Siegel Modular Forms
Lecture $\#6$

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Baily-Borel Compactification

Also called Satake compactification. Recall we have a topological compactification

$$\Gamma_g \backslash \mathcal{H}_g^* = \sqcup_{r=0}^{g} \Gamma_r \backslash \mathcal{H}_r$$

We define a sheaf of rings $\mathcal{F}$: for U open set

$$\mathcal{F}(U) := \{ f : U \to \mathbf{C}, \begin{array}{c} f \text{ continuous} \\ f_{|U \cap (\Gamma_g \backslash \mathcal{H}_g)} \text{holom.} \end{array} \}$$

A ringed space $(X, \mathcal{F})$ is called an analytic space if for all $x \in X$ there is an open nhd U of x and an isomorphism of ringed spaces

$$(U, \mathcal{F}|U) \cong (Y, \mathcal{O}_Y)$$

with $Y \subset \mathbf{C}^n$ the zero set of finitely many analytic functions.

We have the following situation: X a locally compact Hausdorff space which is stratified $X = \cup_i X_i$ together with a sheaf of rings $\mathcal{F}$ such that each X_i is a normal analytic space and

1) X_0 is dense in X with $\dim X_i \leq \dim X$ with $=$ only for $i = 0$. $\cup_{i:\dim X_i \leq d} X_i$ is closed.

2) Every $x \in X$ has a basis of open nhds $\{U_j\}$ such that $U_j \cap X_0$ is connected.

3) $\mathcal{F}_{|X_i} = \mathcal{O}_{X_i}$

4) every $x \in X$ has a nhd U_x such that elements of $\mathcal{F}(U)$ separate points.

Then (Fact): $(X, \mathcal{F})$ is a normal analytic space and for each d the set $X^{\leq d} = \cup_{\dim X_i \leq d}$ is an analytic subspace.

We apply this to $\Gamma_g \backslash \mathcal{H}_g^*$. Checking the conditions using induction is not difficult.

Conclusion: $\Gamma_g \backslash \mathcal{H}_g^*$ is a normal analytic space.

We want to see that it is an algebraic variety.

Lemma 1. *The ring* $R = \oplus_k M_k(\Gamma_g)$ *is integrally closed.*

Proof. A graded ring is integrally closed if every quotient of homogeneous elements that is integral over R is in R.

If f_1 and f_2 are two modular forms (of weights k_1 and k_2) and $f = f_1/f_2$ is integral over R then f is holomorphic since the ring of holomorphic functions on $\mathcal{H}_g$ is integrally closed. So f is a holomorphic modular form of weight $k_1 - k_2$. Q.e.d.

We prove that

$$\Gamma_g\backslash\mathcal{H}_g^* = \Gamma_g\backslash\mathcal{H}_g \sqcup \Gamma_{g-1}\backslash\mathcal{H}_{g-1}^*$$

is algebraic by induction. The cases $g = 0$ and $g = 1$ are clear. We show that $\Gamma_g\backslash\mathcal{H}_g^*$ can be embedded in $\mathbf{P}^N$ by modular forms.

We assume by induction that there are modular forms

$$f_0, \ldots, f_n$$

of the same weight (say in $M_k(\Gamma_{g-1})$) that embed $\Gamma_{g-1}\backslash\mathcal{H}_{g-1}^*$ into projective space via:

$$x \mapsto (f_0(x) : \cdots : f_n(x))$$

Now use that

$$\Phi : M_k(\Gamma_g) \to M_k(\Gamma_{g-1})$$

is surjective for k large. If necessay replace f_i by their powers so that these are in the image of Φ. Say

$$\Phi(F_i) = f_i\,.$$

Put $Z =$ set of common zeros of $(F_0, \dots, F_n)$. The F_i have no common zeros in $\Gamma_{g-1}\backslash\mathcal{H}^*_{g-1}$, so Z is relatively compact in $\Gamma_g\backslash\mathcal{H}_g$.

We know we can construct a modular form F_{n+1} not vanishing on all of Z. By compactness and after finitely many steps we have $F_0, \dots, F_{n+m}$ without common zeros in $\Gamma_g\backslash\mathcal{H}_g$. We may assume that these have the same weight (take powers if necessary).

