Siegel Modular Forms
Lecture $\#3$

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Let $\rho : \mathrm{GL}(g, \mathbf{C}) \to \mathrm{GL}(W)$ be a non-trivial irreducible representation.

Lemma 1. *Let $f \in M_\rho(\Gamma_g)$ and $f \neq 0$. If ℓ is a linear form on W such that $\ell(f)$ is constant, then $\ell = 0$.*

Proof. After choosing appropriate coordinates and conjugating by an elt of $\mathrm{GL}(W)$ we may assume that the first r coordinates of f are constant with $1 \leq r < g$. Using

$$a(u'nu) = \rho(u')a(n)$$

for $u \in \mathrm{GL}(g, \mathbf{Z})$ we get

$$\rho_{ij}(u') = 0 \qquad \text{for } i \leq r \text{ and } j > r$$

But $\mathrm{GL}(g,\mathbf{Z})$ lies Zariski dense in $\mathrm{GL}(g,\mathbf{C})$, hence ρ_{ij} is zero identically for $i \leq r$ and $j > r$. But this contradicts the irreducibility. Q.e.d.

Theorem 1. *Let ρ be an irrep of $\mathrm{GL}(g,\mathbf{C})$ of highest weight $\lambda_1 \geq \cdots \geq \lambda_g)$. Then $M_\rho(\Gamma_g) = (0)$ for $\lambda_g < 0$.*

Proof. Follows later.

Remark 1. *If $\rho \neq \det^{\mathrm{even}}$ then for $f = \sum a(n)q^n \in M_\rho(\Gamma_g)$ we have $a(0) = 0$.*

Proof. We have $a(u'nu) = \rho(u')a(n)$ for $u \in \mathrm{GL}(g,\mathbf{Z})$, so $a(0) = \rho(u')a(0)$. If $\rho(u') \neq \mathrm{Id}$ then $a(0) = 0$. Q.e.d.

The Siegel Operator

For $f \in M_\rho(\Gamma_g)$ we define

$$\Phi(f)(\tau) = \lim_{t \to \infty} f \begin{pmatrix} \tau & 0 \\ 0 & it \end{pmatrix}$$

with $\tau \in \mathcal{H}_{g-1}$, $t \in \mathbf{R}_{>0}$. Since f is bounded on $\{\tau \in \mathcal{H}_g : \mathrm{Im}(\tau) > c 1_g\}$ with $c > 0$, hence converges uniformly. The limit can be taken term by term in the Fourier series

$$\Phi(f)(\tau) = \sum_{n \geq 0} a \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix} e^{2\pi i \mathrm{Tr}(n\tau)}$$

and we get a holomorphic map $\mathcal{H}_{g-1} \to W$. The image generates a subspace $W' \subset W$

invariant under

$$\{\begin{pmatrix} 1_{g-1} & b \\ 0 & 1 \end{pmatrix}\}$$

and preserved under

$$\{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(g,\mathbf{C}) : a \in \mathrm{GL}(g-1,\mathbf{C})\}$$

If ρ has highest weight $(\lambda_1 \geq \cdots \geq \lambda_g)$ then the restriction ρ' of ρ to the subrepresentation of $\mathrm{GL}(g-1,\mathbf{C})$ has highest weight $(\lambda_1 \geq \cdots \geq \lambda_{g-1})$. Embed Γ_{g-1} into Γ_g via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 1 \\ c & 0 & d & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

This gives

$$\Phi(f)(\gamma(\tau)) = \rho'(c\tau + d)\Phi(f)(\tau)\,.$$

Definition 1. *The Siegel operator $\Phi = \Phi_{g,\rho}$ is the linear map*

$$M_\rho(\Gamma_g) \to M_{\rho'}(\Gamma_{g-1}), \quad f \mapsto \Phi(f)$$

Example. $g = 2$. If $\rho = \mathrm{Sym}^j(\mathrm{St}) \otimes \det(\mathrm{St})^{\otimes k}$, then $(\lambda_1, \lambda_2) = (j + k, k)$ and we find $\rho' = j + k$, so we land in $M_{j+k}(\Gamma_1)$.

Definition 2. *An element $f \in \ker(\Phi)$ is called a cusp form.*

We will suppress indices

$$\Phi^r = \Phi_{g-r} \cdots \Phi_{g-1}\Phi_g$$

We write $S_k(\Gamma_g) \subseteq M_k(\Gamma_g)$ and $S_\rho(\Gamma_g) \subseteq M_\rho(\Gamma_g)$ for the spaces of cusp forms.

