Siegel Modular Forms
Lecture #9

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Eigenforms for the Hecke algebra

Recall that the Hecke algebra $H(\Gamma, G)$ is a commutative algebra of operators acting on $S_\rho(\Gamma_g)$ and $M_\rho(\Gamma_g)$; moreover the operators T are hermitian with respect to the Petersson product

$$(Tf_1, f_2) = (f_1, Tf_2)$$

for $f_1, f_2 \in S_\rho(\Gamma_g)$. By standard linear algebra we thus find a basis of $S_\rho(\Gamma_g)$ of common eigenforms, that is eigenforms for the whole algebra. By induction and using the fact that Φ sends eigenforms to eigenforms we find a basis of eigenforms for $M_\rho(\Gamma_g)$.

Example. We take $\chi_{10} \in S_{10}(\Gamma_2)$ with $\dim S_{10}(\Gamma_2) = 1$.

In 1978 H. Saito and Kurokawa (independently) calculated some eigenvalues $\lambda(p)$ of the Hecke operator $T(p)$ on χ_{10}.

p	$\lambda(p)$ on $S_{10}(\Gamma_2)$
2	240
3	21960
5	1317900
7	49344400

They realized that in these cases:

$$\lambda(p) = p^8 + p^9 + a(p)$$

with the $a(p)$ Fourier coefficients of $f =$

$\sum a(n)q^n \in S_{18}(\Gamma_1)$ with

$$f = q - 528\,q^2 - 4283\,q^3 + 147712\,q^4 + \ldots$$

Some other examples. For $\chi_{35} \in S_{35}(\Gamma_2)$:

$$\lambda_2(\chi_{35}) = -25073418240,$$

$$\lambda_3(\chi_{35}) = -11824551571578840,$$

and $-\lambda_{37}(\chi_{35})$ is given by

$$-\lambda_{37}(\chi_{35}) = 4778858564154594803526 \\ 785949392620832705065697 1703460.$$

Or examples of a vector-valued ones; for $\chi_{6,8}$ we find

p	$\lambda(p)$ on $S_{6,8}$
2	0
3	-27000
5	2843100
7	-107822000
11	3760397784
13	9952079500
17	243132070500
19	595569231400

Eigenvalues of Hecke operators on $S_{8,8}(\Gamma_2)$

We have $\dim S_{8,8}(\Gamma_2) = 1$.

p	$\lambda(p)$ on $S_{8,8}$
2	$2^6 \cdot 3 \cdot 7$
3	$-2^3 \cdot 3^2 \cdot 89$
5	$-2^2 \cdot 3 \cdot 5^2 \cdot 13^2 \cdot 607$
7	$2^4 \cdot 7 \cdot 109 \cdot 36973$
11	$2^3 \cdot 3 \cdot 4759 \cdot 114089$
13	$-2^2 \cdot 13 \cdot 17 \cdot 109 \cdot 3404113$
17	$2^2 \cdot 3 \cdot 17 \cdot 41 \cdot 1307 \cdot 168331$
19	$-2^3 \cdot 5 \cdot 74707 \cdot 9443867$

Here is one for $g = 3$:

$$\chi_{4,0,8} \in S_{4,0,8}(\Gamma_3).$$

Here $\rho = \mathrm{Sym}^4(St) \otimes \det(St)^8$ and $\dim S_{4,0,8}(\Gamma_3) = 1.$ Recall that $\chi_{4,0,8}$ appeared in the Taylor development of the Schottky form $\phi_8 \in S_8(\Gamma_4)$ along $\mathcal{H}_1 \times \mathcal{H}_3$. We have

$$\lambda(p) = \tau(p)(p^5 + \tau(p) + p^6)$$

with

$$\Delta = \sum_n \tau(n) q^n \in S_{12}(\Gamma_1)$$

More examples can be found on the website http://smf.compositio.nl/ Enough examples.