These forms $F_0, \ldots, F_{n+m}$ define an analytic map

$$\Gamma_g \backslash \mathcal{H}_g^* \to \mathbf{P}^N$$

It is well-defined, but maybe not injective and maybe not an embedding.

Suppose we have two points (in $\Gamma_g \backslash \mathcal{H}_g$) with the same image. Then we can find a modular form separating these points. Add this form to the F_i. After repeating this finitely many steps we have forms $F_0, \ldots, F_N$ that define an injective map

$$\Gamma_g \backslash \mathcal{H}_g^* \to \mathbf{P}^N$$

The image Y is compact and is by a theorem of Grauert an analytic subset of $\mathbf{P}^N$, hence by a theorem of Chow it is an algebraic variety.

Any polynomial vanishing on Y is a modular form vanishing on $\Gamma_g\backslash\mathcal{H}_g$, hence is zero. So the ideal of Y is given by the ideal of relations between the $F_0,\ldots,F_N$.

Replace now the ring generated by $F_0,\ldots,F_N$ by its integral closure. This is a finitely generated ring (say by $G_0,\ldots,G_M$) and defines an algebraic variety Y' together with an analytic map

$$\Gamma_g\backslash\mathcal{H}_g^* \to Y'$$

$$x \mapsto (G_0(x):\ldots:G_M(x))$$

that is bijective. Both spaces are normal analytic spaces, therefore this must be an isomorphism of analytic spaces. Q.e.d.

Corollary 1. *The ring* $R_g = \oplus_k M_k(\Gamma_g)$ *is a finitely generated ring.*

We can write

$$\mathcal{A}_g^* = \mathrm{Proj}(\oplus_k M_k(\Gamma_g)) \, .$$

The line bundle $\mathcal{L} = \det(\mathbf{E})$ is an ample line bundle on $\mathcal{A}_g^*$. The finite dimensionality of $M_k(\Gamma_g)$ follows immediately and also the bound $\dim M_k(\Gamma_g) = O(k^{g(g+1)/2})$.

If we consider the $f_0, \ldots, f_N \in M_k(\Gamma_g)$ on the rank 1- partial compactification $\Gamma_g \backslash \mathcal{H}_g \sqcup D$ with D the divisor at infinity with

$$D \sim \mathcal{X}_{g-1}/\langle \pm 1 \rangle$$

then it is clear that the (pull backs of the) f_i are constant on the fibres of

$$\mathcal{X}_{g-1} \to \Gamma_{g-1} \backslash \mathcal{H}_{g-1}$$

Indeed, we look for $f = \sum_m \varphi_m(\tau_1, z) q_2^m$ at $\varphi_0(\tau, z)$ and for φ_m the factor is

$$e^{-2\pi i(m\xi'\tau_1\xi + 2m\xi' z)}$$

So under the map $\Gamma_g \backslash \mathcal{H}_g \sqcup D \to \mathbf{P}^N$ these fibres are blown down (contracted). This shows that for $g \geq 2$ the Satake compactification is very singular.

It is convenient to have a smooth compactification of $\Gamma_g \backslash \mathcal{H}_g$. Igusa constructed a partial desingularization by 'blowing up the ideal of cusp forms'. For $g = 2$ it leads to a non-singular model of $\mathcal{A}_g^*[n]$ for $n \geq 3$.

The rank 1 partial compactification often is a good substitute for a smooth model. Mumford's theory of toroidal compactifications allows one to construct smooth compactifications (say of $\mathcal{A}_g[n]$ for $n \geq 3$) but in general these depend on choices.

Constructing Siegel Modular Forms

Theta Characteristics. Consider Riemann's theta function on $\mathcal{H}_g \times \mathbf{C}^g$

$$\theta(\tau, z) = \sum_{m \in \mathbf{Z}^g} e^{\pi i m' \tau m + 2\pi i m' z}.$$

Note that such a series converges very rapidly. For τ fixed this is a section of $\mathcal{O}_{X_\tau}(\Theta)$ on X_τ. But for $z = 0$ the function $\theta(\tau, 0)$ is a "modular form of weight $1/2$". The transformation behavior was studied classically.

