Siegel Modular Forms
Lecture $\#5$

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Recall that we embed

$$\mathcal{H}_1 \times \mathcal{H}_1 \hookrightarrow \mathcal{H}_2, \quad (\tau_1, \tau_2) \mapsto \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$$

A modular form $f \in M_k(\Gamma_2)$ is invariant under translations $\tau_2 \mapsto \tau_2 + 1$, hence we have a Fourier expansion

$$f = \sum_{l=0}^{\infty} \varphi_l(\tau_1, z) q_2^l$$

where φ_l satisfies for all $\xi \in \mathbf{Z}$:

1) $\varphi_l(\tau_1, z + \xi) = \varphi_l(\tau_1, z)$,

2) $\varphi_l(\tau_1, z+\xi\tau_1) = e^{-2\pi i \mathrm{Tr}(l\xi^2\tau_1+2l\xi z)}\varphi_l(\tau_1, z)$

Hence for fixed τ_1 the function $\varphi_l(\tau_1, z)$ is a theta function of order $2l$ on the elliptic curve $\mathbf{C}/\Lambda_{\tau_1}$.

We can develop a modular form $f \in M_k(\Gamma_2)$ around $\mathcal{H}_1 \times \mathcal{H}_1$.

To $f \in M_k^{\geq 2n}(\Gamma_2)$ we associate the $2n$th Taylor term (with $z = \tau_{12}$)

$$f = f_0(\tau_1, \tau_2) z^{2n} + \cdots$$

We saw that $f_0 \in M_{k+2n}(\Gamma_1) \otimes M_{k+2n}(\Gamma_1)$ and that

1) $f(\tau_1, \tau_2) \in M_{k+2n}(\Gamma_1) \otimes M_{k+2n}(\Gamma_1)$

2) $f(\tau_2, \tau_1) = (-1)^k f(\tau_1, \tau_2)$

3) $f(\tau_1, \tau_2) = (-1)^{k+2n} f(\tau_1, \tau_2)$.

Lemma 1. *For even k we have an exact sequence*

$$0 \to M_k^{\geq 2n+2}(\Gamma_2) \to M_k^{\geq 2n}(\Gamma_2)$$

$$\xrightarrow{t} \mathrm{Sym}^2(M_{k+2n}^{\geq n}(\Gamma_1)) \to 0$$

Proof. If $f \in M_k^{\geq 2n}(\Gamma_2)$ then $\varphi_m(\tau, z)$ vanishes with order $\geq 2n$ at $z = 0$, hence we land in Sym^2 of $M_{k+2n}^{\geq n}(\Gamma_1)$. The only point is surjectivity.

Use that $\oplus_k \mathrm{Sym}^2(M_k(\Gamma_1))$ is generated by

$$e_4 \otimes e_4, \quad e_6 \otimes e_6, \quad \Delta \otimes \Delta$$

An elt in $\mathrm{Sym}^2(M_k^{\geq n}(\Gamma_1))$ can be written as

$$(\Delta \otimes \Delta)^n P \otimes P$$

with P an (isobaric) polynomial in e_4, e_6 and Δ of weight $k-12n$. But

$$\chi_{10}^n P(\psi_4, \psi_6, \chi_{12}) \mapsto (\Delta \otimes \Delta)^n P \otimes P$$

and this proves the surjectivity. Q.e.d.

Theorem 1. *(Igusa) The ring* $R_2^{\mathrm{ev}} = \oplus_k M_{2k}(\Gamma_2)$ *is isomorphic to*

$$\mathbf{C}[\psi_4, \psi_6, \chi_{10}, \chi_{12}]\,.$$

Proof. We note

$$M_k^{\geq n}(\Gamma_1) = \Delta^n M_{k-12n}(\Gamma_1)$$

$$M_k^{\geq 2n}(\Gamma_2) = \chi_{10}^n M_{k-10n}(\Gamma_2)$$

The generating series of R_2^{ev} is

$$\sum_{k\equiv_2 0} \dim M_k(\Gamma_2)t^k =$$

$$\sum_{k\equiv_2 0} \sum_{n=0}^{\infty} \dim \operatorname{Sym}^2(M_{k+2n}^{\geq n}(\Gamma_1))t^k$$

