Siegel Modular Forms
Lecture $\#2$

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Fixed Points

$\mathrm{Sp}(2g,\mathbf{R})$ acts transitively on $\mathcal{H}_g$. Suppose $\gamma \in \Gamma_g$ acts with a fixed point τ. With $\alpha \in \mathrm{Sp}(2g,\mathbf{R})$ with $\alpha(\tau) = i_g$ we see

$$\alpha\gamma\alpha^{-1} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in K = U_g$$

Diagonalize to get a diagonal matrix $\mathrm{diag}(\zeta_1, \ldots, \zeta_g, \zeta_1^{-1}, \ldots, \zeta_g^{-1})$.

Lemma 1. *If $\gamma \in \Gamma_g$ has order m in $\Gamma_g/\langle \pm 1 \rangle$ then $\exists\ \zeta_j \in \mathbf{C}^*$ with $\zeta_j^m = 1$ such that action on tangent space at $\tau \in \mathcal{H}_g$ is (in suitable coordinates) by*

$$t_{ij} \mapsto \zeta_i \zeta_j t_{ij}$$

Corollary 1. *The codimension of the locus of fixed points (for $\Gamma_g/\langle\pm\rangle$)*

$$S_g = \{\tau \in \mathcal{H}_g : I_\tau \neq \{\mathrm{Id}\}\}$$

is $g-1$ for $g \geq 2$.

Proof. The characteristic polynomial of $\gamma \in I_\tau$ has rational coefficients, so all conjugates occur. Hence at least $g-1$ products $\zeta_i\zeta_j$ are not 1. Note that $\mathcal{H}_{g-1} \times \mathcal{H}_1$ has codim $g-1$. Q.e.d.

For $g=1$ we saw two fixed points. For $g=2$ Gottschling determined the fixed points of $\mathrm{PSp}(4,\mathbf{Z})$. We find a stratification with 14 strata: two surfaces $(\tau_{12}=0)$ and $(\tau_{11}=\tau_{22})$, their intersections, some other curves like

$$\left\{ \begin{pmatrix} z & 0 \\ 0 & \tau \end{pmatrix} : \tau \in \mathcal{H}_1 \right\} \quad \text{with } z = \rho \text{ or } z = i$$

and their intersection points.

Lemma 2. *The congruence subgroup*

$$\Gamma_g[n] = \{\gamma \in \Gamma_g : \gamma \equiv 1 (\mathrm{mod} n)\}$$

acts freely for $n \geq 3$.

Proof. Say γ fixes τ. The characteristic roots are

$$\zeta = 1 + n\zeta' \quad n\zeta' \text{ alg. integer}$$

May raise to power so that order is a prime p. Then

$$\zeta^p = 1 = (1 + n\zeta')^p$$

that is

$$(n\zeta')^{p-1} + p(n\zeta')^{p-2} + \cdots + p = 0$$

hence $\mathrm{Norm}(n\zeta')$ divides p, hence n^{p-1} divides p, hence $p = 2$, and $n = 2$, a contradiction. Q.e.d.

The index of $\Gamma_g[n]$ in Γ_g is

$$n^{g(2g+1)} \prod_{p|n} \prod_{j=1}^{g} (1 - p^{-2j})$$

The quotient

$$\Gamma_g \backslash \mathcal{H}_g$$

is an orbifold, a global quotient under a finite group of the manifold $\Gamma_g[n] \backslash \mathcal{H}_g$ for $n \geq 3$.

This orbifold is a moduli space, namely of complex principally polarized abelian varieties, that is, of complex tori V/Λ with V a $\mathbf{C}$-vector space of dim g together with a hermitian metric H on V such $\mathrm{Im}(H)$ is integral and unimodular on Λ.