Consider vectors $\epsilon = \begin{bmatrix} \eta \\ \delta \end{bmatrix}$ with η, δ column vectors in $\{0,1\}^g$. Define a parity

$$\eta'\delta \bmod 2\,.$$

We have 2^{2g} such characteristics and $2^{g-1}(2^g-1)$ are odd. We define

$$\theta[\epsilon](\tau) = \sum_{m\in\eta/2+\mathbf{Z}^g} e^{2\pi i m'\tau m + m'\delta/2}$$

Identically zero for ϵ odd. These define modular forms of weight $1/2$ on congruence subgroups $\Gamma_g[4,8]$. That is the subgroup of $\Gamma_g[4]$ consisting of $\gamma=(a,b;c,d)$ with

$$\mathrm{diag}(b) \equiv 0 \equiv \mathrm{diag}(c) \pmod 8)$$

We can use symmetric expressions of these to construct modular forms in level 1:

$g = 1$

$$\left(\theta[{0 \atop 0}]\theta[{0 \atop 1}]\theta[{1 \atop 0}]\right)^8 = 2^8\Delta \in S_{12}(\Gamma_1)$$

$g = 2$

$$\prod_{\epsilon \text{ even}} \theta[\epsilon]^2 = 2^{14}\chi_{10} \in S_{10}(\Gamma_2)$$

Let us pause to remark that Γ_2 has a character:

$$\chi : \mathrm{Sp}(4, \mathbf{Z}) \to \mathrm{Sp}(4, \mathbf{Z}/2\mathbf{Z}) \cong \mathfrak{S}_6 \xrightarrow{\mathrm{sgn}} \{\pm 1\}$$

Reiner showed that

$$[\Gamma_g : \Gamma'_g] = \begin{cases} 12 & g = 1 \\ 2 & g = 2 \\ 1 & g > 2 \end{cases}$$

with Γ'_g the commutator subgroup.

Then one checks

$$\chi_5 = \frac{1}{2^7} \prod_{\epsilon \text{ even}} \theta[\epsilon] \in M_5(\Gamma_2, \chi)$$

with $\chi_5^2 = \chi_{10}$. We put

$$\chi_5 \sum_E \pm \left(\theta[\epsilon_1]\theta[\epsilon_2]\theta[\epsilon_3]\right)^{20} \in S_{35}(\Gamma_2)$$

with E the set of triples $(\epsilon_1, \epsilon_2, \epsilon_3)$ of even characteristics such that the sum is odd.

$g = 3$

$$\prod_{\epsilon \text{ even}} \theta[\epsilon] = \chi_{18} \in S_{18}(\Gamma_3)$$

For $g = 2$ we have 16 theta characteristics; 10 even ones, 6 odd ones. The fourth powers of the ten theta constants $\theta[\epsilon](\tau, 0)^4$ are modular forms of weight 2 and level 2:

$$\theta[\epsilon](\tau, 0)^4 \in M_2(\Gamma_2[2])\,.$$

Call these $x_1, \ldots, x_{10}$. These satisfy linear relations like $x_1 - x_4 - x_6 - x_7 = 0$ and span a space of dimension 5. The group $\mathfrak{S}_6 \cong \mathrm{Sp}(4, \mathbf{Z}/2\mathbf{Z})$ acts on it. The x_i define a map

$$\mathcal{A}_2[2] \to \mathbf{P}^4 \subset \mathbf{P}^9$$

and it extends to an embedding $\mathcal{A}_2^*[2] \hookrightarrow \mathbf{P}^4$. We can determine the equation

$$\left(\sum_{i=1}^{10} x_i^2\right)^2 - 4\sum_{i=1}^{10} x_i^4 = 0$$

and the ring $R^{\mathrm{ev}}[2] = \oplus_k M_{2k}(\Gamma_2[2])$ is generated by the x_i. This variety contains 15 (singular) lines intersecting each other in 15 points that together form the 'boundary'.

There is another model in $\mathbf{P}^4 \subset \mathbf{P}^5$ given in coordinates $y_1, \ldots, y_6$ by

$$\sigma_1 = 0, \quad \sigma_2^2 - 4\,\sigma_4 = 0\,.$$

with $\mathfrak{S}_6$ acting by permutation on the y_i. Then

$$\prod_{i<j}(y_i - y_j)$$

defines a $\mathfrak{S}_6$ anti-invariant modular form χ_{30} of weight 30. Multiplied by χ_5 gives a modular form $\chi_{35} \in M_{35}(\Gamma_2)$.