Let $f = \sum_{n \geq 0} a(n) q^n \in S_\rho(\Gamma_g)$. If n is singular then $\exists u \in \mathrm{GL}(g, \mathbf{Z})$ such that

$$u' n u = \begin{pmatrix} n_1 & 0 \\ 0 & 0 \end{pmatrix}$$

with n_1 of size $g-1$, hence since $\Phi(f) = 0$

$$\lim_{t \to \infty} a \begin{pmatrix} n_1 & 0 \\ 0 & 0 \end{pmatrix} e^{2\pi i \mathrm{Tr}(n_1 \tau_1)} = 0$$

hence $a \begin{pmatrix} n_1 & 0 \\ 0 & 0 \end{pmatrix} = 0$. So cusp forms have $a(n) = 0$ for n singular.

Definition 3. *The co-rank of $f \in M_\rho(\Gamma_g)$ with $f \neq 0$ is the largest $r \in \mathbf{Z}_{\geq 0}$ such that $\Phi^r f \neq 0$.*

Lemma 2. *Let* $f \in M_\rho(\Gamma_g)$ *with* $f \neq 0$. *Then*

$$\mathrm{co}-\mathrm{rank}(f) = g - \min\{\mathrm{rk}(n) : a(n) \neq 0\}$$

Use $a(u'nu) = \rho(u')a(n)$ for $u \in \mathrm{GL}(g, \mathbf{Z})$.

Proposition 1. *Let* $0 \neq f \in M_\rho(\Gamma_g)$ *with* ρ *of highest weight* $(\lambda_1, \ldots, \lambda_g)$. *Then*

$$\mathrm{co}-\mathrm{rank}(f) \leq \{i : 1 \leq i \leq g, \lambda_i = \lambda_g\}$$

Proof. Suppose f has co-rank r. Consider n with rank $g-r$ with $a(n) \neq 0$. May assume

$$n = \begin{pmatrix} n_1 & 0 \\ 0 & 0_r \end{pmatrix}.$$

We have

$$a(u'nu) = \rho(u')a(n)$$

for $u \in \mathrm{GL}(g, \mathbf{Z})$. The subgroup of $\mathrm{GL}(g, \mathbf{C})$ of u with $u'nu = n$ contains a subgroup

$$G = \{\begin{pmatrix} 1_{g-r} & \beta \\ 0 & \delta \end{pmatrix} : \delta \in \mathrm{GL}(r, \mathbf{C})\}$$

Note: $G \cong \mathrm{GL}(r, \mathbf{C}) \ltimes N$ with N the unipotent radical. The Zariski closure of $\mathrm{GL}(g, \mathbf{Z}) \cap G$ contains

$$\mathrm{SL}(r, \mathbf{C}) \ltimes N$$

hence

$$a(u'nu) = \rho(u')a(n) \quad \forall u \in G'$$

with G' a finite index subgroup of G. Look in $\mathrm{GL}(g, \mathbf{C})$ at

$$P = \{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : d \in \mathrm{GL}(r, \mathbf{C}) \}$$

a subgroup of $\mathrm{GL}(g, \mathbf{C})$. It acts on W and on W^N, the N-invariants. So W^N is a representation of

$$P/N = \mathrm{GL}(g - r, \mathbf{C}) \times \mathrm{GL}(r, \mathbf{C})\,.$$

Highest weights

$$(\lambda_1, \ldots, \lambda_{g-r}) \quad \text{and} \quad (\lambda_{g-r+1}, \ldots, \lambda_g)$$

If not $\lambda_{g-r+1} = \cdots = \lambda_g$ then W^N does not contains non-zero $\mathrm{SL}(r, \mathbf{C})$-invariant vectors. Hence $a(n) = 0$ for all n. If $\lambda_{g-r+1} = \cdots = \lambda_g$ then $\dim W^G = 1$. Q.e.d.

Example. $f \in M_\rho(\Gamma_g)$, $\Phi(f)$ not a cusp form. Then $\lambda_{g-1} = \lambda_g$.

So for $g = 2$ we can find non-cusp forms in the image of Φ only for scalar-valued modular forms.