A non-degenerate hermitian form H on a complex vector space determines and is determined by an alternating form $E = \mathrm{Im}(H)$ such that

$$H(z, w) = E(iz, w) + \sqrt{-1}E(z, w)$$

with E satisfying

$$E(iz, iw) = E(z, w),$$

$$E(iz, z) > 0 \quad \text{for } z \neq 0\,.$$

To $\tau \in \mathcal{H}_g$ we associate $\mathbf{C}^g/\Lambda_\tau$ with Λ_τ the lattice spanned by the columns of the $g \times 2g$ matrix

$$(\tau \quad 1_g)$$

Define the positive definite form

$$H(z,w) = z' y^{-1} \bar{w} \,.$$

For $z = \tau m_1 + n_1$ and $w = \tau m_2 + n_2$ we have

$$\mathrm{Im}(H(z,w)) = m_1' n_2 - n_1' m_2 \,.$$

So $(\mathbf{C}^g/\Lambda_\tau), H)$ is a principally polarized complex abelian variety.

Conversely, if (V, Λ, H) gives a p.p.a.v. of dim g then there exists a basis $e_1, \ldots, e_g, f_1, \ldots, f_g$ of Λ such that H is given by

$$J = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$$

Express this symplectic basis in a complex basis $\eta_1, \ldots, \eta_g$:

$$\Omega = \begin{pmatrix} \Omega_1 & \Omega_2 \end{pmatrix} \quad g \times 2g \text{ matrix}$$

such that

$$e_i = \sum_j \Omega_{ji} \eta_j, \quad f_i = \sum \Omega_{j,g+i} \eta_j$$

Claim 1. $f_1, \ldots, f_g$ *is a complex basis.*

Proof. $U = \langle f_1, \ldots, f_g \rangle_{\mathbf{R}}$ the $\mathbf{R}$-vector space. Then $U \cap i(U)$ is a complex vector space on which H vanishes. Hence $U \cap i(U) = (0)$. Q.e.d.

So we can write $\Omega = (\,\tau \quad 1_g\,)$. Note that H is determined by the values on the f-basis:

$$H(z, w) = z' \mathrm{Im}(\tau) \bar{w}\,.$$

The equations

$$E(iz, iw) = E(z, w), \quad E(iz, z) > 0\,(z \neq 0)$$

give

$$\tau' = \tau, \quad \mathrm{Im}(\tau) > 0\,.$$

But we chose a symplectic basis $e_1, \ldots, e_g, f_1, \ldots f_g$. We can change the basis via $\gamma = (a, b; c, d) \in \Gamma_g$. Say we have a new basis $\dot{e}_i, \dot{f}_i$ with

$$\dot{e}_i = \sum a_{ij} e_j + \sum b_{ij} f_f$$

$$\dot{f}_i = \sum c_{ij} e_j + \sum d_{ij} f_f$$

that is, $\dot{e}_i$ is the ith row of $a\tau + b$, $\dot{f}_i$ is the ith row of $c\tau + d$. So the basis is given by the columns of

$$(\,(a\tau + b)' \quad (c\tau + d)'\,)$$

Changing by $(c\tau + d)'^{-1}$ gives a basis coming from

$$(\,(c\tau + d)'^{-1}(a\tau + b)' \quad 1_g\,)$$

which equals

$$(\,(a\tau + b)(c\tau + d)^{-1} \quad 1_g\,)$$

Proposition 1. *The points of* $\Gamma_g \backslash \mathcal{H}_g$ *correspond* $1-1$ *to isomorphism classes of complex p.p.a.v. of dim* g*.*

Note that for p.p.a.v. X we have

$$\#\mathrm{Aut}(X_\tau) = \#\mathrm{Stab}_{\Gamma_g}(\tau)\,.$$

The paramodular groups

We mainly deal with $\Gamma_g = \mathrm{Sp}(2g, \mathbf{Z})$ to avoid technicalities.

Other groups of interest: take instead of $J = (0, 1_g, -1_g, 0)$ the alternating matrix

$$J_D = \begin{pmatrix} 0_g & D \\ -D & 0_g \end{pmatrix}$$

with $D = \mathrm{diag}(d_1, \ldots, d_g)$ with $d_i \in \mathbf{Z}_{\geq 1}$ and $d_i | d_{i+1}$ for $i = 1, \ldots, g-1$. Put

$$\Gamma_D = \mathrm{Aut}(\mathbf{Z}^{2g}, J_D)$$

It is called: paramodular group of type D.

