Siegel Modular Forms

Lecture #4

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The Compact Dual

We generalize the standard embedding $\mathcal{H}_1 \to \mathbf{P}^1$. Let $(L, \langle\ ,\ \rangle)$ be our symplectic lattice. Extend scalars to $\mathbf{C}$: $L_{\mathbf{C}}$ with alternating form $E_{\mathbf{C}}$. We define

$$Q_g = \{W \subset L_{\mathbf{C}} : \dim W = g,\ E_{\mathbf{C}} = 0 \text{ on } W\}$$

Thm of Witt: $\mathrm{Sp}(2g, \mathbf{C})$ acts transitively on Q_g. Moreover, Q_g is homogeneous manifold of dimension $g(g+1)/2$. It is compact.

Look at the subset Q_g^+

$$\{W \in Q_g : -iE_{\mathbf{C}}(w, \bar{w}) > 0 \text{ for } 0 \neq w \in W\}$$

$\mathrm{Sp}(2g, \mathbf{R})$ preserves Q_g^+; stabilizer of a point is isomorphic to U_g. Hence

$$Q_g^+ \cong \mathrm{Sp}(2g, \mathbf{R})/U_g \cong \mathcal{H}_g$$

We thus find an embedding

$$\mathcal{H}_g \hookrightarrow Q_g.$$

Note: for an abelian variety $X = \mathbf{C}^g/\Lambda_\tau$ we have

$$H_1(X, \mathbf{Z}) = \Lambda_\tau, \quad H_1(X, \mathbf{R}) = \Lambda_\tau \otimes \mathbf{R},$$

and

$$H_1(X_\tau, \mathbf{C}) = \Lambda_\tau \otimes \mathbf{C}\,.$$

The dual $H^1(X, \mathbf{C})$ contains $H^0(X, \Omega^1_X)$, isomorphic to the cotangent space of X at the origin. This suggests the following.

Interpretation of $\mathcal{H}_g$

To $\tau \in \mathcal{H}_g$ we associated a complex torus $\mathbf{C}^g/\Lambda_\tau$ together with a positive definite hermitean form $H(z,w)$ satisfying

$$H(z,w) = E(\mathcal{J}z, w) + iE(z,w)$$

So $\mathcal{J}$ with $\mathcal{J}^2 = -1$ defines the complex structure on $L_{\mathbf{R}} = \mathbf{R}^{2g}$. It identifies $L_{\mathbf{R}}$ with $\mathbf{C}^g$ such that

$$\tau\xi_1 + \xi_2 \qquad (\xi_i \in \mathbf{R}^g)$$

are complex coordinates. (Here ξ_1 and ξ_2 are column vectors.) We have

$$E(\mathcal{J}z, \mathcal{J}w) = E(z,w), \quad E(\mathcal{J}z, z) > 0 \quad z \neq 0$$

Starting from the other side, given $\mathcal{J}$ on $L_{\mathbf{R}}$ we consider $L_{\mathbf{C}}$ and extend E to $L_{\mathbf{C}}$. Then $E_{\mathbf{C}}$ vanishes on the set

$$\{ix - \mathcal{J}x : x \in L_{\mathbf{R}}\}$$

that is, on the $(-i)$-eigenspace of the linear extension of $\mathcal{J}$. We thus have a Lagrangian subspace W of $L_{\mathbf{C}}$ and identification of $\mathbf{C}$-vector spaces

$$(L_{\mathbf{R}}, \mathcal{J}) \cong L_{\mathbf{C}}/W$$

and also with the $(+i)$-eigenspace of $\mathcal{J}_{\mathbf{C}}$ in $L_{\mathbf{C}}$. The positivity of $E(\mathcal{J}z, z)$ gives

$$-iE_{\mathbf{C}}(w, \bar{w}) > 0 \quad (0 \neq w \in W)$$

Conclusion 1. *$\mathcal{H}_g$ parametrizes complex structures $\mathcal{J}$ on $L_{\mathbf{R}}$ such that $H(z, w) =$*

$E(\mathcal{J}z, w) + iE(z, w)$ *is a positive definite hermitean form.*

Boundary Components

We have $\mathcal{H}_g \xrightarrow{\sim} Q_g^+$ with Q_g^+ the set of Lagrangian subspaces $W \subset L_{\mathbf{C}}$ such that $-iE_{\mathbf{C}}(w, \bar{w}) > 0$ for $w \neq 0$. Recall $\mathrm{Sp}(2g, \mathbf{R})$ acts on Q_g^+.

