_c(\mathcal{A}_2, \mathbf{V}_{a+1,b+1}) \\ +e_c(\mathcal{A}_2, \mathbf{V}_{a+1,c}) \\ -e_c(\mathcal{A}_2, \mathbf{V}_{b,c}) \\ -e_{2,\mathrm{extra}}(a+1, b+1) \otimes S[c+2] \\ +e_{2,\mathrm{extra}}(a+1, c) \otimes S[b+3] \\ -e_{2,\mathrm{extra}}(b, c) \otimes S[a+4] \end{aligned}$$

Here $S[a-b, b-c, c+4]$ *is a hypothetical motive or bookkeeping device such that* $\mathrm{Tr}(T(p), S_{a-b,b-c,c+4}(\Gamma_3))$ *is equal to*

$$\mathrm{Tr}(F_p, S[a-b, b-c, c+4])\,.$$

This was found by using the Torelli map $t : \mathcal{M}_3 \to \mathcal{A}_3$ and calculating Frobenius eigenvalues for curves over finite fields. We have a stratification of $\mathcal{A}_3$:

$$t(\mathcal{M}_3^0) \sqcup t(H_3) \sqcup (t(\mathcal{M}_2) \times \mathcal{A}_1) \sqcup \mathcal{A}_{1,1,1}$$

and $\mathcal{M}_3$ is stratified by $\mathcal{M}_3^0 \sqcup H_3$, non-hyperelliptic (ternary quartics) and hyperelliptic. Thus all abelian varieties corresponding to points of $\mathcal{A}_3$ can be described by products of Jacobians of smooth curves.

For each isomorphism class of an abelian variety X of dimension g over $\mathbf{F}_q$ we can calculate the eigenvalues of Frobenius acting on the Tate module:

$$\alpha_1(X), \ldots, \alpha_g(X), \overline{\alpha}_1(X), \ldots, \overline{\alpha}_g(X)$$

By the Lefschetz trace formule we see that

$$\mathrm{Tr}(F_q, e_c(\mathcal{A}_g \otimes \overline{\mathbf{F}}_q), \mathbf{V}_\lambda)$$

equals

$$\sum_{[X] \in \mathcal{A}_g(\mathbf{F}_q)} \frac{s_\lambda(\alpha_1(X), \ldots, \alpha_g(X))}{\#\mathrm{Aut}_{\mathbf{F}_q}(X)}$$

where s_λ is a symmetric polynomial in the $\alpha_i(X)$ and $\bar{\alpha}_i(X)$, a Schur polynomial.

We carried out this counting for $g = 3$ and $p \leq 17$. This means we can calculate (conjecturally) the trace of $T(p)$ on $S_{a-b,b-c,c+4}$ for $p \leq 17$ and all $a \geq b \geq c \geq 0$.

The evidence is overwhelming.

Some arguments:

1) results are integers

2) dimensions fit (cf. Taïbi's work).

3) the numerical Euler characteristic of $S[a-b, b-c, c=4]$ is divisible by 8

4) consistency with other results

5) Harder type congruences

Two examples:

$$e_c(\mathcal{A}_3, \mathbf{V}_{11,5,2}) = S[6,3,6] - S[12]L^3 + L^7 - L^3 + 1$$

We know $\dim S_{6,3,6}(\Gamma_3) = 1$. Assuming this we calculate the eigenvalues

$$\lambda(2) = 0,\ \lambda(3) = -453600, \ldots,$$

$$\lambda(17) = -107529004510200$$

We can construct the eigenform generating $S_{6,3,6}$ using concomitants and check a few eigenvalues.

Another example:

$$e_c(\mathcal{A}_3, \mathbf{V}_{10,6,4}) = S[4,2,8] + L^8 - S[4,10] - 1$$

p	$\lambda(q)$ on $S_{6,3,6}$	$\lambda(q)$ on $S_{4,2,8}$
2	0	9504
3	−453600	970272
4	10649600	89719808
5	−119410200	−106051896
7	12572892800	112911962240
8	0	1156260593664
9	−29108532600	5756589166536
11	−57063064032	44411629220640
13	−25198577349400	209295820896008
16	341411782197248	−369164249202688
17	−107529004510200	1230942201878664
19	1091588958605600	51084504993278240

Liftings

For $g = 2$ we have the Saito-Kurokawa lifting for odd a

$$S_{2a+4}(\Gamma_1) \to S_{0,a+3}(\Gamma_2), \quad f \mapsto \tilde{f}$$

e.g. $f_{18} \mapsto \chi_{10}$. Hecke eigenvalue for $T(p)$

$$\lambda_{\tilde{f}}(p) = p^{a+1} + a(p) + p^{a+2}$$

Motive: $L^{a+1} + S[2a+4] + L^{a+2}$. But only $S[2a+4]$ fits in $H^3_!(\mathcal{A}_2, \mathbf{V}_{a,a})$;

For $g = 1$ the space $S_k(\Gamma_1)$ is conjectured to be irreducible as Hecke module (Maeda conjecture). For $g \geq 2$ we see submodules.