The quotient $\Gamma_D \backslash \mathcal{H}_g$ describes the moduli of complex abelian varieties with a polarization of type D.

For a fundamental domain take a finite index normal subgroup Γ in $\Gamma_g \cap \Gamma_D$. A fundamental domain for Γ is of the form $\cup_i \gamma_i(\mathcal{F}_g)$ with γ_i representatives of Γ_g/Γ; divide out action of Γ_D/Γ.

But we will stick mostly to Γ_g.

We let $\mathbf{Z}^{2g}$ act on $\mathcal{H}_g \times \mathbf{C}^g$ via

$$(\tau, z) \mapsto (\tau, z + \tau m + n)$$

where m, n are column vectors in $\mathbf{Z}^g$. We then let Γ_g act via

$$(\tau, z) \mapsto (\gamma(\tau), {(c\tau + d)'}^{-1} z)$$

This gives an action of $\Gamma_g \ltimes \mathbf{Z}^{2g}$. The subgroup

$$\Gamma_g[n] \ltimes \mathbf{Z}^{2g}$$

acts freely on $\mathcal{H}_g \times \mathbf{C}^g$ for $n \geq 3$. The quotient space

$$\mathcal{X}_g[n] = \Gamma_g[n] \ltimes \mathbf{Z}^{2g} \backslash \mathcal{H}_g \times \mathbf{C}^g$$

is a complex manifold and a family of complex abelian varieties of dimension g and

$$\mathcal{X}_g = \Gamma_g \ltimes \mathbf{Z}^{2g} \backslash \mathcal{H}_g \times \mathbf{C}^g$$

is an orbifold quotient of $\mathcal{X}_g[n]$.

The action of Γ_g on Lie, that is on $\mathbf{C}^g$, is by $(c\tau + d)'^{-1}$, hence by $(c\tau + d)$ on the space of differentials.

Therefore we define an (orbifold) vector bundle

$$\Gamma_g \backslash \mathcal{H}_g \times \mathbf{C}^g$$

using the action of $(c\tau + d)'^{-1}$ or that of $(c\tau + d)$. The latter vector bundle is called the Hodge bundle

$$\mathbf{E}_g, \quad \text{a rank } g \text{ orbifold vector bundle}$$

living over $\mathcal{A}_g = \Gamma_g \backslash \mathcal{H}_g$. The fibre of $\mathbf{E}_g$ over

X_τ is $H^0(X_\tau, \Omega^1)$.

Using the orbifold map

$$\pi : \mathcal{X}_g \to \mathcal{A}_g$$

we have

$$\mathbf{E}_g = \pi_*(\Omega^1_{\mathcal{X}_g/\mathcal{A}_g})\,.$$

In level $n \geq 3$ we have an actual bundle $\mathbf{E}_g[n]$ on $\mathcal{A}_g[n]$ via

$$\pi : \mathcal{X}_g[n] \to \mathcal{A}_g[n]$$

with

$$\mathbf{E}_g[n] = \pi_*(\Omega^1_{\mathcal{X}_g[n]/\mathcal{A}_g[n]})$$

Invariant Metric

For $g = 1$ we have the Poincaré metric

$$\frac{dx^2 + dy^2}{y^2}, \quad \omega = \frac{1}{4\pi i}\frac{d\tau \wedge d\bar{\tau}}{y^2}$$

This generalizes

$$ds^2 = \mathrm{Tr}(y^{-1} d\tau\, y^{-1} d\bar{\tau}) \quad \text{with } d\tau = (d\tau_{ij})$$

This is a $\mathrm{Sp}(2g, \mathbf{R})$-invariant metric.