It can be a bit laborious work to calculate the eigenvalues. For example $T(p)$ for $g = 2$ has degree $p^3 + p^2 + p + 1$. Right coset representatives are

$$\Gamma_2 \begin{pmatrix} p\,1_2 & \\ & 1_2 \end{pmatrix} + \sum_{0 \le a,b,c \le p-1} \Gamma_2 \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & b & c \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

$$+ \sum_{0 \le a \le p-1} \Gamma_2 \begin{pmatrix} 0 & -p & 0 & 0 \\ 1 & 0 & a & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & p & 0 \end{pmatrix} +$$

$$\sum_{0 \le a,m \le p-1} \Gamma_2 \begin{pmatrix} p & 0 & 0 & 0 \\ -m & 1 & 0 & a \\ 0 & 0 & 1 & m \\ 0 & 0 & 0 & p \end{pmatrix}$$

Local Hecke Algebra

Fix a prime p. Recall that we looked at the local algebra H_p generated by double cosets $\Gamma\gamma\Gamma$ with

$$\gamma = \mathrm{diag}(p^{a_1}, \ldots, p^{a_g}, p^{d_1}, \ldots, p^{d_g})$$

with

$$a_1 \leq \cdots \leq a_g \leq d_g \leq \cdots \leq d_1,$$

and

$$p^{a_i+d_i} = \eta(\gamma).$$

We have the subalgebra H_p^0 generated by such γ that are integral.

$$H_p = H_p^0[p^{-1}1_{2g}]$$

Recall that we can define a map

$$\Psi : {}^{g}H_p^0 \to {}^{g-1}H_p^0$$

where the green index indicates the degree.

Recall that we have operators $T(p)$ and $T_i(p)$ defined by double cosets with representatives

$$T(p) : \quad \begin{pmatrix} 1_g & \\ & p1_g \end{pmatrix},$$

and for $i = 0, \ldots, g$

$$T_i(p^2) : \quad \begin{pmatrix} 1_{g-i} & & & \\ & p1_i & & \\ & & p^2 1_{g-i} & \\ & & & p1_i \end{pmatrix}$$

and that these generate the ring H_p^0. The kernel of the map $\Psi : {}^{g}H_p^0 \to {}^{g-1}H_p^0$ is generated by $T_g(p^2)$.

Remark. We saw that $\mathrm{GSp}(2g, \mathbf{Q}_p) = \sqcup_\gamma K\gamma K$ with γ running though

$$\mathrm{diag}(p^{a_1}, \ldots, p^{a_g}, p^{d_1}, \ldots, p^{d_g})$$

with $a_1 \leq a_2 \leq \cdots \leq d_g \leq d_{g-1} \leq \cdots \leq d_1$ and $a_i + d_i = c$. This describes a positive Weyl chamber in the co-character group of a maximal torus T in a Borel subgroup B.

Another view of H_p

Look at the group $\mathrm{GSp}(2g, \mathbf{Q}_p)$ The maximal compact subgroup $K = \mathrm{GSp}(2g, \mathbf{Z}_p)$ acts both on the left and the right.

Consider the ring of $\mathbf{Z}$-valued locally constant functions on $\mathrm{GSp}(2g, \mathbf{Q}_p)$ with compact support which are invariant under the actions of K: $f(k_1 x k_2) = f(x)$ for $k_1, k_2 \in K$. The multiplication in this ring is convolution

$$f_1 \cdot f_2 = \int_{\mathrm{GSp}(2g,\mathbf{Q}_p)} f_1(\xi) f_2(\xi^{-1}\eta) d\xi$$

with $d\xi$ the Haar measure with $\mathrm{vol}(K) = 1$.

We look at the $\mathbf{Q}$-algebra obtained by tensoring with $\mathbf{Q}$. We can identify the algebra H_p with this algebra. The identification is

$$K\gamma K \leftrightarrow \text{charact. function of } K\gamma K$$

A compactly supported function in H_p is constant on double cosets and its support is a linear combination of characteristic functions of double cosets.

We consider the subgroups

$$\mathbf{T} \subset M \subset \mathrm{GSp}(2g, \mathbf{Q})$$

with $\mathbf{T}$ the diagonal torus and with M the so-called Levi subgroup

$$M = \{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \mathrm{GSp}(2g, \mathbf{Q}) \}$$

of the Borel subgroup

$$B = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{GSp}(2g, \mathbf{Q})\}$$

that stabilizes the first summand $\mathbf{Z}^g$ of our symplectic lattice.

Similarly for the group over $\mathbf{Q}_p$. We have

$$\mathbf{T} \cong \mathbf{G}_m^{g+1} \quad \text{and} \quad M \cong \mathrm{GL}(g) \times \mathbf{G}_m$$