If we have elliptic modular forms $f_i \in S_{k_i}(\Gamma_1)$ for $i = 1, \ldots, m$, eigenforms with Satake parameters at p

$$(\alpha_{p,0}(f_i), \alpha_{p,1}(f_i))$$

we can define an L-series

$$L(\otimes_{i=1}^{m} \mathrm{Sym}^{r_i}(f_i), s)$$

by an Euler product with p-factors. For example for $f \in S_k(\Gamma_1)$ with Euler p-factor of $L(f, s)$ given by

$$\big((1-\alpha\, p^{-s})(1-\bar{\alpha}\, p^{-s})\big)^{-1}$$

the Euler p-factor of $L(\mathrm{Sym}^2(f), s)$ is

$$\big((1-\alpha^2 p^{-s})(1-\alpha\bar{\alpha}\, p^{-s})(1-\bar{\alpha}^2 p^{-s})\big)^{-1}$$

and if $h \in S_l(\Gamma_1)$ has the Euler p-factor

$$\big((1-\beta\, p^{-s})(1-\bar{\beta}\, p^{-s})\big)^{-1}$$

the inverse of the Euler p-factor of $L(f \otimes h, s)$ is

$$(1-\alpha\beta p^{-s})(1-\alpha\bar{\beta}\, p^{-s})(1-\bar{\alpha}\bar{\beta} p^{-s})(1-\bar{\alpha}\bar{\beta} p^{-s})$$

$g = 3$

We find experimentally and conjecture generally the following motives/liftings in $S[a-b, b-c, c+4]$:

1) For $a > b > c > 0$ we find

$$s_{a-c+3} S[b+3] \otimes S[a+c+5]$$

That is, for eigenforms $f \in S_{b+3}(\Gamma_1)$, $g \in S_{a+c+5}(\Gamma_1)$ and $h \in S_{a-c+3}(\Gamma_1)$ there is $F \in S_{a-b,b-c,c+4}(\Gamma_3)$ with

$$L(F, s) = L(f \otimes g, s)\, L(f \otimes h, s-c-1)\,.$$

2) For $b = c$ we find

$$S[a+4] \otimes S[2b+4]$$

For eigenforms $f \in S_{a+4}(\Gamma_1)$ and $g \in S_{2b+4}(\Gamma_1)$ there is $F \in S_{a-b,0,b+4}(\Gamma_3)$ with

$$L(F,s) = L(f,s-b-1)L(f,s-b-2)L(f \otimes g,s)$$

3) For $a = b$ and $c > 0$ we find

$$S[c+2] \otimes S[2a+6]$$

For eigenforms $f \in S_{c+2}(\Gamma_1)$ and $g \in S_{2a+6}(\Gamma_1)$ there is $F \in S_{0,a-c,c+4}(\Gamma_3)$ with

$$L(F,s) = L(f,s-a-2)L(f,s-a-3)L(f \otimes g,s)$$

Example. Take $(a, b, c) = (8, 8, 8)$. Miyawaki observed for the generator $F \in S_{12}(\Gamma_3)$ that

$$L(F, s) = L(\Delta, s-9)L(\Delta, s-10)L(\Delta \otimes f_{20}, s)$$

with $f_{20} \in S_{20}(\Gamma_1)$. This was proven by Ikeda.

Miyawaki conjectured the cases 2) and 3) for $a = b = c$.

Example. $\lambda = (12, 12, 12)$. We have $\dim S_{0,0,16}(\Gamma_3) = 3$. There is supposedly a 2-dimensional space of lifts. There should be a 'genuine' form in $S_{0,0,16}(\Gamma_3)$ with eigenvalues

$$\lambda(2) = -115200,\ \lambda(3) = 14457333600,$$

$$\lambda(17) = -8464399250968010566002060 0.$$

One may check for example that these eigenvalues grow like $p^{(a+b+c+6)/2}$.

The conjectures says that there should be 123 cases with a 1-dim. space of ‘genuine’ eigenforms (for $a+b+c \leq 60$).

On the website smf.compositio.nl one can find numerical values of the traces of Hecke operators.