Now relax the condition $-iE_{\mathbf{C}}(w, \bar{w}) > 0$ to ≥ 0 on W. Choose $U \subset L_{\mathbf{R}}$ of dim r and define

$$F(U) =$$

$$\{W \in Q_g : -iE_{\mathbf{C}}(w, \bar{w}) \geq 0, \, W \cap \bar{W} = U_{\mathbf{C}}\}$$

We have

$$F(U) \cong Q_{g-r}^+$$

via

$$W \mapsto W/U \in L_{\mathbf{R}}/U \, .$$

The closure of $F(U)$ in $\overline{Q_g^+}$ is the set of $W \in Q_g$ where H degenerates further.

A flag of subspaces

$$U_g \supset U_{g-1} \supset \cdots \supset U_1 \qquad (\dim U_i = i)$$

corresponds to a flag of boundary components with

$$F(U_i) \subset \overline{F(U_{i-1})}$$

so that we get

$$\overline{Q_g^+} \supset \overline{F(U_1)} \supset \cdots \supset F(U_g)$$

$\mathrm{Sp}(2g, \mathbf{R})$ acts on Q_g, Q_g^+ and $\overline{Q_g^+}$. For $0 \leq r \leq g$ we define a standard boundary component F_r as $F_r = F(U)$ with $U = \langle e_1, \ldots, e_r \rangle$ (with $e_1, \ldots, e_g, f_1, \ldots, f_g$ the basis of L).

Lemma 1. *The action of Γ_g on isotropic flags $U_0 \subset U_1 \subset \cdots \subset U_g$ in $L_{\mathbf{Q}}$ with $\dim U_i = i$ and $E|U_i = 0$ is transitive.*

Proof. If U_g is a rational isotropic subspace in $L_{\mathbf{Q}}$ then we can find a complement $U_g^\vee$. Choose a basis of $U_g \cap L$. It is given by the columns of a $2g \times g$-matrix

$$(a\, b)'$$

Make the pair coprime: $(a_1\, b_1)$. The isotropy condition says $a_1 b_1' - a_1' b_1 = 0$, so $a_1 b_1'$ is symmetric. Then we can find c_1, d_1 such that $(a_1, b_1; c_1, d_1) \in \Gamma_g$. This transforms the basis in the standard one $(1_g, 0_g)$. Finish by induction. Q.e.d.

Corollary 1. *The action of Γ_g on the set of* *rational boundary components of given degree is transitive.*

Stabilizer of a boundary component $P_r \subseteq \mathrm{Sp}(2g, \mathbf{R})$

$$P_r = \left\{ \begin{pmatrix} a & 0 & b & * \\ * & u & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & {u'}^{-1} \end{pmatrix} \right\}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2r, \mathbf{R}), \quad u \in \mathrm{GL}(g-r, \mathbf{R})$$

We have

$$P_r \cong \mathrm{Sp}(2r, \mathbf{R}) \times \mathrm{GL}(g-r, \mathbf{R}) \ltimes R_r$$

with R_r the unipotent radical

$$\left\{ \begin{pmatrix} 1_r & 0 & 0 & n \\ m' & 1_{g-r} & n' & b \\ 0 & 0 & 1_r & -m \\ 0 & 0 & 0 & 1_{g-r} \end{pmatrix} \right\}$$

satisfying

$$n'm + b = mn' + b'$$

An element of R_r is given by a triple (m, n, b). Product of (m_1, n_1, b_1) and (m_2, n_2, b_2) corresponds to

$$(m_1 + m_2, n_1 + n_2, m_1' n_2 - n_1' m_2 + b_1 + b_2)$$

hence we get a Heisenberg group. The center of R_r are the elements $(0, 0, b)$, hence the center can be identified with the space C_{g-r} of symmetric matrices.

Eisenstein Series

Fix g. Let $f \in S_k(\Gamma_r)$ with k even and $0 \le r < g$. We have a map $P_r \to \mathrm{Sp}(2r, \mathbf{R})$ denotes $\gamma \mapsto \tilde{\gamma}$. Similarly we have a map $\mathcal{H}_g \to \mathcal{H}_r$

$$\tau = \begin{pmatrix} \tilde{\tau} & * \\ * & * \end{pmatrix} \mapsto \tilde{\tau}$$

Then

$$\widetilde{\gamma(\tau)} = \tilde{\gamma}(\tilde{\tau})$$

We define a function

$$E_g(f) = \sum_{P_r \backslash \Gamma_g} f(\tilde{\gamma}(\tilde{\tau})) \det(c\tau + d)^{-k}$$

where $\gamma = (a, b; c, d)$ runs through $P_r \backslash \Gamma_g$.