Let

$$Y = \mathrm{Hom}(\mathbf{G}_m, \mathbf{T})$$

be the co-character group of $\mathbf{T}$. We have

$$Y \cong \mathbf{Z}^{g+1}$$

A basis of Y is given by

$$\chi_0 : t \mapsto \mathrm{diag}(1, \ldots, 1, t, \ldots, t),$$

$$\chi_1 : t \mapsto \mathrm{diag}(t, 1 \ldots, 1, t^{-1}, 1 \ldots, 1)$$

$$\vdots$$

$$\chi_g : t \mapsto \mathrm{diag}(1, \ldots, 1, t, 1 \ldots, 1, t^{-1})$$

We have a local Hecke algebra at p for the pair $(\mathbf{T}, \mathbf{T}_{\mathbf{Q}})$. It is the $\mathbf{Q}$-algebra of $\mathbf{Q}$-valued locally constant functions with compact support on $\mathbf{T}(\mathbf{Q}_p)$ invariant under left and right action by $\mathbf{T}(\mathbf{Z}_p)$. But $\mathbf{T}$ is commutative, so a double coset is one right coset. So the local Hecke algebra for $\mathbf{T}$ is

$$\mathbf{Q}[Y],$$

the group algebra over $\mathbf{Q}$ of Y. We have an exact sequence

$$0 \to \mathbf{T}(\mathbf{Z}_p) \to \mathbf{T}(\mathbf{Q}_p) \xrightarrow{\nu} Y \to 0$$

$$\nu(\mathrm{diag}(t^{a_1},\ldots,t^{a_g},t^{d_1},\ldots,t^{d_g})) = \chi_0^c \prod_{i=1}^{g} \chi_i^{a_i}$$

where $c = a_i + d_i$. We can identify $H_p(\mathbf{T}) \otimes \mathbf{Q}$ with

$$\mathbf{Q}[(u_1/v_1)^{\pm 1},\ldots,(u_g/v_g)^{\pm 1},(u_1\cdots u_g)^{\pm 1}]$$

where $u_i/v_i \longleftrightarrow \chi_i$. The Weyl group

$$W_G = N(\mathbf{T})/\mathbf{T} \cong \mathfrak{S}_g \ltimes (\mathbf{Z}/2\mathbf{Z})^g$$

(semi-direct product) acts.

We have the involutions

$$\epsilon_i : a_i \leftrightarrow d_i \qquad (i = 1,\ldots,g)$$

The symmetric group $\mathfrak{S}_g$ permutes the a_i and

d_i simultaneously. So ϵ_i acts on the χ_i:

$$\chi_0 \mapsto \chi_0\chi_i, \qquad \chi_i \mapsto \chi_i^{-1}$$

Fact:

$$H_p(G) = H_p(\mathbf{T})^{W_G}$$

We can also do the same thing for the

$$\mathbf{T} \subset M \subset \mathrm{GSp}(2g, \mathbf{Q})$$

with M the so-called Levi subgroup (defined by $b = 0$) of the Borel subgroup

$$\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{GSp}(2g, \mathbf{Q})\}$$

This induces maps

$$H_p(\Gamma, G) \to H_p(M) \to H_p(\mathbf{T})$$

and these maps can be given by integration over p-adic groups, the so-called Satake homomorphism. For example, the map S_{GT} can be described by

$$S_{GT}(\phi)(t) = e^{2\rho}(t) \int_N \phi(tn)dn$$

where N is the unipotent radical of B, dn is a Haar measure on N with $\mathrm{vol}(N \cap K) = 1$ and $e^{2\rho}$ is given by sending

$$\mathrm{diag}(\alpha_1, \ldots, \alpha_g, \delta_1, \ldots, \delta_g)$$

to

$$\eta^{-g(g+1)/2} \prod_{i=1}^{g} \alpha_i^{2g+2-2i}$$

Here 2ρ corresponds to the sum of the positive roots (that occur in the representation on

$\mathrm{Lie}(B)$). It appears here naturally because $B = T \cdot N$ and

$$d(bnb^{-1}) = |e^{\rho}(b)| \cdot dn$$

with $|p| = 1/p$.

(There is an issue here whether one uses $e^{2\rho}$ or e^{ρ}; in the latter case one must use $\mathbf{Q}(\sqrt{p})$-coefficients.)

The Weyl group of M is $\mathfrak{S}_g$.

Theorem 1. *This defines isomorphisms of* $\mathbf{Q}$*-algebras*

$$H_p(G) \cong H_p(\mathbf{T})^{W_g}, \quad H_p((M) \cong H_p(\mathbf{T})^{W_M}$$

The map $S_{GT} : H_p(\Gamma, G) \to H_p(\mathbf{T})$ can be calculated explicitly: if

$$\gamma = \mathrm{diag}(p^{a_1}, \ldots, p^{c-a_g})$$

then

$$K\,\gamma K \mapsto p^{c\,g(g+1)/4}(v_1 \cdots v_g)^c \prod_{i=1}^{g} (u_i/p^i v_i)^{a_i}$$

We can use the map $H_p(\Gamma, G) \to H_p(M)$ to understand better the structure of the Hecke algebra.