On $\mathcal{M}_3$

We can also carry out the counting on $\mathcal{M}_3$ for the cohomology

$$e_c(\mathcal{M}_3, \mathbf{V}'_\lambda)$$

where $\mathbf{V}'_\lambda$ is the pullback under $t : \mathcal{M}_3 \to \mathcal{A}_3$. For example, we found heuristically for $e_c(\mathcal{M}_3, \mathbf{V}'_{5,5,5})$ the result

$$L^7+2L^6+5L^5+6L^4+4L^3+3L^2+2L+1+S[4,10]$$

and the Hodge weight 21-part should be the Hodge weight 21-part of $S[4,10]$ and this should be generated by Teichmüller modular form χ_9.

But it is a mystery which new motives (of 'level 1') will turn up in this cohomology.

For example, in $e_c(\mathcal{M}_3, \mathbf{V}'_{11,3,3})$ we see a 6-dimensional motive of weight 23 first identified by Chenevier-Renard. It has Hodge degrees $0, 5, 9, 14, 18, 23$. It cannot come from Siegel modular form. It should correspond to a Teichmüller modular form of weight $(8, 0, 7)$. We can construct such a form.

L-series of Modular Forms

If $f = \sum a_n q^n$ is a normalized eigen cusp form of weight k we set

$$L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s} \qquad \mathrm{Re}(s) > k/2 + 1,$$

or better, we take

$$\Lambda(f,s) = \frac{\Gamma(s)}{(2\pi)^s} L(f,s) = \int_0^{\infty} f(iy) y^{s-1} dy$$

$\Lambda(f,s)$ can be extended holomorphically to all s and satisfies

$$\Lambda(f,s) = i^k \Lambda(f, k-s).$$

We can look at the *critical values*

$$\Lambda(f, k-1), \Lambda(f, k-2), \ldots, \Lambda(f, k/2).$$

MANIN AND VISHNIK: there are two real numbers ω_+ and ω_- (*the periods*) such that the numbers

$$\frac{\Lambda(f, k-1)}{\omega_-}, \frac{\Lambda(f, k-2)}{\omega_+}, \ldots, \frac{\Lambda(f, k/2)}{\omega_\pm}$$

lie in $\mathbf{Q}_f$ with $\mathbf{Q}_f = \mathbf{Q}(a_1, a_2, \ldots)$, the field generated by the Fourier coefficients of f.

Example. Take $f = \Delta$. We get

$$(\Lambda(f, 10) : \Lambda(f, 8) : \Lambda(f, 6)) = (48 : 25 : 20)$$

In general we find only small primes as entries. But occasionally a 'large' prime divides these values: for $f \in S_{22}$ we get

$$(\Lambda(f, 20) : \ldots : \Lambda(f, 14) : \Lambda(f, 12))$$

equals

$$(2^5 \cdot 3^3 \cdot 5 \cdot 19 : 2^3 \cdot 7 \cdot 13^2 : 3 \cdot 5 \cdot 7 \cdot 13 : 2 \cdot 3 \cdot 41 : 2 \cdot 3 \cdot 7)$$

and we see $41 \mid \Lambda(f_{22}, 14)$. Question: What is the meaning of such primes?

Example: Large Primes ℓ (or their norms) dividing critical values $\Lambda(f, t)$

weight k	t	ℓ
22	14	41
24	19	73
24	17	179
26	19	29
26	21	97
26	23	43
28	23	367
28	22	647
28	21	4057
28	19	2027
28	20	157
⋮	⋮	⋮
38	36	67 and 83

Harder's Conjecture. Let $a > b$ and let $f = \sum a(n)q^n \in S_{a+b+4}(\Gamma_1)$ be an eigenform. If an *ordinary* prime ℓ (in $\mathbf{Q}_f$) divides a critical value $\Lambda(f, a+3)$ then there exists an eigenform form $F \in S_{a-b,b+3}(\Gamma_2)$ with eigenvalue $\lambda(p)$ for $T(p)$ such that for all primes p

$$\lambda(p) \equiv p^{a+2} + a(p) + p^{b+1} \pmod{\ell}.$$

Ordinary means that that if ℓ lies above the prime $l \in \mathbf{Z}$ then ℓ does not divide the eigenvalue $a(l)$.

Example. Let $f \in S_{22}(\Gamma_1)$, $a = 11$, $b = 7$.

$$f = q - 288\,q^2 - 128844\,q^3 - 2014208\,q^4 \\ +21640950\,q^5 + \ldots$$

and $41|\Lambda(f, 14)$ and thus $\ell = 41$. There should exist an eigenform $F \in S_{4,10}$ with eigenvalues $\lambda(p)$ such that for all p

$$\lambda(p) \equiv p^8 + a(p) + p^{13} \pmod{41}.$$

We find $\dim S_{4,10}(\Gamma_2) = 1$ and we have

p	2	3	5
$a(p)$	-288	-128844	21640950
$\lambda(p)$	-1680	55080	-7338900

and one checks

$$2^8 - 288 + 2^{13} = -1680 + 9840$$

with

$$9840 = 2^4 \cdot 3 \cdot 5 \cdot 41$$

We checked it for all $p \leq 37$. This example was later proved by Chenevier and Lannes using unimodular lattices of rank 24.