The odd theta characteristics are zero for $z = 0$: $\theta[\epsilon](\tau, 0) = 0$.

Consider for odd ϵ the gradient of $\theta[\epsilon](\tau, z)$

$$\partial\,\theta[\epsilon] := (\frac{\partial}{\partial z_1}\theta[\epsilon], \frac{\partial}{\partial z_2}\theta[\epsilon])_{|(\tau,0)}$$

This is a vector-valued modular form of "weight $(1, 1/2)$" , that is,

$$\rho(\gamma) = (c\tau + d) \cdot \sqrt{\det(c\tau + d)}$$

on a congruence subgroup; the group is $\Gamma_2[4, 8]$.

More generally: the transposed gradients

$$(\partial\theta[\epsilon]/\partial z_1, \ldots, \partial\theta[\epsilon]/\partial z_g)$$

defines a holomorphic section of $\mathbf{E} \otimes \det(\mathbf{E})^{1/2}$ on the congruence subgroup

$\Gamma_g[4,8]$ of $\gamma = (a,b;c,d)$ satisfying

$$\gamma \equiv 1_{2g} \mod 4,\ \mathrm{diag}(b) \equiv 0 \equiv \mathrm{diag}(c) \mod 8$$

Then one finds

$$\chi_{6,3} = \prod_{\epsilon\,\mathrm{odd}} \partial\,\theta[\epsilon] \in S_{6,3}(\Gamma_2, \chi)$$

with χ a character and

$$\chi_{6,8} = \chi_5 \cdot \chi_{6,3} \in S_{6,8}(\Gamma_2)$$

Here $\rho(\gamma) = \mathrm{Sym}^6(c\tau + d)\,\det(c\tau + d)^8$.

Let me give the beginning of the Fourier series:

$$\chi_{6,3} = \begin{pmatrix} 0 \\ 0 \\ (R - R^{-1}) \\ (2R + 2R^{-1}) \\ (R - R^{-1}) \\ 0 \\ 0 \end{pmatrix} Q_1 Q_2 + \cdots$$

where $Q_1 = e^{\pi i \tau_{11}}$, $Q_2 = e^{\pi i \tau_{22}}$ and $R = e^{\pi i \tau_{12}}$.

Theta Functions of Unimodular Lattices

Let L be an even unimodular lattice of rank $2k$ with $(\,,\,)$ the bilinear form. Define for $l = (l_1, \ldots, l_g) \in L^g$ the matrix

$$G_l := (l_i, l_j)_{1 \le i,j \le g}$$

and

$$\vartheta_L^{(g)}(\tau) := \sum_{l \in L^g} e^{\pi i \mathrm{Tr}(G_l \tau)}, \qquad \tau \in \mathcal{H}_g$$

We have $\vartheta_L^{(g)} = \sum a(n) q^n$ with

$$a(n) = \#\{l \in L^g : G_l = 2\,n\}$$

Fact: $\vartheta_L^{(g)} \in M_k(\Gamma_g)$.

Proof: Poisson summation formula. In its simplest form this formula says that, if $f : \mathbf{R}^n \to \mathbf{C}$ and $\Lambda \subset \mathbf{R}^n$ is a lattice, we have (provided all expressions converge)

$$\sum_{l \in \Lambda} f(\lambda) = \frac{1}{\mathrm{vol}(\mathbf{R}^n/\Lambda)} \sum_{\mu \in \Lambda^\vee} \hat{f}(\mu)$$

with $\hat{f}(y) = \int_{\mathbf{R}^n} f(x) e^{-2\pi i x \cdot y} dx$ with $\Lambda^\vee$ the dual lattice.

Remark 1. *We have* $2k \equiv 0 \pmod 8$.