$$= \sum_{k\equiv_2 0} \sum_{n} \dim \operatorname{Sym}^2(M_{k-10n}(\Gamma_1))t^k$$

$$= \frac{1}{1-t^{10}} \sum_{k} \dim \operatorname{Sym}^2(M_k(\Gamma_1))t^k$$

$$= \frac{1}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}\,.$$

This shows that ψ_4, ψ_6, χ_{10} and χ_{12} generate and are algebraically independent. Q.e.d.

Alternatively we can say

$$R_2^{\text{ev}} = \mathbf{C}[\psi_4, \psi_6, \psi_{10}, \psi_{12}]\,.$$

If we assume the existence of a cusp form of weight 35 (we will show its construction later) we can treat odd weights too: write $f \in M_k^{\geq 2n+1}(\Gamma_2)$ as

$$f = \sum_m \varphi_m(\tau_1, z) q_2^m ,$$

where the functions φ_m are all odd (as function of z), hence vanish at the points of order 2 of the elliptic curve X_{τ_1}. Hence φ_m is zero unless

$$2m \geq (2n+1) + 3$$

Now the first term in the Taylor expansion is anti-symmetric. So we get an estimate

$$\dim M_k(\Gamma_2) \leq \sum_{n \geq 0} \dim \wedge^2 M_{k+2n+1}^{\geq n+2}(\Gamma_1)$$

and

$$M^{\geq n+2}_{k+2n+1}(\Gamma_1) = \Delta^{n+2} M_{k-10n-23}(\Gamma_1)$$

hence $\wedge^2 = 0$ for $k < 35$.

The upper bound shows

$$\sum_{k \,\mathrm{odd}} \dim M_k(\Gamma_2) t^k \leq$$

$$\frac{t^{35}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$$

Now the existence of χ_{35} shows that this upper bound is reached. So we have the result

Theorem 2. *(Igusa) The ring R_2 is isomorphic to*

$$\mathbf{C}[\psi_4, \psi_6, \chi_{10}, \chi_{12}, \chi_{35}]/(\chi_{35}^2 - \cdots)$$

where χ_{35}^2 *is equal to a polynomial in* $\psi_4, \psi_6, \chi_{10}, \chi_{12}$.

Igusa used Eisenstein series to prove this result and also established the relation with invariant theory of binary sextics. We will come back to this later. Igusa also used theta functions to construct these modular forms and prove his result.

Topological Compactification

Consider the embedding $\mathcal{H}_g \hookrightarrow B_r$ with B_r given by

$$\left\{ \begin{pmatrix} \tau_1 & z \\ z' & \tau_2 \end{pmatrix} \in \mathrm{SymMat}(g \times g, \mathbf{C}) : \tau_1 \in \mathcal{H}_r \right\}$$

We have

$$B_r \cong \mathcal{H}_r \times \mathrm{Mat}(r \times (g-r), \mathbf{C}) \times \mathcal{C}_{g-r} \otimes \mathbf{C}$$

with $\mathcal{C}_l$ the space of real symmetric matrices of size l. Recall the stabilizer P_r of $F(U) = F_r$. The center Z_r of P_r acts by

$$\tau \mapsto \tau + \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$$

for b symmetric of size $g-r$. For $\tau \in B_r$ we write

$$\mathrm{Im}(\tau) = \begin{pmatrix} y_1 & v \\ v' & y_2 \end{pmatrix}$$

Make a real-analytic map

$$\psi_r : B_r \to \mathcal{C}_{g-r}, \quad \tau \mapsto y_2 - v' y_1^{-1} v$$

This generalizes the map $\tau \mapsto y$ for $g = 1$, $r = 0$. Recall that P_r is a semi-direct product $\mathrm{Sp}(2r, \mathbf{R}) \times \mathrm{GL}(g-r) \ltimes R_r$ with R_r

$$\left\{ \begin{pmatrix} 1_r & 0 & 0 & n \\ m' & 1_{g-r} & n' & b \\ 0 & 0 & 1_r & -m \\ 0 & 0 & 0 & 1_{g-r} \end{pmatrix} \in P_r \right\}$$

with the triple (m, n, b) satisfying $n'm + b = m'n + b'$.

