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GENERATORS FOR MODULES OF VECTOR-VALUED PICARD MODULAR FORMS

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Abstract. We construct generators for modules of vector-valued Picard modular forms on a unitary group of type $(2,1)$ over the Eisenstein integers. We also calculate eigenvalues of Hecke operators acting on cusp forms.

§1. Introduction

Modular forms on unitary groups have been studied intensively in the theory of automorphic forms (see, e.g., [14] and [11]), but explicit examples have been scarce. Shintani [18] considered vector-valued Picard modular forms in an unpublished manuscript; in particular, he determined a criterion for such a modular form to be a Hecke eigenform in terms of the Fourier–Jacobi series. Explicit (scalar-valued) Picard modular forms were considered for $F = \mathbb{Q}(\sqrt{-1})$ by Resnikoff and Tai (see [12], [13]) and for $F = \mathbb{Q}(\sqrt{-3})$ by Shiga, Holzapfel, Feustel, and Finis (see [15], [6], [7], [4], [5], resp.); in particular, for the latter case, Holzapfel [6] and Feustel [4] determined a presentation of the ring of scalar-valued Picard modular forms on the congruence subgroup

$$\Gamma_1[\sqrt{-3}] = \{g \in \Gamma_1 : g \equiv 1 \pmod{\sqrt{-3}}\},$$

while Finis [5] computed a number of Hecke eigenvalues for low-weight (less than or equal to 12) scalar-valued forms. Vector-valued Picard modular forms have not attracted much attention so far.

There is another approach to calculating Hecke eigenvalues of modular forms via the study of the cohomology of local systems on moduli spaces of algebraic curves. This approach, which uses counting of points over finite

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fields, was pioneered by Faber and the second author in [3] and continued with Bergström in [1]. It has provided a lot of explicit examples and gathered information that led to conjectures on vector-valued Siegel modular forms of genus 2 and 3.

The Picard modular surface underlying the work of Feustel, Holzapfel, and Finis can be interpreted as a moduli space of degree 3 Galois covers of genus 3 of the projective line (see early work of Shimura [16]). Using this interpretation, Bergström and the second author set out to calculate traces of Hecke operators on spaces of vector-valued modular forms by calculating the trace of Frobenius on the étale cohomology of local systems over finite fields. This approach is carried out in [2]. It naturally led to the question of constructing the corresponding Picard modular forms directly. That constitutes our starting point here.

The purpose of this article is to construct vector-valued (eigen)forms on the Picard modular group in question, that is, the congruence subgroup $\Gamma_1[\sqrt{-3}]$. The weight of a modular form is a pair (j,k), with the case $(0,k)$ corresponding to scalar-valued modular forms of weight k. We will denote by $M_{j,k}(\Gamma[\sqrt{-3}],\det^\ell)$ (resp., $S_{j,k}(\Gamma[\sqrt{-3}],\det^\ell)$) the vector space of modular forms (resp., cusp forms) of weight (j,k) with character $\det^\ell$ with $0 \le \ell \le 2$ (see Section 2 for precise definitions). Then

$$\mathcal{M}_j^\ell = \bigoplus_k M_{j,k}\big(\Gamma[\sqrt{-3}],\det^\ell\big) \qquad \text{and} \qquad \Sigma_j^\ell = \bigoplus_k S_{j,k}\big(\Gamma[\sqrt{-3}],\det^\ell\big)$$

are modules over the ring $M = \mathcal{M}_0^0 = \bigoplus M_{0,k}(\Gamma[\sqrt{-3}])$ of scalar-valued modular forms. Generators for the ring M of scalar-valued modular forms are known by work of Feustel and Holzapfel; in fact, $M = \mathbb{C}[\varphi_0,\varphi_1,\varphi_2]$ with φ_i of weight 3. We give a presentation for the M-modules $\mathcal{M}_j^\ell$ and Σ_j^ℓ for $j=1$ and 2. We also give the structure for $\mathcal{M}_3^0$. To illustrate this, we give one example.

