Siegel Modular Forms
Lecture #7

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Harmonic Theta Series

L even unimodular of rank n with quadratic form $Q = (\ ,\ )$. Choose elements

$$e_1, \dots, e_g$$

that span a totally isotropic subspace of

$$V = L \otimes_{\mathbf{R}} \mathbf{C}$$

with the property that

$$Q(e_i, \overline{e}_j) > 0, \quad 1 \leq i, j \leq g$$

For a vector $l = (l_1, \dots, l_g) \in L^g$ we consider $g \times g$ matrix

$$Q(l_i, e_j)$$

Fact:

$$\vartheta_{L,e,k} = \sum_{l \in L^g} \det(Q(l_i, e_j))^k \, e^{\pi i \mathrm{Tr}(G_l \tau)}$$

is a Siegel modular form of degree g and weight $n/2 + k$.

Theorem 1. *(Böcherer) Every cusp form of degree* $g \geq 1$ *and weight* $n/2 + k$ *with* $n > 2g$, $k > 0$ *and* $n \equiv 0 (\bmod 8)$ *is a linear combination of theta series of the form* $\vartheta_{L,e,k}$ *with* L *an even unimodular lattice of rank* n.

We have a variant for vector-valued modular forms.

Example. Take the lattice

$$\Lambda = \{x \in \mathbf{Z}^{16} \cup (1/2 + \mathbf{Z})^{16} : \sum x_i \in 2\mathbf{Z}\}$$

Put $a = (2, i, i, i, i, 0, \ldots, 0) \in \mathbf{C}^{16}$. Define a vector-valued function $(F_0, \ldots, F_6)$ with F_j given by

$$\sum_{x,y\in\Lambda} (x, a)^{6-j}(y, a)^j \, e^{\pi i((x,x)\tau_{11}+2(x,y)\tau_{12}+(y,y)\tau_{22})}$$

Ibukiyama proved that this is a non-zero cusp form of weight $(6, 8)$, that is for $\rho = \mathrm{Sym}^6(St) \otimes \det(St)^8$.

We know $\dim S_{6,8}(\Gamma_2) = 1$, hence this form is a multiple of $\chi_{6,8}$.

Restricting Modular Forms

Witt restricted modular forms of degree 2 to the 'diagonal' $\mathcal{H}_1 \times \mathcal{H}_1 \hookrightarrow \mathcal{H}_2$. We saw that the first Taylor term of $f \in M_k(\Gamma_2)$ in

$$f = f_0(\tau_1, \tau_2) z^n + \dots$$

defines an element of $M_{k+n}(\Gamma_1) \otimes M_{k+n}(\Gamma_1)$. We generalize this.

$L = (\mathbf{Z}^{2g}, \langle \, , \, \rangle)$ our standard symplectic lattice. If we write

$$L = L_1 \oplus L_2$$

an orthogonal product of two standard symplectic lattices of rank $2g_1$ and $2g_2$ with

$2g = 2g_1 + 2g_2$ then this gives an embedding

$$\mathrm{Sp}(2g_1, \mathbf{Z}) \times \mathrm{Sp}(2g_2, \mathbf{Z}) \to \mathrm{Sp}(2g, \mathbf{Z})$$

and noting that $\mathcal{H}_j = \mathrm{Sp}(2j, \mathbf{R})/U(j)$ it defines an embedding $\mathcal{H}_{g_1} \times \mathcal{H}_{g_2} \to \mathcal{H}_g$. This induces a map

$$\Gamma_{g_1} \backslash \mathcal{H}_{g_1} \times \Gamma_{g_2} \backslash \mathcal{H}_{g_2} \to \Gamma_g \backslash \mathcal{H}_g \,,$$

that is generically of finite degree.

Take $f \in M_k(\Gamma_g)$ and pull it back to $\mathcal{H}_{g_1} \times \mathcal{H}_{g_2}$. We get a form

$$f_1 \otimes f_2 \in M_k(\Gamma_{g_1}) \otimes M_k(\Gamma_{g_2})$$

It may happen that f vanishes on the image of $\mathcal{H}_{g_1} \times \mathcal{H}_{g_2}$. We write

$$\tau = \begin{pmatrix} \tau_1 & z \\ z' & \tau_2 \end{pmatrix}$$

and we consider the Taylor series of f in the coordinates of $z \in \mathrm{Mat}(g_1 \times g_2, \mathbf{C})$. The action of $\Gamma_{g_1} \times \Gamma_{g_2}$ is via