Lemma 2. *The action of P_r on $\mathcal{H}_g$ extends to an action on B_r and the map $\psi_r : B_r \to \mathcal{C}_{g-r}$ is equivariant for this action of P_r on B_r and $\mathrm{GL}(g-r, \mathbf{R})$ on $\mathcal{C}_{g-r}$.*

Moreover, the image of $\mathcal{H}_g$ in B_r maps under ψ_r to $\mathcal{C}^+_{g-r}$.

Here $\mathcal{C}^+_{g-r}$ is the cone of positive definite matrices.

Proof. P_r is a semi-direct product. The center Z_r of R_r acts by translations. The centralizer of Z_r is generated by $\mathrm{Sp}(2r, \mathbf{R})$ and R_r. Here $\mathrm{Sp}(2r)$ is embedded via

$$\gamma_1 = (a, b; c, d) \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 1 \\ c & 0 & d & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

and acts via

$$\begin{pmatrix} \tau_1 & z \\ z' & \tau_2 \end{pmatrix} \mapsto \begin{pmatrix} \gamma_1(\tau_1) & (c\tau_1+d)'^{-1}z \\ * & \tau_2 - z'(c\tau_1+d)'^{-1}cz \end{pmatrix}$$

Then $u \in \mathrm{GL}(g-r, \mathbf{R}) \subset P_r$ acts by

$$\tau \mapsto \begin{pmatrix} \tau_1 & zu \\ uz' & u\tau_2 u' \end{pmatrix}$$

The group R_r acts via sending (τ_1, z, τ_2) to

$$\begin{pmatrix} \tau_1 & z + \tau_1 m + n \\ * & \tau_2 + m'\tau_1 m + m'z + z'm + n'm + b \end{pmatrix}$$

For the positivity use the identity

$$\begin{pmatrix} 1 & 0 \\ -v'y_1^{-1} & 1 \end{pmatrix} \begin{pmatrix} y_1 & v \\ v' & y_2 \end{pmatrix} \begin{pmatrix} 1 & -y_1^{-1}v \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} y_1 & 0 \\ 0 & y_2 - v'y_1 v \end{pmatrix}$$

to see that if $y_1 > 0$ then

$$\mathrm{Im}(\tau) > 0 \iff y_2 - v' y_1 v > 0$$

Q.e.d.

The cone $\mathcal{C}_r^+$ of positive definite real matrices carries a quadratic form

$$(\eta_1, \eta_2) \mapsto \mathrm{Tr}(\eta_1 \eta_2)$$

and $\mathcal{C}_r^+$ is self-dual:

$$\mathcal{C}_r^+ = \{\eta_1 \in \mathcal{C}_r : \mathrm{Tr}(\eta_1 \eta_2) > 0 \quad \forall \eta_2 \in \mathcal{C}_r^+\}$$

Define now

$$\mathcal{H}_g^* = \mathcal{H}_g \cup (\cup_U F(U))$$

with the union over all totally isotropic subspaces $U \subseteq L_{\mathbf{Q}}$, that is $\mathcal{H}_g$ union all the rational boundary components. This is the analogue of $\mathcal{H}_1 \sqcup \mathbf{P}^1(\mathbf{Q}) \subset \mathbf{P}^1(\mathbf{C})$.