The local Hecke algebra of $\mathrm{GL}(g)$, that is for the pair $(\Gamma, G) = (\mathrm{GL}(g, \mathbf{Z}_p), \mathrm{GL}(g, \mathbf{Q}_p))$, is generated by the elements

$$\phi_i = p^{i(i+1)/2} \Gamma \begin{pmatrix} 1_{g-i} & 0 \\ 0 & p1_i \end{pmatrix} \Gamma$$

for $i = 1, \ldots, g$. When one maps this Hecke algebra to the Hecke algebra of the diagonal torus $T = \mathbf{G}_m^g$ we get a map

$$H_{\mathrm{GL}(g)} \cong \mathbf{Q}[y_1, \ldots, y_g]^{\mathfrak{S}_g}, \quad \phi_i \mapsto \sigma_i$$

with σ_i the ith elementary symmetric function. In this way we see under $H_p(G) \to H_p(M)$

$$T(p) \to \sum_{i=0}^{g} \phi_i$$

and under $H_p(M) \to H_p(\mathbf{T})$

$$\phi_i \mapsto (v_1 \cdots v_g)\, \sigma_i(u_1/v_1, \ldots, u_g/v_g)$$

with σ_i the ith elementary symmetric function. Furthermore

$$T_i(p^2) \mapsto \sum_{j,k: j+i \leq k} m_{k-j}(i) p^{-\binom{k-j+1}{2}} \phi_j \phi_k$$

with $m_h(i)$ the number of elements

$$\#\{A \in \mathrm{Mat}(h \times h, \mathbf{F}_p) : A' = A, \mathrm{corank}(A) = i\}$$

Shimura and Satake studied this algebra and the maps. Andrianov studied this Hecke algebra in detail.

Satake parameters

Recall that Deligne proved in 1968 that for a normalized eigenform $f = \sum a(n)q^n \in S_k(\Gamma_1)$ we have

$$a(p) = \beta + \overline{\beta}$$

with $\beta\overline{\beta} = p^{k-1}$. We can associate to f the tuple $(\alpha_0, \alpha_1) = (\beta, \overline{\beta}/\beta)$ or $(\overline{\beta}, \beta/\overline{\beta})$. To the normalized Eisenstein series of weight k we can associate $(\alpha_0, \alpha_1) = (1, p^{k-1})$.

More generally, if $f \in M_\rho(\Gamma_g)$ is an eigenform then it defines a character

$$H(\Gamma, G) \to \mathbf{C}^*$$

and also for the local Hecke algebra

$$\mathrm{Hom}(H_p, \mathbf{C}^*) = (\mathbf{C}^*)^{g+1}/W_G$$

An element on the right hand side is represented by a $(g+1)$-tuple

$$(\alpha_0, \alpha_1, \ldots, \alpha_g),$$

a W_G-orbit of non-zero complex numbers with

$$\alpha_i \quad \text{the image of } u_i/v_i$$

and

$$\alpha_0 \quad \text{the image of } v_1 \cdots v_g$$

The $g+1$-tuple $(\alpha_0, \ldots, \alpha_g)$ is called the tuple of p-Satake parameters of the eigenform f.

We know

$$\alpha_0^2\alpha_1\cdots\alpha_g = p^{-g(g+1)/2+\sum_i \lambda_i}$$

This is a consequence of the fact that $T_g(p^2)$ (given by the double coset of $p{\cdot}1_{2g}$) is mapped to

$$p^{-g(g+1)/2}(v_1\cdots v_g)^2\prod_i (u_i/v_i)$$

For an eigenvorm we can express the eigenvalues $\lambda(p)$ and $\lambda_i(p^2)$ for the Hecke operators $T(p)$ and $T_i(p^2)$ for $i=0,\ldots,g$ as

$$\lambda(p) = \alpha_0(1+\sigma_1+\cdots+\sigma_g)$$

where σ_i is the ith elementary symmetric function of the α_i $(i = 1, \ldots, g)$ and

$$\lambda_i(p^2) = \sum_{j,k \geq 0, j+i \leq k}^{g} m_{k-j}(i) p^{-\binom{k-j+1}{2}} \alpha_0^2 \sigma_j \sigma_k$$

with the $m_h(i)$ appearing in the formula for $S_{G,M}(T_i(p^2))$.