THEOREM 1.1. *The M-module $\mathcal{M}_1^0$ is generated by three vector-valued cusp forms Φ_0, Φ_1, and Φ_2 of weight $(1,7)$ satisfying a single relation $\varphi_0\Phi_0 + \varphi_1\Phi_1 + \varphi_2\Phi_2 = 0$.*

We can calculate the Fourier–Jacobi expansions of our forms and thus can calculate the Hecke eigenvalues of these modular forms. We give Hecke eigenvalues for a number of generators. The corresponding Galois representations are of dimension 1, 2, or 3. These results agree with the cohomological results of Bergström and van der Geer [2]. The results of [2] on the S_4-equivariant numerical Euler characteristics of the local systems predicted where to look for generators of these modules of modular forms. We hope these results will make Picard modular forms more tangible than they have been so far and that the presence of such explicit examples can help lead to the discovery of new phenomena.

The case of Picard modular forms over the Gaussian integers will be treated in a forthcoming publication; also we intend to treat cases of modular forms on higher-rank unitary groups.

§2. The Picard modular group

Let F be an imaginary quadratic field of discriminant D with ring of integers O_F. We consider a 3-dimensional F-vector space V that contains an O_F-lattice L with a nondegenerate Hermitian form h that is O_F-valued on L and of signature $(2,1)$. This determines an algebraic group G of unitary similitudes

$$\left\{g \in \mathrm{GL}(3,F) : h(gz,gu) = \eta(g)h(z,u)\right\}$$

defined over $\mathbb{Q}$. We have $\eta(g)^3 = N_{F/\mathbb{Q}}(\det(g)) \in \mathbb{Q}_{>0}$, and η defines a homomorphism $\eta : G \to \mathbb{G}_m$, called the *multiplier*. The kernel $G^0 := \ker \eta$ is the usual unitary group, and $G^0 \cap \ker \det$ is the special unitary group of signature $(2,1)$. The base change of G to F is isomorphic to $\mathrm{GL}(3,F) \times \mathbb{G}_m$, where the latter factor corresponds to η.

We choose an embedding $\sigma : F \to \mathbb{C}$, and we identify $F \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{C}$. Then $V' = V \otimes_\sigma \mathbb{R}$ becomes a 3-dimensional complex vector space, and we can look at

$$B := \left\{U \subset V' : \dim(U) = 1, h \,\middle|\, U < 0\right\} \subset \mathbb{P}(V') = \mathbb{P}^2,$$

the set of complex lines on which h is negative definite. The group $G^+ = \{g \in G(\mathbb{R}) : \det(g) > 0\}$ acts on the Grassmannian $\mathrm{Gr}(1,V')$ and on B. We can identify B with the complex 2-ball in $\mathbb{P}^2$.

A standard choice for the Hermitian form is

$$h(z,z) = \frac{1}{\sqrt{D}} z_1 \bar{z}_3 + z_2 \bar{z}_2 - \frac{1}{\sqrt{D}} z_3 \bar{z}_1$$

on the lattice $O_F^3 \subset \mathbb{C}^3$. This is a maximal lattice in the sense of Shimura. Note that z_1 and z_3 do not vanish in B, and by taking $u = z_1/z_3$ and

$v = z_2/z_3$, the set of negative complex lines can be identified with the Siegel domain

$$\left\{(u,v) \in \mathbb{C}^2 : \frac{2}{\sqrt{|D|}}\,\mathrm{Im}(u) + |v|^2 < 0\right\}$$

embedded in $\mathbb{P}^2(\mathbb{C})$ via $(u,v) \mapsto (u : v : 1)$. If we identify $G(\mathbb{Q})$ with the matrix group

$$\{g \in \mathrm{GL}(3,F) : \bar{g}^t H g = \eta(g) H\},$$

with H the skew-Hermitian matrix

$$H = \begin{pmatrix} & & 1/\sqrt{D} \\ & 1 & \\ -1/\sqrt{D} & & \end{pmatrix},$$

then the action of $g = (g_{ij}) \in G$ on B is given by

$$(u,v) \mapsto \Big(\frac{g_{11}u + g_{12}v + g_{13}}{g_{31}u + g_{32}v + g_{33}}, \frac{g_{21}u + g_{22}v + g_{23}}{g_{31}u + g_{32}v + g_{33}}\Big).$$