Finite-Dimensionality

There is the Minkowski inequality for *reduced* y: there exist a $c_g > 0$ such that

$$\prod_k y_{kk} \leq c_g \det(y) \,.$$

Conjugating y by $\mathrm{diag}(1/\sqrt{y_{11}}, \ldots, 1/\sqrt{y_{gg}})$ we get a matrix $\tilde{y}$ with eigenvalues l_i with trace g, hence $l_i < g$ for all i. But

$$l_1 \cdots l_g = \frac{\det(y)}{y_{11} \cdots y_{gg}} \geq 1/c_g \,.$$

This gives

$$l_i \geq \frac{1}{c_g} \prod_{j \neq i} l_j^{-1} \geq \frac{1}{g^{g-1} c_g}$$

hence

$$y \geq \frac{1}{g^{g-1}c_g}\mathrm{diag}(y_{11}, \ldots, y_{gg})$$

We know $y_{kk} \geq \sqrt{3}/2$. Put

$$c'_g = (\sqrt{3}/2)/g^{g-1}c_g\,.$$

So Minkowski reduction implies

$$F_g \subset \{\tau \in \mathcal{H}_g : y > c'_g 1_g, \mathrm{Tr}(y^{-1}) \leq g/c'_g\}$$

First do $M_k(\Gamma_g)$.

Proposition 2. *For $f = \sum a(n)q^n \in S_k(\Gamma_g)$ we have: if $a(n) = 0$ for all n with*

$$\mathrm{Tr}(n) < kg/4\pi c'_g$$

then $f = 0$.

Proof. Assume $f \neq 0$. Fix $\tau \in \mathcal{H}_g$. Put $\delta = c'_g$.

Consider for $z \in \mathbf{C}$ with $\mathrm{Im}(z) > -\delta$

$$F(z) = f(\tau + z \cdot 1_g);$$

it is holomorphic in z; invariant under $z \mapsto z + 1$. Hence we have a Fourier series

$$F(z) = \sum_{m=0}^{\infty} \left(\sum_{n:\mathrm{Tr}(n)=m} a(n) q^n \right) w^m$$

with $w = e^{2\pi i z}$. First non-zero term is

$$w^{m_0} \quad \text{with } m_0 \geq [kg/2\pi c'_g]$$

so $w^{-m_0} F(z)$ is holomorphic in disc $|w| < e^{2\pi\delta}$. Apply maximum principle: maximum of

$w^{-m_0}F(z)$ on disc $|w| \leq e^{2\pi\epsilon}$ with $0 \leq \epsilon < \delta$ taken on boundary, say for $w = e^{2\pi i z_\epsilon}$ with $\mathrm{Im}(z_\epsilon) = -\epsilon$. Thus we get

$$|f(\tau)| \leq e^{-2\pi m_0 \epsilon}|f(\tau + z_\epsilon 1_g)|$$

On the other hand, $|f(\tau)|\det(y)^{k/2}$ is Γ_g-invariant, zero in the cusp, takes its max on F_g, say in τ_0.

$$|f(\tau_0 + z_\epsilon 1_g)|\det(y_0 - \epsilon 1_g)^{k/2}$$

$$\leq |f(\tau_0)|\det(y_0)^{k/2}$$

Combining the red and blue inequalities we get

$$\det(y_0)^{-k/2}\det(y_0 - \epsilon 1_g)^{k/2} \leq e^{-2\pi m_0 \epsilon}$$

that is

$$\det(1_g - \epsilon y_0^{-1})^{k/2} < e^{-2\pi m_0 \epsilon}$$

for small ϵ. Comparing linear terms gives

$$\mathrm{Tr}(y_0^{-1}) \geq 4\pi m_0/k$$

But $m_0 > kg/4\pi c'_g$ and this contradicts $\mathrm{Tr}(y_0^{-1}) < g/c'_g$. Hence $f(\tau_0) = 0$. Q.e.d.

Corollary 1. *We have* $\dim M_k(\Gamma_g) < \infty$.

Proof. We see $\dim S_k(\Gamma_g) < \infty$. Using Φ we get

$$\dim M_k(\Gamma_g) \leq \dim S_k(\Gamma_g) + \dim M_k(\Gamma_{g-1})\,.$$

Then use induction. Q.e.d.

