Siegel Modular Forms
Lecture $\#8$

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Hecke Operators

Recall that for $g = 1$ we have an algebra of operators acting on spaces of modular forms on $\Gamma_1 = \mathrm{SL}(2, \mathbf{Z})$. These are defined by first associating to $f \in M_k(\Gamma_1)$ a function F on the set $\mathcal{R}$ of lattices in $\mathbf{C}$ via

$$f \mapsto F, \quad F(\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2) = \omega_2^{-k} f(\omega_1/\omega_2)$$

This F is homogeneous of degree $-k$:

$$F(\lambda L) = \lambda^{-k} F(L)$$

Then we define for $n \in \mathbf{Z}_{\geq 1}$ an (sort of averaging) operator on the free abelian group of $\mathcal{R}$ by

$$T(n):\ L \mapsto \sum_{[L:L']=n} L'$$

and a scaling operator for $0 \neq m \in \mathbf{Z}$

$$R_m : L \mapsto L_m = \{m\,\omega : \omega \in L\}$$

These operators commute and for prime p and integers m, n these satisfy

$$T(m)T(n) = T(mn) \quad \text{if } (m,n) = 1$$

$$T(p^{n+1}) = T(p^n)T(p) - p\,T(p^{n-1})R_p$$

We let these operators act on functions on lattices (of given degree $-k$)):

$$(T(n)F)(L) = n^{k-1} \sum_{[L:L']=n} F(L')$$

A lattice $L = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ has as sublattices of index n the lattices $\alpha \cdot L$ with

$$\alpha = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad ad = n, \, 0 \leq b < d.$$

Then

$$T(n)f = n^{k-1} \sum_{ad=n, 0 \leq b < d} d^{-k} f(\frac{a\tau + b}{d})$$

If $f = \sum a(n)q^n$ and $T(m)f = \sum_n b(n)q^n$ then

$$b(n) = \sum_{d|(m,n)} d^{k-1} a(mn/d^2)$$

These operators preserve $M_k(\Gamma_1)$ and $S_k(\Gamma_1)$.

If $f = \sum a(n)q^n$ is an eigenform with

$$T(n)\, f = \lambda(n) f$$

then $\lambda(n)a(1) = a(n)$.

The Petersson product and the fact that

$$\langle T(n)f, g\rangle = \langle f, T(n)g\rangle$$

implies that we can find a basis of eigenforms.

From the fact that the $T(n)$ stabilize $M_k(\Gamma_1)(\mathbf{Z})$=

$$\{f = \sum a(n)q^n \in M_k(\Gamma_1) : a(n) \in \mathbf{Z}\}$$

we see that $a(n)$ is an algebraic integer for a normalized $(a(1) = 1)$ eigenform.

Geometric Interpretation

Consider the modular curve $X_0(N)$; analytically $X_0(N) = \Gamma_0(N)\backslash\mathcal{H}_1^*$. It parametrizes isomorphism classes of triples (E_1, E_2, φ)

$$E_1 \xrightarrow{\varphi} E_2$$

with φ a cyclic isogeny of degree N. This comes with two projections, q_1 and q_2

$$q_i : X_0(N) \to \mathcal{A}_1, \quad (E_1 \xrightarrow{\varphi} E_2) \mapsto E_i$$

and these extend to the compactified curve $X_0(N)$. This makes $X_0(N)$ into a correspondence on $\mathcal{A}_1^* \times \mathcal{A}_1^*$. This acts on differential forms and cohomology via:

$$q_{1*} \circ q_2^*$$

Hecke operators for $g \geq 1$

Let $f : X_1 \to X_2$ be an isogeny of principally polarized abelian varieties of dimension g. We write $X_i = V/\Lambda_i$, V a complex vector space with a non-degenerate hermitean form and Λ_i a lattice.