The denominator $j_1(g,u,v) := g_{31}u + g_{32}v + g_{33}$ defines a factor of automorphy for this action. The Jacobian $J(g,u,v)$ of the action of G on B defines a second factor of automorphy:

$$J(g,u,v) = j_1(g,u,v)^{-2} \begin{pmatrix} G_{23}v + G_{22} & G_{13}v + G_{12} \\ -G_{23}u + G_{21} & -G_{13}u + G_{11} \end{pmatrix},$$

where G_{ij} is the minor of g_{ij}. One checks that

$$\det J(g,u,v) = j_1(g,u,v)^{-3} \det(g). \tag{1}$$

We thus have two factors of automorphy:

$$j_1(g,u,v) := g_{31}u + g_{32}v + g_{33}$$

and

$$j_2(g,u,v) := \det(g)^{-1} \begin{pmatrix} -G_{13}u + G_{11} & -G_{13}v - G_{12} \\ G_{23}u - G_{21} & G_{23}v + G_{22} \end{pmatrix}.$$

Note that $\det(j_2(g,u,v)) = j_1(g,u,v)/\det(g)$. (See also [17, Section 1] for the general case.)

Our normalization. Some authors use different normalizations. Via the coordinate change $w_1 = z_1/\sqrt{D}, w_2 = z_3, w_3 = z_2$, we get the Hermitian form

$$w_1\bar{w}_2 + w_2\bar{w}_1 + w_3\bar{w}_3$$

used, for example, by Finis [5, Section 1]. The symmetric domain is given by

$$B = \{(u,v) \in \mathbb{C}^2 : 2\,\mathrm{Re}\, v + |u|^2 < 0\},$$

with $u = w_3/w_2$, $v = w_1/w_2$, and the action is given by

$$(u,v) \mapsto \Big(\frac{g_{31}v + g_{32} + g_{33}u}{g_{21}v + g_{22} + g_{23}u}, \frac{g_{11}v + g_{12} + g_{13}u}{g_{21}v + g_{22} + g_{23}u}\Big).$$

The factors of automorphy are then

$$j_1(g,u,v) = g_{21}v + g_{22} + g_{23}u$$

and

$$j_2(g,u,v) := \det(g)^{-1}\begin{pmatrix} G_{32}u + G_{33} & G_{32}v + G_{31} \\ G_{12}u + G_{13} & G_{12}v + G_{11} \end{pmatrix}, \tag{2}$$

so that we have

$$j_2(g,u,v)^{-1} = j_1(g,u,v)\big(J(g,u,v)\big)^t. \tag{3}$$

Since we are using some of Finis's calculations, we use this normalization later in this article.

For a pair (j,k) of integers and $g \in G(\mathbb{R})$, we define a slash operator on functions $f : B \to \mathrm{Sym}^j(\mathbb{C}^2)$ via

$$(f|_{j,k}g)(u,v) = j_1(g,u,v)^{-k}\,\mathrm{Sym}^j\big(j_2(g,u,v)^{-1}\big) f\big(g \cdot (u,v)\big).$$

For a discrete subgroup Γ of $G(\mathbb{R})$ and a character χ of Γ of finite order, we define the space of modular forms of weight (j,k) and character χ as

$$M_{j,k}(\Gamma,\chi) := \{f : B \to \mathrm{Sym}^j(\mathbb{C}^2) : f \text{ holomorphic}, \\ f|_{j,k}g = \chi(g)f \text{ for all } g \in \Gamma\}.$$

It contains a subspace $S_{j,k}(\Gamma,\chi)$ of cusp forms.