Check for $j(\gamma,\tau) = \det(c\tau + d)$ that

$$j(\gamma_1\gamma(\tau)) = j(\gamma_1, \gamma(\tau))j(\gamma,\tau)$$

for $\gamma_1 \in P_r$. Note

$$j(\gamma_1, \gamma(\tau)) = \pm\tilde{j}(\tilde{\gamma}_1, \gamma(\tilde{\tau}))$$

because $\det(u) = \pm 1$ for $u \in \mathrm{GL}(g-r)$. We see

$$E_g(f)_{|\gamma} = E_g(f)$$

In the specific case $r = 0$, $f = 1 \in S_k(\Gamma_0)$ we get

$$E_g(f) = \sum_{\gamma \in P_0 \backslash \Gamma_g} \det(c\tau + d)^{-k}$$

with $P_0 = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma_g \right\}$.

A coset consists of all $(a_1, b_1; c_1, d_1)$ with bottom of the form (uc, ud) with $u \in \mathrm{GL}(g, \mathbf{Z})$.

The condition that (c, d) occurs as bottom row is that c and d are coprime and cd' is symmetric.

Theorem 1. *(Hel Braun) Let* $0 \leq r < g$*,* $k \equiv 0 \pmod 2$ *and* $k \geq g + r + 2$*. For every cusp form* $f \in S_k(\Gamma_r)$ *the series* $E_g(f)$ *converges to a Siegel modular form in* $M_k(\Gamma_g)$ *with*

$$\Phi(E_g(f)) = E_{g-1}(f)\,,$$

and (applying this repeatedly)

$$\Phi^{g-r}(E_g(f)) = f\,.$$

Corollary 1. *The Siegel operator*

$$\Phi : M_k(\Gamma_g) \to M_k(\Gamma_{g-1})$$

is surjective for even $k > 2g$.

Proof. We show that then $M_k(\Gamma_g)$ is generated by cusp forms and Eisenstein series. Induction: clear for $g = 0$ and $g = 1$. Let $F \in M_k(\Gamma_g)$. Then $\Phi(F) = f + \sum_{i=1}^{t} E_{g-1}(h_i)$. So $F - E_g(f) - \sum_i E_g(h_i)$ is a cusp form. Eisenstein series and cusp forms are in the image of Φ. Q.e.d.

One can do some things for vector-valued modular forms (work of Weissauer). For $g = 1$ and k even and $k \geq 4$ we saw the Fourier series

$$E_k = 1 - \frac{2k}{B_k} \sum_{n \geq 0} \sigma_{k-1}(n) q^n$$

Maass determined the Fourier coefficients for $g = 2$. Katsurada generalized this for $g \geq 3$ but the formulas do not allow easy expressions.

For $g = 2$ we have $E_{2,k} = \sum_{n \geq 0} a(n) q^n$ with for $n = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ which we write as $n = [a, b, c]$

$$a(n) = \sum_{d|(a,b,c)} d^{k-1} H(k-1, \frac{4ac - b^2}{d^2})$$

where $H(k-1, D)$ is the Cohen function given by

$$H(k-1,D) = L_{-D}(2-k)$$

if $D > 0, D \equiv 0$ or $3(\mathrm{mod} 4)$ and else for $D \not\equiv 0, 3(\mathrm{mod} 4)$

$$H(k-1,D) = \begin{cases} \zeta(3-2k) & D = 0 \\ 0 & D \equiv 1,2(\mathrm{mod} 4) \end{cases}$$

To give an example: (with $q_j = e^{2\pi i \tau_{jj}}$ and $\zeta = e^{2\pi i \tau_{12}}$)

$$\psi_4 = 1 + 240(q_1 + q_2) + 2160(q_1^2 + q_2^2) +$$

$$240(\zeta^{-2} + 56\zeta^{-1} + 126 + 56\zeta + \zeta^2)q_1 q_2 +$$
$$4320(7\zeta^{-2} + 32\zeta^{-1} + 42 + 32\zeta + 7\zeta^2)(q_1^2 q_2 + q_1 q_2^2)$$
$$+ \cdots$$

Singular Modular Forms

A form $f = \sum a(n)q^n \in M_k(\Gamma_g)$ is called singular if $a(n) = 0$ for non-singular n.

Such forms can be characterized by a differential equation. Consider the $g \times g$-matrix

$$\left(\frac{1+\delta_{ij}}{2}\frac{\partial}{\partial\tau_{ij}}\right)$$

and let $\partial_{\det}$ be the determinant.