We can arrange so that the lift $\tilde{f} : V \to V$ of f is the identity and $\Lambda_1 \subset \Lambda_2$. After choosing bases of Λ_1 and Λ_2 the isogeny is given by an integral matrix

$$\gamma \in \mathrm{GSp}(2g, \mathbf{Q})$$

with $\mathrm{GSp}(2g, \mathbf{Q})$ = the group

$$\{\gamma \in \mathrm{GL}(2g, \mathbf{Q}) : \gamma' J \gamma = \eta(\gamma) J\}$$

the group of symplectic similitudes. Here

$$J = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}.$$

Our γ defining the isogeny is integral and satisfies:

$$\gamma' J \gamma = n\, J$$

It follows that $\deg(\gamma) = n^g$ by looking at the Pfaffian.

Changing bases of Λ_1 and Λ_2 results in multiplication of γ on the left and right by elements of Γ_g. So we are looking at a double coset

$$\Gamma_g\, \gamma\, \Gamma_g$$

We call this double coset the type of the isogeny.

Consider the graph of

$$\mathcal{H}_g \to \mathcal{H}_g, \quad \tau \mapsto \gamma(\tau)$$

given in $\mathcal{H}_g \times \mathcal{H}_g$ by

$$\tau_2(c\tau_1 + d) - (a\tau_1 + b) = 0$$

This defines a correspondence T_γ:

$$\left(\gamma^{-1}\Gamma_g\gamma' \cap \Gamma_g\right) \backslash \mathcal{H}_g \longrightarrow (\Gamma_g \backslash \mathcal{H}_g)^2$$

Now $\Gamma_g \backslash \mathcal{H}_g = \mathcal{A}_g$ and we can identify

$$T_\gamma \cap \text{horizontal fibre over } X_1$$

with

$$\{X_2 \overset{\varphi}{\longrightarrow} X_1 : \text{isogeny type of } \varphi \text{ is } \Gamma_g\gamma\Gamma_g\}$$

In other words, if

$$\Gamma_g \gamma \Gamma_g = \sum_{i=1}^{r} \Gamma_g \alpha_i$$

then the X_2 in this fibre

$$T_\gamma \cap \text{horizontal fibre over } X_1$$

are of the form

$$V/\alpha_i(\Lambda_1)$$

for $i = 1, \ldots, r$. We can normalize.

Lemma 1. *For $\gamma \in \mathrm{GSp}(2g, \mathbf{Q})$ with γ integral and $\det(\gamma) > 0$ the double coset $\Gamma_g \gamma \Gamma_g$ has a unique integral representative of the form*

$$\gamma_0 = \mathrm{diag}(\alpha_1, \ldots, \alpha_g, \delta_1, \ldots, \delta_g)$$

with

$$\alpha_i > 0, \quad \alpha_i | \alpha_{i+1} \quad (j = 1, \ldots, g-1),$$

and $\alpha_g | \delta_g$, *and finally*

$$\alpha_j \delta_j = \eta(\gamma) \text{ for all } j.$$

Proof. By induction. We show that we can bring γ in the form

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & & & \\ & a_1 & & b_1 \\ & & \delta & \\ & c_1 & & d_1 \end{pmatrix}$$

with $\alpha, \delta \in \mathbf{Z}$ and $\alpha\delta = n$. Namely, we choose a representative γ such that

$$\min\{|\gamma_{ij}| : \gamma_{ij} \neq 0\}$$

is minimal for the double coset. Assume this is γ_{11}. Then multiply on the left by

$$\begin{pmatrix} u & 0 \\ 0 & {u'}^{-1} \end{pmatrix}$$

to make the other entries of the first column and row vanish. Then use $(1, S; 0, 1)$ with S symmetric integral to reduce the first row of b modulo γ_{11}, hence by the minimality of γ_{11} it is zero. Similarly, use elements $(1, 0; S, 0)$ to make the first column of c zero. Now ab' and $a'c$ are symmetric, hence first column of b and first row of c vanish. Then the matrix has the shape that we wanted where

$$\gamma_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

fulfills the requirements of the Lemma for $g - 1$.

Using $\mathrm{GL}(g, \mathbf{Z})$ we can change a to uav with $u, v \in \mathrm{GL}(g, \mathbf{Z})$ Then $\alpha = \gamma_{11}$ must be the smallest elementary divisor of a. But then α must divide the elementary divisors of a_1. Q.e.d.