Define a topology. For $\sigma \in \Sigma_g$ with Σ_g the set of coprime pairs (b, d) with bd' symmetric we define a sphere of influence V_σ^*. Do this first for $\sigma = \infty = (1_g, 0_g)$: Define V_∞^* as

$$\cup_{r=0}^{g} \{\tau \in F_r : \mu_r(\sigma, \tau) < \mu_r(\infty, \tau)\, \forall \sigma \in \Sigma_r\}$$

the union of all $V_\infty^{(r)}$ for all $r = 0, \ldots, g$. This set is preserved by

$$P_0 = \mathrm{Stab}(\infty) = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{Sp}(2g, \mathbf{R})\}$$

Via the transitive action of Γ_g we get

$$V_\sigma^* = \gamma^{-1}(V_\infty^*)\,, \qquad \gamma(\sigma) = \infty$$

We need a topology on $\mathcal{H}_g^*$. For $g = 1$ the basis of nhds of ∞ is

$$\{\tau \in \mathcal{H}_1 : \mathrm{Im}(\tau_1) > c\}$$

for real $c > 1$. Proceed by induction. Suppose we have a topology on $\mathcal{H}_{g-1}^*$ such that $\mathrm{Sp}(2g-2, \mathbf{Q})$ acts via homeomorphisms. If U^* is an open subset of $\mathcal{H}_{g-1}^*$ we put

$$U = V_\infty^* \cap U^* \qquad \text{in } \mathcal{H}_{g-1}$$

and

$$U_c = \{ \begin{pmatrix} \tau_1 & z \\ z' & \tau_2 \end{pmatrix} \in B_{g-1} : \tau_1 \in U, \psi_1(\tau) > c\}$$

where $\psi_1(\tau) = y_2 - v' y_1^{-1} v$ and put

$$U_c^* = (U^* \cap V_\infty^*) \cup U_c \,.$$

Take the weakest topology that is $\mathrm{Sp}(2g,\mathbf{Q})$-invariant and for which all U_c^* are open. This gives a topology and $\mathrm{Sp}(2g,\mathbf{Q})$ acts by homeomorphisms.

Theorem 3. *The group Γ_g acts by homeomorphisms on $\mathcal{H}_g^*$ and the quotient space is compact. It contains $\Gamma_g\backslash\mathcal{H}_g$ as an open dense subset.*

Proof. $\Gamma_g\backslash\mathcal{H}_g^*$ is a disjoint union $\cup_{r=0}^{g}\Gamma_r\backslash\mathcal{H}_r$. Let $\{\tau_n \bmod \Gamma_g, \tau_n \in \mathcal{H}_g : n = 1, 2, \ldots\}$ be an arbitrary set. We must show it has an accumulation point. May assume $\tau_n = x_n + iy_n \in F_g$, the fundamental domain. Then

$$(y_n)_{kk} \le (y_n)_{ll} \quad \text{for } k \le l$$

$$(y_n)_{kl} \le (y_n)_{kk}$$

Either all $(y_n)_{rr}$ remain bounded, then ready, or there exists $0 \leq r \leq g-1$ such that $(y_n)_{rr}$ remains bounded, but $(y_n)_{r+1,r+1}$ does not. Since $\tau_n \in F_g$ we know $|x_n|$ bounded. Write

$$\tau_n = \begin{pmatrix} \tau_n^{(1)} & z_n \\ z_n' & \tau_n^{(2)} \end{pmatrix} \text{with} \quad \begin{matrix} \tau_n^{(1)} \in \mathcal{H}_r \\ z_n \in \mathbf{C}^{r \times (g-r)} \end{matrix}$$

Here $\tau_n^{(1)}$ bounded, z_n bounded; can find a subsequence τ_m with $\tau_m^{(1)} \to \tau^{(1)}$ and $z_m \to z$ and $\mathrm{Im}(\tau_m^{(2)}) \to \infty$. Then

$$\tau^{(1)} \in V_\infty^{(r)}$$

and with $v_n = \mathrm{Im}(z_n)$ we have

$$y_m^{(2)} - v_m' {y^{(1)}}^{-1} v_m \to \infty$$

so $\tau^{(1)}$ is the accumulation point by the definition of the topology.