Lemma 2. *$f \in M_\rho(\Gamma_g)$ is singular if and only if $\partial_{\det} f = 0$.*

Indeed,

$$\partial_{\det} f = (2\pi i)^g \sum_{n \geq 0} a(n)\det(n)q^n\,.$$

Proposition 1. *If $f \in M_k(\Gamma_g)$ is singular then $k < g/2$.*

Proof. We know k must be even. Let f be of co-rank $g-r$ with $0 < r < g$; so $\Phi^{g-r}(f) \neq 0$. Then there is a m of size r with $\det(m) \neq 0$ such that

$$a \begin{pmatrix} 0 & 0 \\ 0 & m \end{pmatrix} \neq 0$$

We may assume $\det(m)$ is minimal. Restrict f to $\mathcal{H}_{g-r} \times \mathcal{H}_r$. So

$$h(\tau_1, \tau_2) = f_{|\mathcal{H}_{g-r} \times \mathcal{H}_r}$$

Write h as Fourier series on $\mathcal{H}_r$ with $(q_2 =$

$e^{2\pi i\tau_2}$ for $\tau_2 \in \mathcal{H}_r$)

$$h = \sum_{n_2 \geq 0} a(n_2) q_2^{n_2}$$

the transformation behavior implies

$$a(n_2) \in M_k(\Gamma_{g-r})$$

Consider $n_2 = m$. We have $a(m) = \sum b(n_1) q_1^{n_1}$ with

$$b(n_1) = \sum_{\ell} a(\begin{matrix} n_1 & \ell \\ \ell' & m \end{matrix})$$

But these matrices have rank r. By conjugation with $u \in \mathrm{GL}(g, \mathbf{Z})$ we can bring these in the form $(\begin{matrix} 0 & 0 \\ 0 & v \end{matrix})$ with $\det(v) =$

$\det(m)$. So using $a(u'un) = \det(u)^k a(n)$ we get

$$a(m) = a\begin{pmatrix} 0 & 0 \\ 0 & m \end{pmatrix} \sum_{\nu} q_1^{n-1}$$

Here ν runs through all half-integral $\begin{pmatrix} n_1 & \ell \\ \ell' & m \end{pmatrix}$ which conjugation by unimodular u can bring in the form $\begin{pmatrix} 0 & 0 \\ 0 & m \end{pmatrix}$. Then $u = \begin{pmatrix} 1_{g-r} & 0 \\ w & 1_r \end{pmatrix}$ and

$$a(m) = a\begin{pmatrix} 0 & 0 \\ 0 & m \end{pmatrix} \vartheta(m, \tau_1)$$

with

$$\vartheta(m, \tau_1) = \sum_{w} e^{2\pi i \mathrm{Tr}(w\tau_1 w')}$$

with w running through all $r \times (g-r)$-matrices. Such a theta series (associated to a lattice) of rank r is a modular form of weight $r/2$, hence $k = r/2 < g/2$. (We will come back to such series later.) Q.e.d.

Later we show

Theorem 2. *If $0 \neq f \in M_\rho(\Gamma_g)$ is singular, then $2\lambda_g < g$.*

Modular Forms of Degree Two

We consider now $R_2 = \oplus_k M_k(\Gamma_2)$. We have the Eisenstein series for even $k \geq g+2 = 4$.

$$\sum_{(c,d)} \det(c\tau + d)^{-k}$$

with the sum over non-associated coprime symmetric matrices. Let $\mathcal{H}_1 \times \mathcal{H}_1 \subset \mathcal{H}_2$ via $(\tau_1, \tau_2) \mapsto \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$. We have for the normalized Eisenstein series ψ_k with Fourier coefficient $a(0) = 1$:

$$\psi_k | \mathcal{H}_1 \times \mathcal{H}_1 = e_k \otimes e_k$$

with e_k the Eisenstein series for $g = 1$.

Write

$$q_j = e^{2\pi i \tau_j} \quad (j = 1, 2), \quad \zeta = e^{2\pi i \tau_{12}}\,.$$

Fourier series of ψ_4

$$\psi_4 = 1 + 240(q_1 + q_2) + 2160(q_1^2 + q_2^2) +$$

$$240(\zeta^{-2} + 56\zeta^{-1} + 126 + 56\zeta + \zeta^2)q_1 q_2 + \ldots$$

Fourier series of ψ_6

$$\psi_6 = 1 - 504(q_1 + q_2) - 16632(q_1^2 + q_2^2) +$$

$$504(\zeta^{-2} + 88\zeta^{-1} + 330 + 88\zeta + \zeta^2)q_1 q_2 + \ldots$$

Under Φ we have $\psi_{10} \mapsto e_{10} \otimes e_{10}$, but $e_{10} = e_4 e_6$. So