Consider again ϕ_0, given by the representative $(1_g, 0; 0, p1_g)$.

The element $S_{MT}(\phi_0) = v_1 \cdots v_g$ and this element is invariant under the action of $\mathfrak{S}_g$, but not under any other element of W_G. So it is a root of the polynomial

$$\prod_{w \in (\mathbf{Z}/2)^g} (X - w(\phi_0))$$

that is,

$$\prod_{I \subset \{1,\ldots,g\}} (X - \prod_{i \in I} u_i \prod_{j \notin I} v_j)$$

Therefore the element generates an extension of degree 2^g. (Fraction field of $H_p(M)$ over $H_p(G)$).

Example $g = 1$; here ϕ_0 is a root of

$$X^2 - T(p)X + pT_1(p^2)$$

Recall that $S_{GM}(T_1(p^2)) = \phi_0\phi_1/p$.

For $g = 2$ we find a polynomial

$$X^4 - T(p)X^3 + (T(p)^2 - T(p^2) - p^2T_2(p^2))X^2$$
$$-p^3T(p)T_2(p^2)X + p^6T_2(p^2)^2$$

The element ϕ_0 is called *Frob*.

The reason is that if we look at isogenies of elliptic curves of degree p in characteristic $p > 0$ that these are either inseparable and then these are equal to Frobenius or their dual is inseparable. In fact, the multiplication by p map $[p] : E \to E$ decomposes

$$E \xrightarrow{F} E^{(p)} \to E$$

Therefore the correspondence defining $T(p)$ decomposes in two components $F + F'$.

A further remark is that the representation ring of the dual group $\hat{G}$ of $G = \mathrm{GSp}(2g)$ can be identified with the invariants $\mathbf{Z}[Y]^W$ under the Weyl group with Y the co-character group of T. In our case the dual group $\hat{G}$ is the spinor group.

L-Series

If $f \in M_k(\Gamma_1)$ is a (normalized) eigenform

$$f = \sum_{n \geq 0} a(n)\, q^n$$

we consider the L-function (for $\mathrm{Re}(s) > k/2 + 1$)

$$L_f(s) = \sum_n a(n) n^{-s}$$

and it has an Euler product

$$L_f(s) = \prod_p (1 - a(p)p^{-s} + p^{k-1-2s})^{-1}$$

with Euler factor

$$1 - a(p)X + p^{k-1}X^2 = (1 - \beta X)(1 - \overline{\beta} X)$$

The Spinor L-Function

This L-function is defined as an Euler product

$$Z_f(s) = \prod_p Z_{f,p}(p^{-s})^{-1}$$

where the Euler factor $Z_{f,p}(t)$ is given by

$$Z_{f,p}(t) = \prod_I (1 - \alpha_0 \alpha_I \, t),$$

with I running over the 2^g subsets of $\{1, \ldots, g\}$ and α_I stands for

$$\alpha_I = \prod_{i \in I} \alpha_i$$

For $g = 1$ and $f \in S_k(\Gamma_1)$ upon writing

$$a(p) = \beta + \overline{\beta} \quad \text{with } \beta\overline{\beta} = p^{k-1}$$

we get

$$(1 - \alpha_0 t)(1 - \alpha_0\alpha_1 t) = (1 - \beta t)(1 - \overline{\beta} t)\,,$$

the usual L-factor.

The meromorphic continuation and functional equation are known for $g = 1, 2$ and $g = 3$, but this is open for $g \geq 4$.

There is a compatibility with the Siegel operator. If $f \in M_k(\Gamma_g)$ is an eigenform, then $\Phi(f) \in M_k(\Gamma_{g-1})$ is an eigenform and if $\Phi(f) \neq 0$ then

$$Z_f(s) = Z_{\Phi(f)}(s)\, Z_{\Phi(f)}(s + g - k)$$

Example. The Eisenstein series of weight k and degree g has p-Satake parameters

$$(1, p^{k-1}, \ldots, p^{k-g})$$

For $g = 2$ and even weight $k \geq 4$ we have

$$Z_{\psi_k}(s) = \zeta(s)\zeta(s-k+1)\zeta(s-k+2)\zeta(s-2k+3)$$

and $\Phi(\psi_k) = E_k$, the Eisenstein series of degree 1 with L-function $\zeta(s)\zeta(s-k+1)$, so

$$Z_{\psi_k}(s) = L(E_k, s)L(E_k, s-k+2)\,.$$

Literature

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website for traces of Hecke operators for degree $g = 2$ and $g = 3$:

http://smf.compositio.nl/