We thus have with

$$\Gamma_g\backslash\mathcal{H}_g^* = \sqcup_{j=0}^{g}\Gamma_j\backslash\mathcal{H}_j$$

or put differently

$$\Gamma_g\backslash\mathcal{H}_g^* = \Gamma_g\backslash\mathcal{H}_g \sqcup (\Gamma_{g-1}\backslash\mathcal{H}_{g-1}^*)$$

If $\Gamma \neq \Gamma_g$, but e.g. $\Gamma = \Gamma_g[n]$ we get more boundary components; for example, for $g = 2$ and $n = 2$ we get 15 1-dim. boundary components, each isomorphic to $\Gamma_1[2]\backslash\mathcal{H}_1$ and 15 0-dim. boundary components. Now $\Gamma_1[2]\backslash\mathcal{H}_1$ is compactified by three cusps. In total the boundary components form a configuration $(15_3, 15_3)$ on which $\Gamma_2/\Gamma_2[2] \cong \mathfrak{S}_6$ acts.

The Rank One Partial Compactification

Look again at B_{g-1} with

$$\{\tau = \begin{pmatrix} \tau_1 & z \\ z' & \tau_2 \end{pmatrix} : \tau' = \tau, \tau_1 \in \mathcal{H}_{g-1}\}$$

so that $B_{g-1} \cong \mathcal{H}_{g-1} \times \mathbf{C}^{g-1} \times \mathbf{C}$. It contains $\mathcal{H}_g$. The group P_{g-1} acts as stabilizer of $F(U)$. The center ($\mathbf{R}$ or $\mathbf{Z}$ in Γ_g) acts via

$$\tau_2 \mapsto \tau_2 + b\,.$$

Consider $\mathcal{H}_{g,c}$ defined as

$$= \{\begin{pmatrix} \tau_1 & z \\ z' & \tau_2 \end{pmatrix} : \tau_1 \in \mathcal{H}_{g-1}, y_2 - v' y_1^{-1} v > c\}$$

P_{g-1} acts and preserves this.

Divide first by the action of translations $\tau_2 \mapsto \tau_2 + 1$:

$$\mathcal{H}_g \to \mathcal{H}_{g-1} \times \mathbf{C}^{g-1} \times \mathbf{C}^*, \tau \mapsto (\tau_1, z, q_2)$$

with $q_2 = e^{2\pi i \tau_2}$. Now the action of P_{g-1} induces an action on $\mathcal{H}_{g-1} \times \mathbf{C}^{g-1} \times \mathbf{C}^*$ We can extend this action to an action on

$$\mathcal{H}_{g-1} \times \mathbf{C}^{g-1} \times \mathbf{C}.$$

as follows. The group R_{g-1} acts by sending (τ, z, q_2) to

$$(\tau_1, z + \tau_1 m + n, e^{2\pi i (m' \tau_1 m + 2m' z)} q_2)$$

Furthermore, $\gamma \in \Gamma_{g-1}$ acts by sending

(τ_1, z, q_2) to

$$(\gamma(\tau_1), (c\tau_1 + d)'^{-1} z, e^{-2\pi i (z'(c\tau_1+d)^{-1} z)} q_2)$$

These actions extend over $\mathbf{C}^* \subset \mathbf{C}$. What do we get for $q_2 = 0$? This is

$$\mathcal{H}_{g-1} \times \mathbf{C}^{g-1} \times \{0\}$$

under the action:

$$(\tau_1, z) \mapsto (\tau_1, z + \tau_1 m + n)$$

and

$$(\tau_1, z) \mapsto (\gamma(\tau_1), (c\tau_1 + d)'^{-1} z) \,.$$

That gives the (orbifold) universal family $\mathcal{X}_{g-1} \to \Gamma_{g-1} \backslash \mathcal{H}_{g-1}$. But we still have to take into account the action of $u \in \mathrm{GL}(1,\mathbf{Z}) = \{\pm 1\}$ via $(\tau_1, z) \mapsto (\tau_1, uz)$.

For large $c \in \mathbf{R}_{>0}$ the action of Γ_g on $\mathcal{H}_{g,c}$ reduces to the action of $\Gamma_g \cap P_{g-1}$ in the following sense: if V is bounded and open in $\mathcal{H}_{g-1}$ and we put

$$U(V,c) = \{\begin{pmatrix} \tau_1 & z \\ z' & \tau_2 \end{pmatrix} : \tau_1 \in V, \psi_1(\tau) > c\}$$

with $\psi_1 = y_2 - v' y_1^{-1} v$, then for suitably large c the action is reduced to P_{g-1}.