Lemma 3. *For* $\gamma \in \mathrm{Sp}(2g, \mathbf{R})$

$$d(\gamma(\tau)) = (d'a - b'c)(c\tau + d)'^{-1} d\tau (c\tau + d)^{-1}$$

Corollary 1. ds^2 *is* $\mathrm{Sp}(2g, \mathbf{R})$*-invariant and positive.*

Indeed, $\Sigma = y^{-1} d\tau \, y^{-1} d\bar{\tau}$ changes to

$$(c\tau + d)'^{-1} \Sigma (c\tau + d)'$$

hence the trace Tr is invariant. We can check the positivity at $\tau = i_g$:

$$\sum_k dx_{kk}^2 + 2 \sum_{k<l} dx_{kl}^2 + \sum_k dy_{kk}^2 + 2 \sum_{k<l} dy_{kl}^2$$

Lemma 4. *Let* S_g *be the set of symmetric matrices of size* g*. The Jacobian of the map* $S_g \to S_g$*,* $x \mapsto a'xa$ *with* $a \in \mathrm{GL}(g)$ *has determinant* $\det(a)^{g+1}$*.*

Corollary 2. *For* $\gamma \in \mathrm{Sp}(2g, \mathbf{R})$ *we have*

$$\det(\partial\gamma/\partial\tau) = \det(c\tau + d)^{-(g+1)}$$

Lemma 5. *The cotangent (orbifold) bundle of $\mathcal{A}_g$ is $\mathrm{Sym}^2(\mathbf{E}_g)$.*

Proof. We have

$$(c\tau + d)' d\gamma(\tau)(c\tau + d) = d\tau$$

So the action is as on symmetric matrices of size g. Q.e.d.

For $n \geq 3$ we have

$$\Omega^1_{\mathcal{A}_g[n]} = \mathrm{Sym}^2(\mathbf{E}_g[n])\,.$$

Differential geometry tells us that we can normalize the invariant volume form ω_g such that for torsion free $\Gamma \subset \Gamma_g$ the integral

$$\int_{\Gamma\backslash\mathcal{H}_g} \omega_g$$

gives the Euler number of $\Gamma\backslash\mathcal{H}_g$.

A priori only for Γ with compact quotient, but by a theorem of Harder for all Γ that are discrete and torsion-free.

Siegel calculated the volume of $\mathcal{F}_g$, the fundamental domain for Γ_g:

$$\mathrm{vol}(\Gamma_g\backslash\mathcal{H}_g) = \zeta(-1)\zeta(-3)\cdots\zeta(1-2g)$$

with $\zeta(s)$ the Riemann zeta function.

Recall that

$$\zeta(1-2n) = -\frac{B_{2n}}{2n}\,.$$

Some values:

$$\zeta(-1) = -\frac{1}{12},$$
$$\zeta(-3) = \frac{1}{120}, \quad \zeta(-5) = -\frac{1}{252}$$

g	$\mathrm{vol}(\Gamma_g \backslash H_g)$
1	$-1/12$
2	$-1/1440$
3	$1/362880$
4	$1/87091200$
5	$-1/11496038400$
6	$-691/376610217984000$
7	$691/4519322615808000$
8	$2499347/3687767254499328000$
9	$-109638854849/52971088843628347392000$

But Γ_g is not without torsion, so

$$e(\Gamma_g\backslash\mathcal{H}_g) \neq \mathrm{vol}(\Gamma_g\backslash\mathcal{H}_g)$$

e.g. for $g = 1$

$$e(\Gamma_1\backslash\mathcal{H}_1) = 1 = 2\zeta(-1) + \frac{1}{2} + \frac{2}{3}$$

We note for $g = 1,\ 2,\ 3$:

$$\mathrm{vol}(\Gamma_g[2]\backslash\mathcal{H}_g) = [-1/2, -1/2, 4]$$

and

$$\mathrm{vol}(\Gamma_g[3]\backslash\mathcal{H}_g) = [-2, -36, 25272]$$

For example

$$e(\overline{\Gamma_1[3]\backslash\mathcal{H}_1}) = e(\Gamma_1[3]\backslash\mathcal{H}_1) + 4 = 2$$

Modular Forms

A holomorphic $f : \mathcal{H}_g \to \mathbf{C}$ such that

$$f((a\tau + b)(c\tau + d)^{-1}) = \det(c\tau + d)^k f(\tau)$$

for all $\gamma \in \Gamma_g$ is called a (scalar-valued) Siegel modular form of weight k for $g \geq 2$. For $g = 1$ we have to ask for holomorphicity at ∞: $f = \sum_{n \geq 0} a(n) q^n$.