$$\begin{pmatrix} \tau_1 & z \\ z' & \tau_2 \end{pmatrix} \mapsto \begin{pmatrix} \gamma_1(\tau_1) & \tilde{z} \\ \tilde{z}' & \gamma_2(\tau_2) \end{pmatrix}$$

with

$$\tilde{z} = (c_1\tau_1 + d_1)'^{-1} z (c_2\tau_2 + d_2)^{-1}$$

Or in more invariant terms writing $X_{\tau_i} = X_i$ for the abelian varieties we have for the deformation space of X_τ

$$\mathrm{Def}(X_\tau) \cong \mathrm{Sym}^2(T_{X_\tau}) = \mathrm{Sym}^2(T_{X_1} \oplus T_{X_2})$$

and this can be written

$$= \mathrm{Sym}^2(T_{X_1}) \oplus (T_{X_1} \otimes T_{X_2}) \oplus \mathrm{Sym}^2(T_{X_2})$$

hence the normal space is

$$T_{X_{\tau_1}} \otimes T_{X_{\tau_2}}$$

with coordinates $z_{ij} = \zeta_i \otimes \eta_j$, hence the conormal bundle is

$$p_1^*(\mathbf{E}_{g_1}) \otimes p_2^*(\mathbf{E}_{g_2})$$

with $p_i : \Gamma_{g_1} \backslash \mathcal{H}_{g_1} \times \Gamma_{g_2} \backslash \mathcal{H}_{g_2} \to \Gamma_{g_i} \backslash \mathcal{H}_{g_i}$ the projection.

So we write our $f \in M_k(\Gamma_g)$ as

$$f = t_0(f) + t_1(f) + \cdots$$

where $t_d(f)$ is the sum of the terms of total degree d in the coordinates z_{ij} of z.

If $t_0(f) = 0$ then $t_1(f)$ is a section of

$$N^\vee \otimes \det(\mathbf{E})^k$$

with

$$N^\vee = p_1^*(\mathbf{E}_{g_1}) \otimes p_2^*(\mathbf{E}_{g_2})$$

the conormal bundle. So if $t_0(f) = 0$ then $t_1(f)$ is a section of

$$\det(\mathbf{E}_{g_1})^k \otimes \det(\mathbf{E}_{g_2})^k \otimes N^\vee$$

hence a vector-valued modular form

$$t_1(f) = f_1 \otimes f_2$$

with f_i a section of

$$\det(\mathbf{E}_{g_1})^k \otimes \mathbf{E}_{g_1}$$

If $t_0(f) = \cdots = t_l(f) = 0$ then $t_{l+1}(f)$ gives a section of

$$\det(\mathbf{E}_{g_1})^k \otimes \det(\mathbf{E}_{g_2})^k \otimes \mathrm{Sym}^{l+1}(N^\vee)$$

These are vector-valued modular forms.

Examples

Consider the 'diagonal' embedding

$$\mathcal{H}_1 \times \mathcal{H}_{g-1} \hookrightarrow \mathcal{H}_g \, .$$

If $f \in M_k(\Gamma_g)$ vanishes along $\mathcal{H}_1 \times \mathcal{H}_{g-1}$ with order $r-1$ then

$$t_r(f) \in M_{k+r}(\Gamma_1) \otimes M_\rho(\Gamma_{g-1})$$

with

$$\rho = \mathrm{Sym}^r(St) \otimes \det(St)^k$$

because $\mathrm{Sym}^r(N^\vee) \otimes \det(\mathbf{E})^k$ is equal to

$$\mathbf{E}_1^{r+k} \otimes \mathrm{Sym}^r(\mathbf{E}_{g-1}) \otimes \det(\mathbf{E}_{g-1})^k \, .$$

In other words, $t_r(f)$ is of degree r in the coordinates and Γ_1 and Γ_{g-1} act on the row vector $z \in \mathbf{C}^g$ by

$$z \mapsto (c_1\tau_1 + d)^{-1}\, z\, (c_2\tau_2 + d_2)^{-1}$$

Here $\tau_1 \in \mathcal{H}_1$, $\tau_2 \in \mathcal{H}_{g-1}$.

Restricting $\phi_8 \in S_8(\Gamma_4)$

The Schottky form $f = \phi_8$ vanishes on the image of $\mathcal{H}_2 \times \mathcal{H}_2$ and of $\mathcal{H}_1 \times \mathcal{H}_3$ because $\dim S_8(\Gamma_i) = 0$ for $g < 4$. We give a table for $\dim S_k(\Gamma_g)$.