$$\Phi(\psi_{10} - \psi_4\psi_6) = 0$$

hence $\psi_{10} - \psi_4\psi_6$ is a cusp form. It is not zero. We thus have $\chi_{10} \in S_{10}(\Gamma_2)$defined as

$$\chi_{10} = \frac{43867}{2307916800}(\psi_4\psi_6 - \psi_{10})\,.$$

We have

$$\chi_{10}(\tau) = (1/\zeta - 2 + \zeta)q_1q_2-$$

$$(2/\zeta^2 + 16/\zeta - 36 + 16\zeta + 2\zeta^2)(q_1^2q_2 + q_1q_2^2)+$$

Locally near $\mathcal{H}_1^2$

$$(-4\pi^2\,\tau_{12}^2 + \ldots)(q_1q_2 + \ldots)$$

It vanishes with multiplicity 2 along $\mathcal{H}_1^2$. Similarly, define χ_{12} by normalizing a suitable linear combination of E_4^3, E_6^2 and E_{12}:

$$\chi_{12} = \frac{77683}{3081722112000}(441\,E_4^3 + 250\,E_6^2 - 691\,E_{12})$$

We find

$$\chi_{12} = (\zeta^{-1} + 10 + \zeta)\, q_1 q_2 + \cdots$$

If $F \in M_k(\Gamma_2)$ we develop F along $\mathcal{H}_1 \times \mathcal{H}_1$ in $\mathcal{H}_2$ (given by $z = \tau_{12} = 0$) in Taylor series:

$$F = f(\tau_1, \tau_2) z^n + \text{higher order terms}$$

with $f(\tau_1, \tau_2) \not\equiv 0$. We then have

1) $f(\tau_1, \tau_2) \in M_{k+n}(\Gamma_1) \otimes M_{k+n}(\Gamma_1)$

2) $f(\tau_2, \tau_1) = (-1)^k f(\tau_1, \tau_2)$

3) $f(\tau_1, \tau_2) = (-1)^{k+n} f(\tau_1, \tau_2)$

To see this we embed $\Gamma_1 \times \Gamma_1 \to \Gamma_2$ via

$$((a_1, b_1; c_1, d_1), (a_2, b_2; c_2, d_2)) \mapsto$$

$$\begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix}$$

For 2) use

$$\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$

with action $(\tau_1, z, \tau_2) \mapsto (\tau_2, z, \tau_1)$ and for 3) use $\mathrm{diag}(1, -1, 1, -1) \in \Gamma_2$ with $(\tau_1, z, \tau_2) \mapsto (\tau_1, -z, \tau_2)$.

Action of $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_1 \subset \Gamma_2$ on (τ_1, z, τ_2):

$$\tau_1 \mapsto \gamma_1(\tau_1), \quad \tau_2 \mapsto \gamma_2(\tau_2)$$

and

$$z \mapsto z\,(c_1\tau_1 + d)^{-1}(c_2\tau_2 + d_2)^{-1}$$

Define

$$M_k^{\geq n}(\Gamma_1) = \{f \in M_k(\Gamma_1) : \mathrm{ord}_\infty(f) \geq n\}$$

and

$$M_k^{\geq n}(\Gamma_2) = \{f \in M_k(\Gamma_2) : \mathrm{ord}_{\mathcal{H}_1 \times \mathcal{H}_1}(f) \geq n\}$$

Note that χ_{10} satisfies

$$\chi_{10} \sim (\Delta \otimes \Delta)\, z^2 + O(z^4)$$

Since $f \in M_k(\Gamma_2)$ is invariant under translations $\tau_2 \mapsto \tau_2 + 1$ we have a Fourier series

$$f = \sum_{l=0}^{\infty} \varphi_l(\tau_1, z) q_2^l$$

where φ_l satisfies for all $\xi \in \mathbf{Z}$:

1) $\varphi_l(\tau, z + \xi) = \varphi_l(\tau, z)$,

2) $\varphi_l(\tau, z+\xi\tau) = e^{-2\pi i \mathrm{Tr}(l\xi^2\tau+2l\xi z)} \varphi_l(\tau, z)$

Hence for fixed τ_1 the function $\varphi_l(\tau_1, z)$ is a theta function of order $2l$ on the elliptic curve $\mathbf{C}/\Lambda_{\tau_1}$.

In particular for $f \in M_k^{\geq n}(\Gamma_2)$ the term φ_l starts with $z^{\geq n}$ at the origin $z = 0$ of $\mathbf{C}/\Lambda_{\tau_1}$. Now $\mathrm{div}(\varphi_m)$ has degree $2m$, hence $n > 2m$ implies $\varphi_m = 0$.

We will treat such developments $f = \sum_l \varphi_l(\tau_1, z) q_2^l$ in detail later.