More generally, consider a complex representation of finite dimension

$$\rho : \mathrm{GL}(g, \mathbf{C}) \to \mathrm{GL}(W)$$

Look at holomorphic $f : \mathcal{H}_g \to W$ such that

$$f((a\tau + b)(c\tau + d)^{-1}) = \rho(c\tau + d) f(\tau)$$

for all $\gamma \in \Gamma_g$ $(g \geq 2)$.

We get a vector space $M_k(\Gamma_g)$ and $M_\rho(\Gamma_g)$.

The functional equation implies

$$f(\tau + s) = f(\tau) \quad \forall \quad \text{symmetric integral } s$$

A half-integral matrix (of size g) is a symmetric $g \times g$ matrix n such that $2n$ is integral and even on the diagonal.

Such a n defines a linear form

$$\mathrm{Tr}(n\tau) = \sum_{i=1}^{g} n_{ii}\tau_{ii} + 2\sum_{i<j} n_{ij}\tau_{ij}$$

with *integral* coefficients. Every such linear form in the τ_{ij} with $\mathbf{Z}$-coefficients is of this form.

Because of $f(\tau + s) = f(\tau)$ for symmetric integral s an element $f \in M_\rho(\Gamma_g)$ admits a Fourier expansion

$$f = \sum_n a(n) e^{2\pi i \mathrm{Tr}(n\tau)}$$

with $a(n) \in W$ and n running through the half-integral matrices. We have

$$a(n) = \int_{x \bmod 1} f(\tau) e^{-2\pi i \mathrm{Tr}(n\tau)} dx$$

with the integral over the x-coordinates $|x_{ij}| \leq 1/2$ and dx the usual Euclidean volume. We put

$$q^n = e^{2\pi i \mathrm{Tr}(n\tau)}$$

and write

$$f = \sum_n a(n) q^n$$

summing over half-integral n.

We may restrict to irreducible representations ρ of $\mathrm{GL}(g, \mathbf{C})$. These correspond 1-1 to g-tuples of integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g)$$

with $\lambda_i \in \mathbf{Z}$. Such an irreducible representation contains a 1-dim subspace on which $\mathrm{diag}(a_1, \ldots, a_g)$ acts by multiplication by

$$\prod_{i=1}^{g} a_i^{\lambda_i}$$

The λ are called the highest weight.

The standard representation corresponds to $(1, 0, \ldots, 0)$ and the determinant to $(1, \ldots, 1)$.

Lemma 6. *For* $f = \sum_n a(n)q^n \in M_\rho(\Gamma_g)$ *we have*

$$a(u'nu) = \rho(u')a(n) \quad \forall\, u \in \mathrm{GL}(g, \mathbf{Z})$$

Proof. Use $\gamma = (u, 0; 0, {u'}^{-1}) \in \Gamma_g$.

$$a(u'nu) = \int_{x \bmod 1} f(\tau)e^{-2\pi i \mathrm{Tr}(u'nu\tau)}dx$$

Using $f(u\tau u') = \rho(u')^{-1}f(\tau)$ this gives

$$\rho(u')\int f(u\tau u')e^{-2\pi i \mathrm{Tr}(u'nu\tau)}dx$$

and using $\mathrm{Tr}(u'nu\tau) = \mathrm{Tr}(nu\tau u')$ we get with $\sigma = u\tau u'$

$$\rho(u')\int_{x \bmod 1} f(\sigma)e^{-2\pi i \mathrm{Tr}(n\sigma)}dx$$

Q.e.d.