$g\backslash k$	< 8	8	9	10	11	12
1	0	0	0	0	0	1
2	0	0	0	1	0	1
3	0	0	0	0	0	1
4	0	1	0	1	0	2
5	0	0	0	0	0	2
6	0	0	0	1	0	3
7	0	0	0	0	0	3
8	0	0	0	≥ 1	0	≥ 4

Later we shall treat results on the dimension of spaces of Siegel modular cusp forms (results of Taïbi and Chenevier).

Consider restriction of ϕ_8 to $\mathcal{H}_1 \times \mathcal{H}_3$. Then $t_0(f) = 0$. Furthermore

$$t_i(f) = 0 \quad \text{for } i \text{ odd}$$

Also $t_2(f) = 0$ because $t_2(f) = f' \otimes f''$ with $f' \in S_{10}(\Gamma_1) = (0)$. Then

$$t_4(f) = f' \otimes f'' \in S_{12}(\Gamma_1) \otimes S_{4,0,8}(\Gamma_3)$$

with $M_{4,0,8}(\Gamma_3)$ the space of sections of

$$\mathrm{Sym}^4(\mathbf{E}_3) \otimes \det(\mathbf{E}_3)^8$$

In other words $\rho = \mathrm{Sym}^4(St) \otimes \det(St)^8$.

One checks:

$$t_4(\phi_8) = \Delta \otimes \chi_{4,0,8}$$

is a non-zero cusp form. In some sense both $\Delta \in S_{12}(\Gamma_1)$ and $\chi_{4,0,8} \in S_{4,0,8}(\Gamma_3)$ are the 'first' cusp form for Γ_1 and the 'first' vector-valued cusp form for Γ_3 that one finds.

Restricting ϕ_8 to $\mathcal{H}_2 \times \mathcal{H}_2$.

We write $M_{j,k}(\Gamma_2) = M_\rho(\Gamma_2)$ for

$$g = 2 \quad \text{and } \rho = \mathrm{Sym}^j(St) \otimes \det(St)^k$$

Now restrict ϕ_8 to $\mathcal{H}_2 \times \mathcal{H}_2$;

We have $t_0(f) = 0$, $t_i(f) = 0$ for i odd.

$$t_2(f) = f' \otimes f''$$

with

$$f' = f'' \in S_{2,8}(\Gamma_2) = (0)$$

For t_4 we have to decompose for two vector spaces V', V'' of dimension 2

$$\mathrm{Sym}^4(V' \otimes V'')$$

we get three terms

$$\mathrm{Sym}^4(V') \otimes \mathrm{Sym}^4(V''),$$

and

$$\mathrm{Sym}^2(V') \otimes \wedge^2(V') \otimes \mathrm{Sym}^2(V'') \otimes \wedge^2(V'')$$

and

$$\wedge^2(V') \otimes \wedge^2(V') \otimes \wedge^2 (V') \otimes \wedge^2(V')$$

But $t_4(f)$ is symmetric under interchanging the factors $\mathcal{H}_2 \times \mathcal{H}_2$. So $t_4(f)$ lands in

$$\mathrm{Sym}^2(M_{4,k}(\Gamma_2)) \oplus \mathrm{Sym}^2(M_{2,k+1}(\Gamma_2)) \\ \oplus \mathrm{Sym}^2(M_{0,k+2}(\Gamma_2))$$

for $k = 8$. Indeed, $t_4(f)$ gives only the term

$$\mathrm{Sym}^2(\chi_{10}) \in \mathrm{Sym}^2(S_{10}(\Gamma_2))$$

Similarly, projection of t_6 gives

$$\chi_{6,8} \otimes \chi_{6,8} \in \mathrm{Sym}^2(S_{6,8}(\Gamma_2))$$

(It does not interfere with 4th order terms..)

Again χ_{10} and $\chi_{6,8}$ are the first scalar-valued and the first vector-valued cusp forms for $g = 2$.