We note

$$\Phi(\vartheta_L^{(g)}) = \vartheta_L^{(g-1)}\,.$$

Example:

$$E_8 = \{v \in \mathbf{Z}^8 + (\mathbf{Z} + \frac{1}{2})^8 : \sum_i v_i = \text{even}\}$$

Then for $g = 1, 2, 3$

$\vartheta_{E_8}^{(g)}$ = weight 4 Eisenstein series of degree g

For $2k = 16$ there are two isomorphism classes of even unimodular lattices $L_1 = E_8 \oplus E_8$ and $L_2 = E_{16}$. We consider

$$\vartheta_{L_1}^{(g)} - \vartheta_{L_2}^{(g)} \in M_k(\Gamma_g)$$

Then this is zero for $g = 1, 2, 3$. But for $g = 4$

$$0 \neq \vartheta_{L_1}^{(g)} - \vartheta_{L_2}^{(g)} \in S_8(\Gamma_4)$$

This form is called the Schottky form ϕ_8.

Consider

$$t : \mathcal{M}_4 \to \mathcal{A}_4 = \Gamma_4 \backslash \mathcal{H}_4, \quad C \mapsto \mathrm{Jac}(C)$$

the Torelli map with $\mathcal{M}_4$ the moduli space of smooth curves of genus 4. Note $\dim \mathcal{M}_4 = 9$, $\dim \mathcal{A}_4 = 10$.

Then ϕ_8 vanishes on the image of t.

In rank 24 there are 24 isomorphism classes of even unimodular lattices: the Niemeier lattices. Hence we get 24 theta functions

$$\vartheta_{L_i}^{(g)}, \qquad i = 1, \ldots, 24.$$

One of these without vectors of length 2 is the Leech lattice.

Others include $24A_1$, $12A_2$,. . ., D_{24}.

The Fourier coefficient $a(n)$ of $\vartheta_{L_i}^{(g)}$ is

$$a(n) = \#\mathrm{Aut}(\mathbf{Z}^g, 2n) \cdot m(L_i, n)$$

with $m(L_i, n)$ = number of sublattices isomorphic to $\mathbf{Z}^g$ with bilinear form $2n$.

One can show that these 24 theta series are linearly dependent for $g < 12$, but independent for $g \geq 12$

Example 1. *Let L_1=Leech, $L_2 = 24A_1$ and $L_3 = 12A_2$. We then have*

$$\Delta = \frac{1}{48}(\vartheta_{L_2}^{(1)} - \vartheta_{L_1}^{(1)})\,.$$

And for $g = 2$:

$$\chi_{12} = \frac{1}{288}(\vartheta_{L_1}^{(2)} - 3\vartheta_{L_2}^{(2)} + 2\vartheta_{L_3}^{(2)})$$

Let $\mathcal{L}_n$ be the set isometry classes of even unimodular lattices of rank n. This is a finite set. Let $\mathbf{C}[\mathcal{L}_n]$ be the vectorspace generated by the elements of $\mathcal{L}_n$.

We get a map

$$\theta_g : \mathbf{C}[\mathcal{L}_n] \to M_{n/2}(\Gamma_g), \quad L \mapsto \vartheta_L^{(g)} .$$

We have

$$\ker \theta_g \subseteq \ker \theta_{g-1}$$

giving a filtration on $\mathbf{C}[\mathcal{L}_n]$. For $g \geq n$ the map θ_g is injective.

Theorem 1. *(Chenevier-Lannes) The map*

$$\theta_g : \mathbf{C}[\mathcal{L}_{24}] \to M_{12}(\Gamma_g)$$

is surjective and induces and isomorphism $\ker\theta_{g-1}/\ker\theta_g \xrightarrow{\sim} S_{12}(\Gamma_g)$ *for* $g \leq 12$. *The dimensions are given in the following table.*

g	1	2	3	4	5	6
$\dim S_{12}(\Gamma_g)$	1	1	1	2	2	3
$\dim M_{12}(\Gamma_g)$	2	3	4	6	8	11
g	7	8	9	10	11	12
$\dim S_{12}(\Gamma_g)$	3	4	2	2	1	1
$\dim M_{12}(\Gamma_g)$	14	18	20	22	23	24

More Literature

J.-I. Igusa: Theta Functions. Springer Verlag

G. Chenevier, J. Lannes: Automorphic Forms and Even Unimodular Lattices. Springer 2019.