Koecher Principle

Theorem 1. *Let* $f = \sum_n a(n)q^n$ *in* $M_\rho(\Gamma_g)$. *Then* $a(n) = 0$ *if* n *is not positive semi-definite.*

Proof. For $g = 1$ this is part of the definition, so assume $g \geq 2$. Now f converges absolutely on $\mathcal{H}_g$. We look at $\tau = i_g$. Then

$$f(i_g) = \sum_n a(n)e^{-2\pi \mathrm{Tr}(n)}$$

There exists a constant $A \in \mathbf{R}_{>0}$ such that $\forall n$ we have

$$|a(n)| \leq A\, e^{2\pi \mathrm{Tr}(n)}$$

where $|\,|$ refers to a norm on W.

Suppose that n is not positive semi-definite. Then $\exists \xi \in \mathbf{Z}^g$ with $\xi' n \xi < 0$. We may assume ξ is primitive. Complete ξ to a unimodular matrix u and replace n by $u' n u$. Recall $a('unu) = \rho(u')a(n)$. Then we may assume $n_{11} < 0$. Consider

$$v = \begin{pmatrix} 1 & m & \\ 0 & 1 & \\ & & \\ & & 1_{g-2} \end{pmatrix}$$

Then $|a(n)| = |\rho(v')^{-1}|\,|a(v'nv)|$.

But $|\rho(v')^{-1}|$ is polynomial in m and

$$|a(v'nv)| \leq A\, e^{2\pi \operatorname{Tr}(v'nv)}$$

We have

$$\operatorname{Tr}(v'nv) = \operatorname{Tr}(n) + n_{11}m^2 + 2n_{12}m$$

with $n_{11} < 0$. Let $m \to \infty$. We get $|a(n)| = 0$. Q.e.d.

We thus get for $f \in M_\rho(\Gamma_g)$

$$f = \sum_{n \geq 0} a(n)\, q^n \,.$$

Observe

$$|a(n)||e^{2\pi i \mathrm{Tr}(n\tau)}| = |a(n)|e^{-2\pi \mathrm{Tr}(ny)} \,.$$

Lemma 7. *Let $f \in M_\rho(\Gamma_g)$. Then each coordinate of f is bounded on any subset*

$$\{\tau \in \mathcal{H}_g : \mathrm{Im}(\tau) > c \cdot 1_g\}$$

with $c \in \mathbf{R}_{>0}$.

Indeed, majorize uniformly by the value at $c \cdot i_g$.

Proposition 2. *$M_k(\Gamma_g) = (0)$ for $k < 0$. Moreover, $M_0(\Gamma_g) = \mathbf{C}$.*

Proof. The function $h = \det(y)^{k/2}|f(\tau)|$ is Γ_g-invariant. The fundamental domain F_g is contained in

$$\{\tau \in \mathcal{H}_g : \mathrm{Im}(\tau) > c \cdot 1_g\}$$

for some $c > 0$. If $k < 0$ then $\det(y)^{k/2}$ is bounded on F_g, and $|f(\tau)|$ is also bounded, so h is bounded on F_g, hence on all of $\mathcal{H}_g$. So $\exists A$

$$|f(\tau)| \leq A \det(y)^{-k/2}$$

We have

$$a(n)e^{-2\pi \mathrm{Tr}(ny)} = \int_{x \bmod 1} f(\tau)e^{-2\pi i \mathrm{Tr}(nx)} dx$$

and this gives a bound for $|a(n)|$:

$$|a(n)|e^{-2\pi \mathrm{Tr}(ny)} \leq \max_{x \bmod 1} |f(x+iy)|$$

$$\leq A \det(y)^{-k/2}$$

Now let $y \to 0$ and get $a(n) = 0$.

For $k = 0$ we see that $|f(\tau)|$ is bounded on $\{\tau \in \mathcal{H}_g : y > c\,1_g\}$ for $c > 0$. This implies that f is bounded on F_g, hence everywhere. But then f is constant by the maximum principle. Q.e.d.

More literature

C.L. Siegel: Symplectic Geometry. In: Collected Works, Vol. II.

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