The Rankin-Cohen Bracket

Let f be a holomorphic function on $\mathcal{H}_g$. We define a $g \times g$ matrix by

$$\partial f(\tau) = \left(\frac{1+\delta_{ij}}{2} \frac{\partial f}{\partial \tau_{ij}} \right)$$

We have the formulas

$$\partial(f \circ \gamma) = (c\tau + d)'^{-1} \{(\partial f) \circ \gamma\} (c\tau + d)^{-1}$$

and

$$\partial(\det(c\tau + d)) = \det(c\tau + d) \cdot (c\tau + d)^{-1}\, c$$

One thus shows for $f \in M_k(\Gamma_g)$:

$$(\partial f)(\gamma(\tau)) = \det(c\tau + d)^k (c\tau + d) \\ \left\{ (c\tau + d)^{-1} k\, c\, f(\tau) + \partial f(\tau) \right\} (c\tau + d)'$$

We put for $F \in M_k(\Gamma_g)$ and $G \in M_l(\Gamma_g)$

$$[F,G](\tau) = \frac{1}{2\pi i}(k\,F\,\partial G - l\,G\,\partial F)(\tau)$$

Let now $\rho = \mathrm{Sym}^2(St)$.

Proposition 1. *For $F \in M_k(\Gamma_g)$ and $G \in M_l(\Gamma_g)$ we have*

$$[F,G] \in M_{\rho\otimes\det^{k+l}}(\Gamma_g)$$

Moreover the bracket satisfies

$$\begin{gathered}[F,G] = -[G,F] \\ F[G,H] + G[H,F] + H[F,G] = 0 \\ [FG,G] = G[F,G]\end{gathered}$$

Proof. Consider

$$\nabla_k f = \frac{k}{2\pi i}(\tau - \overline{\tau})^{-1} f + \frac{1}{2\pi i}\partial f$$

Then

$$[F, G] = kF\,\nabla_l G - l\,G\nabla_k F$$

We know

$$\gamma(\tau) - \overline{\gamma(\tau)} = (c\tau + d)'^{-1}(\tau - \overline{\tau})(c\overline{\tau} + d)^{-1}$$

One checks that $\nabla_k f$ for $f \in M_k(\Gamma_g)$ is a C^∞-function that behaves like a modular form of weight $\rho \otimes \det^{k+l}$.

This implies that $[F, G]$ satisfies the required transformation formula. But it is holomorphic. Q.e.d.

A Variation for $g = 2$

If V is a 2-dimensional vector space then

$$\wedge^2 \mathrm{Sym}^2(V) \cong \mathrm{Sym}^2(V) \otimes \det(V)$$

as $\mathrm{GL}(V)$-representations, hence

$$\wedge^2 \mathrm{Sym}^2(\mathbf{E}_2) \cong \mathrm{Sym}^2(\mathbf{E}_2) \otimes \det(\mathbf{E}_2)$$

Proposition 2. *Let* $f_i \in M_{k_i}(\Gamma_2)$ *for* $i = 1, \ldots 3$*. Then with* $\rho = \mathrm{Sym}^2(St)$ *the expression* $[f_1, f_2, f_3]$ *equal to*

$$k_1 f_1 \partial f_2 \wedge \partial f_3 + k_2 f_2 \partial f_3 \wedge \partial f_1 + k_1 f_3 \partial f_1 \wedge \partial f_2$$

is a form in $M_{\rho \otimes \det^{k_1+k_2+k_3+1}}(\Gamma_2)$*.*

Proposition 3. *Let* $f_i \in M_{k_i}(\Gamma_2)$ *for* $i = 1, \ldots 4$. *Then*

$$[f_1, f_2, f_3, f_4] = \sum_{i=1}^{4} (-1)^i k_i \, f_i(\wedge_{j \neq i} \partial \, f_j)$$

is in $M_{k_1+k_2+k_3+k_4+3}(\Gamma_2)$.

Example: take generators $\psi_4, \psi_6, \chi_{10}, \chi_{12}$ of $R^{\text{ev}} = \oplus_k M_{2k}(\Gamma_2)$. This gives a form χ_{35} of weight 35. We know

$$R = \oplus_k M_k(\Gamma_2) = R^{\text{ev}}[\chi_{35}]/(\chi_{35}^2 - P)$$

with P a polynomial in $\psi_4, \psi_6, \chi_{10}, \chi_{12}$.

Forms of Weight $(2, k)$

Let $\mathcal{M} = \oplus_k M_{2,k}(\Gamma_2)$. This is a module over R_2. We have a variant $\mathcal{M}^{\mathrm{ev}} = \oplus_{k \text{ even}} M_{2,k}(\Gamma_2)$, a module over R_2^{ev}).