Take $U(V,c) \subset \mathcal{H}_{g,c}$ and consider its image under

$$\Gamma_g \cap P_{g-1} \backslash \mathcal{H}_{g,c} \hookrightarrow \Gamma_g \cap P_{g-1} \backslash \mathcal{H}_{g-1} \times \mathbf{C}^{g-1} \times \mathbf{C}$$

and take the closure of the image $\overline{P_{g-1} \backslash U(V,c)}$. The rank-one partial compactification of $\Gamma_g \backslash \mathcal{H}_g$ is the orbifold obtained by gluing for a covering $\{V_\alpha\}$ of $\mathcal{H}_{g-1}$ the sets $\overline{U(V,c)}$ for sufficiently large c over the image of $U(V,c) \cap \mathcal{H}_g$ in $\Gamma_g \backslash \mathcal{H}_g$. Notation

$$\tilde{\mathcal{A}}_g^{(1)}$$

It was introduced by Mumford. It is not compact, but an orbifold, the union of $\mathcal{A}_g = \Gamma_g \backslash \mathcal{H}_g$ and an orbifold divisor D, the quotient of the universal abelian variety $\mathcal{X}_{g-1}$ of dimension $g-1$ over $\mathcal{A}_{g-1}$ by $\{\pm 1\}$:

$$\tilde{\mathcal{A}}_g^{(1)} = \mathcal{A}_g \sqcup D \,.$$

It allows a map to $\mathcal{A}_g^{(1)} = \mathcal{A}_g \sqcup \mathcal{A}_{g-1}$. It also has a moduli interpretation.

Fourier-Jacobi Forms

Start with the embedding

$$\mathcal{H}_{g+1} \to \mathcal{H}_g \times \mathbf{C}^g \times \mathcal{H}_1$$

$$\tau = \begin{pmatrix} \tau_1 & z \\ z' & \tau_2 \end{pmatrix} \mapsto (\tau_1, z, \tau_2)$$

Consider a modular form $f \in M_\rho(\Gamma_{g+1})$. It is invariant under $\tau_2 \mapsto \tau_2 + 1$. So we have a Fourier series:

$$f = \sum_{m=0}^{\infty} \varphi_m(\tau_1, z)\, q_2^m$$

called the Fourier-Jacobi series.

Look at the Jacobi group Γ^J generated by $\Gamma_g \subset \Gamma_{g+1}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & & b & \\ & 1 & & 1 \\ c & & d & \\ & 1 & & 1 \end{pmatrix}$$

and the Heisenberg group R_{g+1} generated by

$$\begin{pmatrix} 1_g & 0 & 0 & n \\ m' & 1 & n' & \beta \\ 0 & 0 & 1_g & -m \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $(m, n, \beta) \in \mathbf{Z}^g \times \mathbf{Z}^g \times \mathbf{Z}$. (This lives in the stabilizer $P_g \subset \Gamma_{g+1}$ of the boundary component.)

Definition: A holomorphic map

$$\varphi(\tau_1, z) : \mathcal{H}_g \times \mathbf{C}^g \to W$$

is called Jacobi-form of weight ρ and index m if

$$f = \varphi(\tau_1, z) q_2^m$$

satisfies

$$f_{|\rho,\gamma} = f$$

for all $\gamma \in \Gamma^J$ (plus holomorphicity at cusps for $g = 1$).

If $f \in M_\rho(\Gamma_{g+1})$ then the φ_m are Jacobi forms.

The relation between the Fourier series

$$f = \sum_{N \geq 0} a(N) q^N$$

and the Fourier-Jacobi series

$$f = \sum_{m} \varphi_m(\tau_1, z) q_2^m$$

is

$$\varphi_m q_2^m = \sum_{N} a(N) q^N$$

with $N \geq 0$ running over $\begin{pmatrix} n_1 & l \\ l' & m \end{pmatrix}$.