We can count dimensions: every 2×2 matrix

$$\begin{pmatrix} n_{ii} & n_{ij} \\ n_{ij} & n_{jj} \end{pmatrix}$$

satisfies $n_{ij}^2 \leq n_{ii} n_{jj}$. The number of n with $\mathrm{Tr}(n) \leq b$ is estimated by $(2b+1)^{g(g+1)/2}$. So for $b = ck$ with c fixed constant this is $O\big(k^{g(g+1)/2}\big)$.

The argument can be refined to give bounds on the dimensions. Later we shall see finite-dimensionality follows from algebraic geometry.

Petersson Product

Let $\rho : \mathrm{GL}(g, \mathbf{C}) \to \mathrm{GL}(W)$ be our complex irrep. Choose a hermitean non-degenerate product (,) on W invariant under $U_g \subset \mathrm{GL}(g, \mathbf{C})$:

$$(\rho(g)v_1, v_2) = (v_1, \rho(g')v_2) \qquad \forall g \in \mathrm{GL}(g, \mathbf{C})$$

This is unique up to a scalar.

Let $f_1, f_2 \in M_\rho(\Gamma_g)$ such that at least one is a cusp form. Define

$$\langle f_1, f_2 \rangle = \int_{F_g} (\rho(y) f_1, f_2) d\mu$$

with

$$d\mu = \det(y)^{-g-1} \prod dx_{ij} dy_{ij}$$

Alternatively, choose a symmetric pos. def. square root $y^{1/2}$ of y and write

$$\int_{F_g} (\rho(y^{1/2})f_1, \rho(y^{1/2})f_2) d\mu$$

This is invariant under $\gamma \in \Gamma_g$ and vanishes at the cusps.

The above product defines a positive definite hermitean product on $S_\rho(\Gamma_g)$. We put

$$\|f\|^2 = \int_{F_g} (\rho(y^{1/2})f, \rho(y^{1/2})f) d\mu$$

We call $f \in M_\rho(\Gamma_g)$ square-integrable if $\|f\|^2 < \infty$.

A cusp form is square-integrable. In order to see whether $f = \sum a(n)q^n$ is square-integrable we have to look at $a(n)$ for n singular.

Theorem 2. *(Weissauer) $f \in M_\rho(\Gamma_g)$ is square-integrable if and only if f is a cusp form or $\operatorname{corank}(f) \leq 2(g - \lambda_g)$.*

Corollary 2. *$M_\rho(\Gamma_g) = (0)$ if $\lambda_g < 0$.*

Proof. If $f \in M_\rho(\Gamma_g)$ then f square-integrable.

$$(\rho(y^{1/2})f, \rho(y^{1/2})f)$$

is Γ_g-invariant. Therefore $\|f(\tau)\|^2$ bounded by $A\,\|\rho(y^{1/2})\|^{-2}$ on F_g for some constant A; hence everywhere on $\mathcal{H}_g$. Now

$$a(n)e^{-2\pi \mathrm{Tr}(ny)} = \int_{x \bmod 1} f(\tau)e^{-2\pi \mathrm{Tr}(nx)}dx$$

Hence

$$|a(n)|^2 e^{-2\pi \mathrm{Tr}(ny)} \le \max_{x \bmod 1} |f(x+iy)|$$

$$\le B|\rho(y^{1/2})|^{-2}$$

for some B.

Take $y = \mathrm{diag}(1, 1, \ldots, 1, t)$ with $t \to 0$; use $\lambda_g < 0$ and get $|a(n)| = 0$. Q.e.d.

Hodge Bundle

Γ_g acts on $\mathcal{H}_g$ and on $\mathcal{H}_g \times \mathbf{C}^g$:

$$(\tau, z) \mapsto (\gamma(\tau), (c\tau + d)z)$$

For $\mathbf{C}^g/\Lambda_\tau$ this is the action on differentials. We thus get an orbifold bundle $\mathbf{E}$ of rank g with

$$\mathbf{E} = \pi_* \Omega^1_{\mathcal{X}_g/\mathcal{A}_g}$$

Similarly, for each irrep ρ we get a bundle $\mathbf{E}_\rho$: quotient under the action of Γ_g on $\mathcal{H}_g \times W$

$$(\tau, w) \mapsto (\gamma(\tau), \rho(c\tau + d)w)\,.$$

Conclusion:

$$M_\rho(\Gamma_g) = \Gamma(\mathcal{A}_g, \mathbf{E}_\rho) \quad \text{for } g > 1$$

We know

$$(c\tau+d)'d(\gamma(\tau))(c\tau+d)=d\tau$$

hence the cotangent (orbifold) bundle of $\Gamma_g\backslash\mathcal{H}_g$ is $\mathrm{Sym}^2(\mathbf{E})$.