We can construct elements here

$$\begin{array}{llllll} \alpha_{10} & = & [\psi_4, \psi_6] & \beta_{14} & = & [\psi_4, \chi_{10}] \\ \gamma_{16} & = & [\psi_4, \chi_{12}] & \delta_{16} & = & [\psi_6, \chi_{10}] \\ \epsilon_{18} & = & [\psi_6, \chi_{12}] & \zeta_{22} & = & [\chi_{10}, \chi_{12}] \end{array}$$

The relation for Cohen-Rankin brackets

$$a[b, c] + b[c, a] + c[a, b] = 0$$

applied to triples of $\psi_4, \psi_6, \chi_{10}, \chi_{12}$ gives rise to relations

$$\begin{aligned} r_{12} : \psi_4\delta_{16} - \psi_6\beta_{14} + \chi_{10}\alpha_{10} = 0, \\ r_{10} : \psi_4\epsilon_{18} - \psi_6\gamma_{16} + \chi_{12}\alpha_{10} = 0, \\ r_6 : \psi_4\zeta_{22} - \chi_{10}\gamma_{16} + \chi_{12}\beta_{14} = 0, \\ r_4 : \psi_6\zeta_{22} - \chi_{10}\epsilon_{18} + \chi_{12}\delta_{16} = 0. \end{aligned}$$

of weights $20, 22, 26$ and 28. We then have the syzygy:

$$s_{32} : \quad \chi_{12}r_{12} - \chi_{10}r_{10} + \psi_6 r_6 - \psi_4 r_4 = 0$$

We claim that this describes the module $\mathcal{M}^{\mathrm{ev}}$. The structure of $\mathcal{M}^{\mathrm{ev}}$ was determined by Satoh in 1986.

Theorem 2. *The module $\mathcal{M}^{\mathrm{ev}}$ is generated over $\mathcal{R}^{\mathrm{ev}}$ by the six forms*

$$\alpha_{10}, \beta_{14}, \gamma_{16}, \delta_{16}, \epsilon_{18}, \zeta_{22}$$

with the relations r_4, r_6, r_{10}, r_{12} *and the syzygy* s_{32}.

Proof. Suppose we have a relation

$$f_4 \partial\psi_4 + f_6 \partial\psi_6 + f_{10} \partial\psi_{10} + f_{12} \partial\psi_{12} = 0$$

with $f_i \in M_{k-j}(\Gamma_2)$. This implies that

$$f_4 \psi_4 + f_6 \psi_6 + f_{10} \chi_{10} + f_{12} \chi_{12} = 0$$

and hence that the form

$$[f_4\psi_4, f_6\psi_6, f_{10}\chi_{10}, f_{12}\chi_{12}]$$

vanishes. Since $[\psi_4, \psi_6, \chi_{10}, \chi_{12}]$, a multiple of χ_{35}, does not vanish it follows that f_4, f_6, f_{10} and f_{12} vanish. Using the relations we find that the subspace of $M_{2,k}(\Gamma_2)$ generated by our six generators consists of the forms

$$a_{4,6}\alpha_{10} + a_{4,10}\beta_{14} + a_{4,12}\gamma_{16}$$
$$+a_{6,10}\delta_{16} + a_{6,12}\epsilon_{18} + a_{10,12}\zeta_{22}$$

with $a_{4,l} \in M_{k-4-l}(\Gamma_2)$, $a_{6,l} \in M_{k-6-l} \cap \mathbf{C}[\psi_6, \chi_{10}, \chi_{12}]$ and $a_{10,12} \in M_{k-22} \cap \mathbf{C}[\chi_{10}, \chi_{12}]$. If we suppose a non-trivial linear combination as above vanishes then we find a relation

$$b_4\partial\psi_4 + b_6\partial\psi_6 + b_{10}\partial\chi_{10} + b_{12}\partial\chi_{12} = 0$$

and the coefficients b_i must vanish. But we have

$$\begin{aligned} 4b_4 = &\quad -a_{4,6}\psi_6 - a_{4,10}\chi_{10} - a_{4,12}\chi_{12} \\ 6b_6 = &\quad a_{4,6}\psi_4 - a_{6,10}\chi_{10} - a_{6,12}\chi_{12} \\ 10b_{10} = &\quad a_{4,10}\psi_4 + a_{6,10}\psi_6 - a_{10,12}\chi_{12} \\ 12b_{12} = &\quad a_{4,12}\psi_4 + a_{6,12}\psi_6 + a_{10,12}\chi_{10} \end{aligned}$$

Using the algebraic independence of ψ_4, ψ_6, ψ_{10} and ψ_{12} one deduces that the coefficients $a_{i,j}$ vanish. Thus we can calculate the dimension of the subspace of $M_{2,k}$ that these forms generate. Indeed, the generating function of the submodule generated by $\alpha_{10}, \ldots, \zeta_{22}$ is

$$\frac{t^{10} + t^{14} + 2\,t^{16} + t^{18} - t^{20} - t^{26} - t^{28} + t^{32}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$$

Now one uses that one knows the dimensions of $M_{2,k}(\Gamma_2)$ or uses the fact that

$$\frac{3}{4 \cdot 6 \cdot 10 \cdot 12} = 3\ \deg(\lambda_1^3)$$

to see that this submodule exhausts $\mathcal{M}^{\rm ev} = \oplus_{k\,{\rm even}} M_{2,k}(\Gamma_2)$.