Corollary 1. *There is only one double coset $\Gamma_g \gamma \Gamma_g$ with $\gamma' J \gamma = p\, J$. A representative is:*

$$\begin{pmatrix} 1_g & 0 \\ 0 & p\, 1_g \end{pmatrix}$$

There are $g+1$ double cosets $\Gamma_g \gamma \Gamma_g$ with

$$\gamma' J \gamma = p^2\, J$$

Representatives are for $i = 0, \ldots, g$

$$\begin{pmatrix} 1_{g-i} & & & \\ & p1_i & & \\ & & p^2 1_{g-i} & \\ & & & p\,1_i \end{pmatrix}$$

When studying isogenies we can restrict to p-power isogenies with p a prime.

In accordance with this, we can decompose the diagonal representative as a product of diagonal matrices each involving only powers of one prime p.

There is a way to compose correspondences: if Z and W are correspondences in $\mathcal{A}_g \times \mathcal{A}_g$ then

$$Z \circ W = p_{13*}(p_{12}^* Z \cdot p_{23}^* W)$$

with $p_{ij} : \mathcal{A}_g^3 \to \mathcal{A}_g^2$ the projection on the i, jth factors.

This corresponds exactly to a product of double cosets:

if Z is defined by

$$\Gamma_g \gamma \Gamma_g = \sum_i \Gamma_g \alpha_i$$

and if W is defined by

$$\Gamma_g \delta \Gamma_g = \sum_j \Gamma_g \beta_j$$

then $Z \circ W$ is given by

$$\sum_{i,j} \Gamma_g \alpha_i \beta_j$$

Given a double coset we write it as a sum of right cosets

$$\Gamma_g \gamma \Gamma_g = \sum_i \Gamma_g \alpha_i$$

Lemma 2. *Each double coset with*

$$\gamma' J \gamma = p^e J$$

can be written as a finite sum of right cosets with α_i of the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with $a'd = p^e 1_g$ and a is upper triangular.

Proof. We can find $\alpha \in \Gamma_g$ such that $\alpha\gamma$ has zero in all but the first entry of the first column. Then one applies induction on g and arrives at $\gamma = (a, b; 0, d)$. By applying suitable $(u, 0; 0, u'^{-1})$ with $u \in \mathrm{GL}(g, \mathbf{Z})$ we get that $a_{ij} = 0$ for $i > j$. Acting by upper diagonal matrices u with 1 on the diagonal we reduce a_{ij} mod a_{ii}. Furthermore, we can reduce b with ab' symmetric modulo βd with β integral. This reduces it to a finite set. (The finiteness also follows from the interpretation as correspondence.)

Recall the slash operator:

$$f_{|\gamma,\rho} = \rho(c\tau + d)^{-1} f(\gamma(\tau))$$

For f holomorphic on $\mathcal{H}_g$ we have: $f \in M_\rho(\Gamma_g) \iff f_{|\gamma,\rho} = f$ (for $g > 1$).

But for a good action (on integral cohomology) one could add a factor

$$\eta(\gamma)^{-g(g+1)/2+\sum_i \lambda_i},$$

where as always $\rho \longleftrightarrow (\lambda_1 \geq \cdots \geq \lambda_g)$. We have

$$f_{|\gamma_1,\rho_{|\gamma_2,\rho}} = f_{|\gamma_1\gamma_2,\rho}$$

Lemma 3. *Let* $M \subset \mathrm{GSp}(2g, \mathbf{Q})$ *be a subset such that*

1) $M = \sqcup_{i=1}^{r} \Gamma_g \gamma_i$*,*

2) $M\,\Gamma_g = M$*.*

Then the map $f \mapsto T_M f = \sum_{i=1}^{r} f_{|\gamma_i,\rho}$ *is a linear operator on* $M_\rho(\Gamma_g)$ *independent of the choice of the representatives* γ_i*.*

Proof. In view of $f_{|\gamma,\rho} = f$ condition 1) implies the choice of the representatives makes no difference. And 2) implies $(T_M f)_{|\gamma,\rho} = T_M f$, so $T_M f \in M_\rho(\Gamma_g)$. Q.e.d.