Note that $19 | \Lambda(f_{22}, 20)$. But $a(19) = -7920788351740$ is divisible by 19.

Many more examples can be checked numerically.

weight r	weight (j,k)	$N(\ell)$
22	$(4,10)$	41
24	$(12,7)$	73
24	$(8,9)$	179
26	$(10,9)$	29
26	$(14,7)$	97
26	$(18,5)$	43
28	$(16,7)$	367
28	$(8,11)$	2027
28	$(20,5)$	193
30	$(10,11)$	97
30	$(24,4)$	97
32	$(4,15)$	61
34	$(28,4)$	103

The philosophy is that direct sum decompositions of rational cohomology stable under the Hecke algebra do not necessarily hold for integral cohomology. This gives rise to congruences.

The best-known congruence is

$$\tau(p) \equiv 1 + p^{11} \pmod{691}$$

which is a congruence between Δ and the Eisenstein series E_{12}. Kurokawa found a first example between eigenforms in $S_{20}(\Gamma_2)$ and a Klingen-type Eisenstein series $E(f_{20})$ with $f_{20} \in S_{20}(\Gamma_1)$.

We found experimentally congruences for $g = 3$.

Harder Type Congruences $g = 3$.

Example. Let $f \in S_{a+4}(\Gamma_1)$, $g \in S_{b+c+4}(\Gamma_1)$ eigenforms. If a 'large' prime ℓ in $\mathbf{Q}_{f,g}$ divides the critical value of

$$\Lambda(\mathrm{Sym}^2(f) \otimes g, a+b+6)$$

then there should be an eigenform $F \in S_{a-b,b-c,c+4}(\Gamma_3)$ such that

$$\lambda_p(F) \equiv \lambda_p(f)(p^{b+2}+\lambda_p(g)+p^{c+1}) \pmod{\ell}$$

for all primes p.

Another example.

Let $f \in S_{c+2}(\Gamma_1)$ and $g \in S_{a+b+6}(\Gamma_1)$ be eigenforms and ℓ an ordinary prime in $\mathbf{Q}_{f,g}$

dividing the critical value

$$\Lambda(\mathrm{Sym}^2(f) \otimes g, a + c + 5)\,.$$

Then there should be a Siegel modular eigenform $F \in S_{a-b,b-c,c+4}(\Gamma_3)$ such that

$$\lambda_F(p) \equiv \lambda_f(p) \left(p^{b+2} + \lambda_g(p) + p^{a+3}\right) \mod \ell$$

for all primes p.

Take $(a, b, c) = (13, 11, 10)$; then $\dim S_{2,1,14}(\Gamma_3) = 1$. We predict that

$$e_c(\mathcal{A}_3, \mathbf{V}_{13,11,10}) = \\ 2(L^{13}S[12] + 1) - 2L^{11} + S[2, 1, 14]$$

We let $f = \Delta$ and $g \in S_{30}(\Gamma_1)$ with

$$g = 1 + (4320 + 96\sqrt{51349})\, q + \cdots$$

The prime $\ell = 199$ divides the critical value $\Lambda(\mathrm{Sym}^2(f) \otimes g, 28)$. (Calculation by Mellit.) One checks that for $F \in S_{2,1,14}(\Gamma_3)$ we have $\lambda_F(2) = -2073600$ and

$$-2073600 + 24(2^{13} + 4320 + 96\sqrt{51349} + 2^{16})$$

has norm

$$-232402452480 \equiv 0 \mod 199$$

These conjectured congruences have been generalized vastly by Bergström and Dummigan in line with the Bloch-Kato conjectures.

Literature

J. Bergström, C. Faber, G. van der Geer: *Siegel modular forms of degree three and the cohomology of local systems.* Selecta Math. 2013.

G. Harder: *A congruence between a Siegel and an elliptic modular form.* In: The 1-2-3 of modular forms. Springer, Berlin (2008)

T. Ikeda: *Pullback of the lifting of elliptic cusp forms and Miyawaki's conjecture.* Duke Math. J. 131, 469-497 (2006).

S. Mizumoto: *Congruences for eigenvalues of Hecke operators on Siegel modular forms of degree two.* Math. Ann. 275(1), 149-161 (1986).

Thank you for your attention