For the $\mathcal{R}_2^{\mathrm{ev}}$-module $\mathcal{M}^{\mathrm{odd}} = \oplus_k M_{2,2k+1}(\Gamma_2)$ we find generators

$$\alpha_{21} = [\psi_4, \psi_6, \chi_{10}], \quad \beta_{23} = [\psi_4, \psi_6, \chi_{12}],$$
$$\gamma_{27} = [\psi_4, \chi_{10}, \chi_{12}], \quad \delta_{29} = [\psi_6, \chi_{10}, \chi_{12}]$$

that satisfy the relation r_{33} of weight 33

$$12\chi_{12}\alpha_{21} - 10\chi_{10}\beta_{23} + 6\psi_6\gamma_{27} - 4\psi_4\delta_{29} = 0\,.$$

Theorem 3. *The $\mathcal{R}_2^{\mathrm{ev}}$-module $\mathcal{M}^{\mathrm{odd}}$ is generated by the forms $\alpha_{21}, \beta_{23}, \gamma_{27}, \delta_{29}$ with one relation r_{33}.*

Similar results were obtained for

$$\mathcal{M}_d^{\mathrm{ev}} = \oplus_{k\,\mathrm{even}} M_{d,k}(\Gamma_2)$$

and similarly $\mathcal{M}_d^{\mathrm{odd}}$ for $d = 4,\ 6,\ 8,\ 10$ by Ibukiyama, Kiyuna, Takemori and van Dorp.

Later we will see a more systematic method to get such results using invariant theory.

The Ring $\oplus_{j,k} M_{j,k}(\Gamma_2)$

In degree 2 one can provide the $\mathcal{R}_2$ module

$$\mathbf{M} = \oplus_{j,k} M_{j,k}(\Gamma_2)$$

with the structure of a ring. The multiplication is defined by using the projection of $\mathrm{GL}(2)$ representations

$$\mathrm{Sym}^m(St) \otimes \mathrm{Sym}^n(St) \to \mathrm{Sym}^{m+n}(St)$$

with St the standard representation, by interpreting $\mathrm{Sym}^j(St)$ as the space of homogeneous polynomials of degree j in two variables, say x_1, x_2 and performing multiplication of polynomials.

Applying this to symmetric powers of the Hodge bundle which is obtained as a quotient of $\mathfrak{H}_2 \times \mathbf{C}^2$ where $\mathrm{GL}(2)$ acts by the standard representation on $\mathbf{C^2}$ gives the required map

$$M_{j_1,k_1}(\Gamma_2)\times M_{j_2,k_2}(\Gamma_2) \to M_{j_1+j_2,k_1+k_2}(\Gamma_2)\,.$$

This ring is not finitely generated as shown by C. Grundh.

Proposition 4. *The ring* $\mathbf{M}$ *is not finitely generated.*

Proof. Suppose that $f_1, \ldots, f_r$ are generators of $\mathbf{M}$ of weights (j_i, k_i) for $i = 1, \ldots, r$. Recall that the corank of $f \in M_{j,k}(\Gamma_2)$ is ≤ 1 if $j + k > k$, that is, if $j > 0$. Therefore $\Phi(f) \in S_{j+k}(\Gamma_2)$ for $j > 0$. If $F \in M_{j,k}$ with $j > \max\{j_1, \ldots, j_r\}$ then we can write F as a $\mathbf{C}$-linear combination of products

$$\prod_{\alpha} f_{\alpha}$$

of the generators and in each product at least two factors have $j_{\alpha} > 0$. Therefore the image of such a product under Φ lies in the ideal (Δ^2) of $\mathcal{R}_1$. But we know that for large enough weight the Siegel operator is surjective. So if we choose an F of weight (j, k) large enough and such that $\Phi(F)$ is divisible by Δ, but not by Δ^2 we see that it cannot be expressed by our $f_1, \ldots, f_r$. Q.e.d.

More literature

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