So suitable sums of right cosets induce an action on spaces of modular forms.

An example of such a set M is

$$M_m = \{\gamma \in \mathrm{Mat}(2g \times 2g, \mathbf{Z}) : \gamma' J \gamma = mJ\}$$

so it induces an operator $T(m)$.

For p prime the set M_p is one double coset. This defines $T(p)$. For $m = p^2$ there are $g+1$ double cosets. This defines

$$T(p^2) = \sum_{i=0}^{g} T_i(p^2)$$

The geometric meaning of the $T_i(p^2)$: we look at isogenies $X_1 \to X_2$ with kernel $H \subset X_1[p^2]$ with

$$\#(H \cap X_1[p]) = p^{g+i}$$

We will now formalize this algebra of operators..

Abstract Hecke Algebra

We have a pair (Γ, G) with

$$G = \mathrm{GSp}(2g, \mathbf{Q}), \quad \Gamma = \Gamma_g$$

We put $G^{+} = \{\gamma \in G : \eta(\gamma) > 0\}$. We define

$$H(\Gamma, G)$$

as the $\mathbf{Q}$-vector space of formal finite sums of double cosets $\Gamma\gamma\Gamma$ with $\gamma \in G^{+}$. Each $\Gamma\gamma\Gamma$ can be written as

$$\Gamma\gamma\Gamma = \sum_i \Gamma\,\alpha_i$$

We let $\mathcal{L}$ be the $\mathbf{Q}$-vector space of formal finite expressions

$$\sum c_i L_i,$$

with $c_i \in \mathbf{Q}$ and $L_i = \Gamma\gamma_i$ a right coset.

We get a map

$$H(\Gamma, G) \hookrightarrow \mathcal{L}$$

The group Γ acts on $\mathcal{L}$ via $\Gamma\gamma_1 \mapsto \Gamma\gamma_1\gamma$.

We thus get

$$H(\Gamma, G) = \mathcal{L}^\Gamma,$$

the invariant part of $\mathcal{L}$.

We define a product for

$$\Gamma\gamma\Gamma = \sum \Gamma\gamma_i, \quad \Gamma\delta\Gamma = \sum \Gamma\delta_j$$

by

$$(\Gamma\gamma\Gamma)\cdot(\Gamma\delta\Gamma) = \sum_{i,j} \Gamma\gamma_i\delta_j$$

The group G admits an involution

$$\gamma \mapsto \gamma^\vee = J\gamma' J^{-1}$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \gamma^\vee = \begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix}$$

Observe $(\alpha\beta)^\vee = \beta^\vee\alpha^\vee$ and since we can choose γ diagonal and $J \in \Gamma$

$$\Gamma\gamma\Gamma = \Gamma J\gamma J^{-1}\Gamma = \Gamma\gamma^\vee\Gamma$$

hence $H(\Gamma, G)$ is commutative.

We can consider the subring $H^0(\Gamma, G)$ of $H(\Gamma, G)$ generated by finite sums of double cosets

$$\sum_{i=1}^{n} m_i \Gamma \gamma_i \Gamma$$

with $m_i \in \mathbf{Z}$ and γ_i integral. There is a degree map

$$\deg : H^0(\Gamma, G) \to \mathbf{Z}$$

that associates to a double coset its number of right cosets.

A double coset $\Gamma_g \gamma \Gamma_g$ defines a correspondence on $\mathcal{A}_g$ and we can take its closure T_γ on $\mathcal{A}_g^*$. If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the correspondence is given by

$$\tau_2\, c\, \tau_1 + \tau_2\, d - a\, \tau_1 - b = 0$$

in $\mathcal{H}_g \times \mathcal{H}_g$.

For later use we want to see the compatibility with the Siegel operator. For that we look at γ that fix a given boundary component, that is, $\gamma \in P_{g-1}(\mathbf{Q})$. Such a γ then has the form

$$\begin{pmatrix} a_1 & 0 & b_1 & * \\ * & u & * & * \\ c_1 & 0 & d_1 & * \\ 0 & 0 & 0 & u^{-1} \end{pmatrix}$$

with u invertible and this defines a T_{γ_1} on $\mathcal{A}^*_{g-1}$.