Take $\rho = \det^k$. Then for fixed τ_1:

$$\varphi_m(\tau_1, z + \xi) = \varphi_m(\tau_1, z)$$

$$\varphi_m(\tau_1, z+\tau_1\xi) = e^{-2\pi i(m\xi'\tau_1\xi+2m\xi' z)}\varphi_m(\tau_1, z)$$

for all $\xi \in \mathbf{Z}^g$.

So for fixed τ_1 the function φ_m is a so-called theta function on the abelian variety

$$X_{\tau_1} = \mathbf{C}^g/\mathbf{Z}^g + \tau_1\mathbf{Z}^g$$

of order $2m$ (section of $\mathcal{O}(2m\,\Theta)$).

We have for given m a basis for the space $(H^0(X_{\tau_1}, O(\Theta)))$ of such functions:

$$\Theta_\epsilon = \Theta_\epsilon^{(m)}$$

defined by

$$\Theta_\epsilon(\tau_1, z) = \sum_{\xi \in \mathbf{Z}^g} e^{\pi i \left((\xi+\epsilon/m)' \tau_1 (\xi+\epsilon/m) + 2(\xi+\epsilon/m \right) z)}$$

Here

$$\epsilon = (\epsilon_1, \ldots, \epsilon_g)$$

with $0 \le \epsilon_i < m$. The dimension of the space: m^g.

Then we let τ_1 vary in $\mathcal{H}_g$. But we do not have the basis in level 1. Pull back f to (appropriate) higher level; then we can write

$$f = \sum_{m=0}^{\infty} \varphi_m(\tau_1, z)\, q_2^m$$

and we can express the φ_m in this basis:

$$\varphi_m(\tau_1, z) = \sum_{\epsilon} h_\epsilon \, \Theta_\epsilon^{(m)}$$

Then the $h_\epsilon = h_\epsilon^{(m)}$ are Siegel modular forms of degree g (of some level) and weight $k-1/2$.

We can view the Fourier-Facobi series as a development along the boundary divisor in the rank one compactification.

This can be extended:

$$\mathcal{H}_g \to \mathcal{H}_{g-r} \times \mathbf{C}^{(g-r)\times r} \times \mathcal{H}_r$$

$$\tau \mapsto (\tau_1, z, \tau_2)$$

and then we can write $f \in M_k(\Gamma_g)$ as

$$f = \sum_m \varphi_m(\tau_1, z) q_2^m$$

Again for fixed τ_1 the φ_m are theta functions for the lattice $\tau_1 \mathbf{Z}^{(g-r)\times r} + \mathbf{Z}^{(g-r)\times r}$ in $\mathbf{C}^{(g-r)\times r}$.

If

$$\det(m) = 0$$

then the φ_m do not depend on some of the z variables.

But for

$$m > 0$$

we can write (in higher level)

$$\varphi_m(\tau_1, z) = \sum h_\epsilon \Theta_\epsilon^{(m)}$$

with now h_ϵ of weight

$$k - r/2$$

(on some congruence subgroup).

Theorem 1. *A modular form $f \in M_\rho(\Gamma_g)$ of weight ρ with $\lambda_g < g/2$ is singular.*

Proof. We do the case of $\rho = \det^k$.

Choose $0 < r < g$ such that we have $k \le r/2$ and r is even. Write

$$f = \sum_m \varphi_m(\tau_1, z)\, q_2^m \quad \text{with} \quad \tau_1 \in \mathcal{H}_{g-r}$$

We have

$$\varphi_m q_2^m = \sum_{N=(n_1,l;l',m)} a(N) q^N$$

If $m > 0$ then $\varphi_m = \sum h_\epsilon \Theta_\epsilon$ with h_ϵ of weight $k - r/2$. If this is negative, then all h_ϵ are zero. So m must be singular if $a(N) \neq 0$. But then f is singular. If $k = r/2$ then the φ_m are theta series, hence singular.

More Literature

J.-I. Igusa: On Siegel modular forms of genus two. Am. J. Math. 84 (1962), p. 612–649.

D. Mumford: On the Kodaira dimension of the Siegel modular variety. In: Algebraic Geometry–Open Problems. SLNM 997 (1983).