Corollary 3. *We have*

$$\det(\Omega^1_{\Gamma_g\backslash\mathcal{H}_g})=\det(\mathbf{E})^{g+1}\,.$$

One can interpret this in term of deformations of abelian varieties. Let $H^1(X,\Theta_X)$ be the deformation space of an abelian variety X. We have $\Theta_X\cong T_{X,0}\otimes\mathcal{O}_X$. Therefore we have

$$H^1(X,\mathcal{O}_X)\otimes T_{X,0}=T_{X^\vee,0}\otimes T_{X,0},$$

but in order to keep the polarization $X \xrightarrow{\sim} X^\vee$ we need symmetric elements, hence we get $\mathrm{Sym}^2(T_{X,0})$.

Poincaré Series

We use the Cayley transform $\mathcal{H}_g \xrightarrow{\sim} \mathcal{D}_g$ with

$\mathcal{D}_g = \{z \in \mathrm{Mat}(g \times g, \mathbf{C}) : z' = z, z \cdot \bar{z} < 1_g\}$

via $\tau \mapsto (\tau - i_g)(\tau - i_g)^{-1}$. On $\mathcal{H}_g$ the Jacobian determinant on Γ_g is

$$j(\gamma, \tau) = \det(c\tau + d)^{-g-1}\,.$$

Recall the slash operator

$$f_{|k,\gamma}(\tau) = \det(c\tau + d)^{-k} f(\gamma(\tau))$$

and

$$f_{|\rho,\gamma}(\tau) = \rho((c\tau + d)^{-1}) f(\gamma(\tau))$$

We have

$$f_{|\rho,\gamma_1\gamma_2} = (f_{|\rho,\gamma_1})_{|\rho,\gamma_2}\,.$$

Then for $g \geq 2$ if f is holomorphic on $\mathcal{H}_g$ we have

$$f \in M_\rho(\Gamma_g) \Longleftrightarrow f_{|\rho,\gamma} = f \quad \forall \gamma \in \Gamma_g$$

On $\mathcal{D}_g$ we have the pullback of the Hodge bundle $\mathbf{E}$. So we have the automorphy factor $j(\gamma, z)$, the Jacobian determinant. We construct modular forms on $\mathcal{D}_g$.

Proposition 3. *For $k \geq 2$ the series $\sum_{\gamma\in\Gamma_g} j(\gamma, z)^k$ converges to a holomorphic function on $\mathcal{D}_g$.*

Proof. Choose for each $z \in \mathcal{D}_g$ a nhd U_z such that $\gamma(U_z) \cap U_z$ is empty or $\gamma(U_z) = U_z$ and

$\gamma \in I_z$. Then

$$\sum_{\gamma\in\Gamma_g} \int_{U_z} |j(\gamma, z)|^2 dz$$

(with dz Euclidean volume) converges since it equals

$$\sum_{\gamma\in\Gamma_g} \int_{\gamma(U_z)} dz \leq (\#I_z) \int_{\mathcal{D}_g} dz = \text{volume} < \infty$$

Now if $\Delta = \Delta(0, r)$ is a polydisc of radius r and f is holomorphic we write $f = \sum_I a_I z^I$ for the Taylor expansion. Then

$$\int_\Delta |f(z)|^2 dz = \sum_I \int_\Delta |a_I z^I|^2 dz \geq \pi^g r^{2g} |a_0|^2$$

because $\int_{|\zeta|\leq r} \zeta^i \bar{\zeta}^j d\zeta = 0$ if $i \neq j$.

But $|a_0| = |f(0)|$ so we find

$$\sum_{\gamma} |j(\gamma, z_0)|^2 \leq c \sum_{\gamma} \int_{U_{z_0}} |j(\gamma, z)|^2 dz$$

Take a compact set, cover it with finitely many U_z as above all contained in $\mathcal{D}_g$. We see that $\sum |j(\gamma, z)|^2$ converges to a holomorphic function. Q.e.d.