For such γ we can define a $\Phi(T_\gamma) = T_{\gamma_1}$ and otherwise $\Phi(T_\gamma) = 0$.

This will ensure a compatibility of the action of Hecke operators and the Siegel operator.

Decomposing a diagonal representative of a double coset as a product of diagonal matrices involving only powers of one prime and composition of correspondences show that we can decompose the Hecke algebra as a product of local algebras

$$H(\Gamma, G) = \boxed{\otimes_p H_p},$$

with

$$H_p = H(\Gamma, G \cap \mathrm{GL}(2g, \mathbf{Z}[\frac{1}{p}]))$$

where we allow in the determinants only powers of one prime. We have a subring H_p^0 of integral matrices and

$$H_p = H_p^0[1/T] \quad \text{with } T = \Gamma_g p 1_{2g} \Gamma_g$$

Local Hecke Algebra

Fix a prime p.

Consider the local algebra H_p generated by double cosets $\Gamma\gamma\Gamma$ with

$$\gamma = \mathrm{diag}(p^{a_1}, \ldots, p^{a_g}, p^{d_1}, \ldots, p^{d_g})$$

with

$$a_1 \leq \cdots \leq a_g \leq d_g \leq \cdots \leq d_1,$$

and

$$p^{a_i+d_i} = \eta(\gamma).$$

We have the subalgebra H_p^0 generated by such γ that are integral.

$$H_p = H_p^0[p^{-1}1_{2g}]$$

Proposition 1. *H_p^0 is generated by $T(p) = \Gamma \begin{pmatrix} 1_g & 0 \\ 0 & p1_g \end{pmatrix} \Gamma$ and $T_i(p^2) = \Gamma\gamma_i\Gamma$ for $i = 1, \ldots, g$ with*

$$\gamma_i = \begin{pmatrix} 1_{g-i} & & & \\ & p1_i & & \\ & & p^2 1_{g-i} & \\ & & & p1_i \end{pmatrix}$$

Proof. By induction. For $g = 1$ we have

$$T(p^n) = T(p^{n-1})T(p) - pT_1(p^2)T(p^{n-2})$$

so $T(p)$ and $T_1(p^2)$ are generators. Assuming the statement for $g-1$ we use the surjectivity of Φ and the knowledge of the kernel $= (T_g(p))$ to conclude that $T(p)$, $T_1(p^2), \ldots, T_{g-1}(p^2)$ generate. $\square$

We get a map

$$\Phi : H_p^0[g] \to H_p^0[g-1]$$

The kernel is generated by $T_g(p^2)$.

Eigenforms for the Hecke algebra

Let $f_1 \in M_\rho(\Gamma_g)$, $f_2 \in M_\rho(\Gamma_g)$ be such that at least one of these is a cusp form. Then we have the Petersson product (f_1, f_2).

Recall that $\gamma \mapsto \gamma^\vee$ defines an anti-involution of $H(\Gamma, G)$. Let $T \in H(\Gamma, G)$.

Proposition 2. *We have*

$$(Tf_1, f_2) = (f_1, T^\vee f_2)$$

$H(\Gamma, G)$ is commutative algebra of operators acting on $S_\rho(\Gamma_g)$; these operators are hermitean for the positive definite hermitean product. Thus we can find a (canonical up to scalars) basis of eigenvectors in $S_\rho(\Gamma_g)$.

Example. We take $\chi_{10} \in S_{10}(\Gamma_2)$.

In 1978 H. Saito and Kurokawa (independently) calculated some eigenvalues $\lambda(p)$ of the Hecke operator $T(p)$ on χ_{10}.

p	$\lambda(p)$ on $S_{10}(\Gamma_2)$
2	240
3	21960
5	1317900
7	49344400

They realized that in these cases:

$$\lambda(p) = p^8 + p^9 + a(p)$$

with the $a(p)$ Fourier coefficients of $f = \sum a(n)q^n \in S_{18}(\Gamma_1)$ with

$$f = q - 528\,q^2 - 4283\,q^3 + 147712\,q^4 + \ldots$$