Corollary 4. *$\sum_{\gamma \in \Gamma_g} j(\gamma, z)^k$ converges for $k \geq 2$ to a holomorphic function.*

Now go back to $\mathcal{H}_g$. It says that

$$\sum_{\gamma_g} \det(c\tau + d)^{-k(g+1)}$$

converges for $k \geq 2$. But then this is a modular form since the chain rule says

$$j(\gamma\gamma_1,\tau)=j(\gamma,\gamma_1(\tau))j(\gamma_1,\tau)$$

hence

$$\sum_{\gamma\in\Gamma_g} j(\gamma,\tau)^k_{|k(g+1),\gamma_1}=\sum_{\gamma\in\Gamma_g} j(\gamma_1,\tau)^k j(\gamma,\gamma_1(\tau))^k$$

and this equals

$$\sum_{\gamma\in\Gamma_g} j(\gamma\gamma_1,\tau)^k=\sum_{\gamma\in\Gamma_g} j(\gamma,\tau)^k\,.$$

More generally, if f is a bounded function on $\mathcal{D}_g$ then

$$\sum_{\gamma} f(\gamma(z))j(\gamma,z)^k \quad (k\geq 2)$$

converges to a holomorphic function on $\mathcal{D}_g$ with

$$f_{|k',g} = f \qquad \forall \gamma \in \Gamma_g$$

with $k' = k(g+1)$. Hence it is a modular form (can be zero).

Write the Cayley transform in matrix form $\alpha = (1_g, -i_g; 1_g, i_g)$

Corollary 5. *The set*

$$\mathbf{F}_g = \{z \in \mathcal{D}_g : |j(\gamma, z)| \leq 1 \quad \forall \gamma \in \alpha\Gamma_g\alpha^{-1}\}$$

meets every orbit.

Proof. Because of the convergence the number of γ with $|j(\gamma, z)| >$ given real number is finite. Hence for every z there is a γ_0 such that $|j(\gamma_0, z)|$ is maximal. But then $\gamma_0(z) \in \mathbf{F}_g$ since

$$|j(\gamma, \gamma_0(z))| = |j(\gamma\gamma_0, z)|/|j(\gamma_0, z)| \leq 1$$

Q.e.d.

Proposition 4. *Let* $\tau_1, \tau_2 \in F_g$ *with* $\tau_1 \notin \Gamma_g \tau_2$. *Then there exists a Poincaré series* f *of some weight such that* $f(\tau_1) = 0$ *and* $f(\tau_2) = 1$

Proof. Work on $\mathcal{D}_g$ with points z_1 and z_2.

1) there are only finitely many $\gamma \in \Gamma_g$ such that $\gamma(z_1) \neq z_1$ and $|j(\gamma, z_1)| = 1$

2) there are only finitely many $\gamma \in \Gamma_g$ such that $|j(\gamma, z_2)| = 1$.

We can find a polynomial P in the coordinates of $\mathcal{D}_g$ with

1) $|P(z_1)| > 1$

2) $P(\gamma(z)) = 0$ $\forall\gamma$ with $\gamma(z_1) \neq z_1$ and $|j(\gamma, z_1)| = 1$

3) $P(\gamma(z_2)) = 0$ $\forall\gamma$ with $|j(\gamma, z_2)| = 1$.

Choose $k \geq 2$ divisible by $\#I_{z_1}$ and consider

$$f = \sum_{\gamma} P(\gamma(z)) j(\gamma, z)^k$$

For γ with $\gamma \notin I_{z_1}$ and $|j(\gamma, z_1)| = 1$ the contribution is 0

For $\gamma \in I_{z_1}$ it gives $\#I_{z_1} \cdot P(z_1)$.

All other γ give $|j(\gamma, z_1)| < 1$.

For large k the contribution from I_{z_1} dominates and for $k \to \infty$ the value $f(z_1)$

tends to $(\#I_{z_1})P(z_1)$. Similarly for z_2 either $P(\gamma(z_2)) = 0$ or $|j(\gamma, z_2)| < 1$ hence value for $k \to \infty$ tends to 0.

We thus find a form f_1 with

$$|f_1(z_1)| > 1, \qquad |f_1(z_2)| < 1$$

Similarly we find a form f_2 with

$$|f_2(z_1)| < 1, \qquad |f_2(z_2)| > 1$$

Raising to some power gives them the same weight. Consider then

$$F(z) = f_1(z_1)f_2(z) - f_2(z_1)f_1(z)$$

it satisfies $F(z_1) = 0$ and $F(z_2) \neq 0$. Q.e.d.

More Literature

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