We will consider in particular the Picard modular group and the special Picard modular group, and we fix Γ as

$$\Gamma = G^0(\mathbb{Z}) \qquad \text{and} \qquad \Gamma_1 = G^0(\mathbb{Z}) \cap \ker\det.$$

The quotient group Γ/Γ_1 is isomorphic to the roots of unity contained in O_F. Note that $N_{F/\mathbb{Q}}(\det(g))$ is a positive integer and a unit; hence, $\det(g)$ is a root of unity in O_F. As characters χ we will consider only powers of $\det(g)$. If $\chi = \mathrm{id}$, then we suppress the notation χ and write $M_{j,k}(\Gamma)$ instead.

We thus have the notions of vector-valued Picard modular forms with character on the groups Γ and Γ_1. We can consider the ring of scalar-valued modular forms

$$\mathcal{M}(\Gamma) = \bigoplus_k M_{0,k}(\Gamma)$$

and, for fixed $j \geq 0$, the $\mathcal{M}(\Gamma)$-module

$$\mathcal{M}_j(\Gamma) := \bigoplus_k M_{j,k}(\Gamma),$$

and similarly for other discrete groups.

§3. The Picard modular group for $F = \mathbb{Q}(\sqrt{-3})$

We now specialize to the case where $F = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\rho)$ with ρ a third root of unity and where $V = F^3$ with Hermitian form given by

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Besides the arithmetic groups

$$\Gamma = G^0(\mathbb{Z}) \qquad \text{and} \qquad \Gamma_1 = G^0(\mathbb{Z}) \cap \ker \det,$$

we consider the two congruence subgroups

$$\Gamma[\sqrt{-3}] = \{ g \in \Gamma : g \equiv 1 \pmod{\sqrt{-3}} \}$$

and

$$\Gamma_1[\sqrt{-3}] = \{ g \in \Gamma_1 : g \equiv 1 \pmod{\sqrt{-3}} \}.$$

Any congruence subgroup Γ' of Γ acts properly discontinuously on B, and the quotient is an orbifold, called a *Picard modular surface.* It is not compact but can be compactified by adding finitely many cusps, that is, the orbits of Γ' on the set $\partial B \cap \mathbb{P}^2(F)$ of rational points in ∂B. It is well known that the action of Γ on $\partial B \cap \mathbb{P}^2(F)$ is transitive (since the class number of $\mathbb{Q}(\sqrt{-3})$ is 1; see [4], [19]), so in this case there is one cusp. The group $\Gamma_1[\sqrt{-3}]$ instead has four cusps; these are represented by $(1:0:0)$, $(0:1:0)$, $(\rho:1:1)$, and $(\rho:1:-1)$ in $B\subset\mathbb{P}^2$. We have an isomorphism

$$\Gamma/\Gamma_1[\sqrt{-3}]\cong S_4\times\mu_6,\qquad g\mapsto\big(\sigma(g),\det(g)\big),$$

with $\sigma(g)$ the permutation of the four cusps. Here S_4 denotes the symmetric group on four objects and μ_6 denotes the group of sixth roots of unity.

The stabilizer in G of the cusp $(1:0:0)$ is the parabolic group P consisting of matrices of the form

$$\begin{pmatrix} t_1 & 0 & 0\\ 0 & t_2 & 0\\ 0 & 0 & t_3\end{pmatrix}\begin{pmatrix} 1 & x & -\bar{y}\\ 0 & 1 & 0\\ 0 & y & 1\end{pmatrix},$$

with $t_i\in F^*$, $x,y\in F$ satisfying

$$t_1\bar{t}_2=t_3\bar{t}_3,\qquad x+\bar{x}=-y\bar{y}.$$

Let T be the corresponding maximal torus, and let U the unipotent radical of P. Then U has a filtration

$$0\to W\longrightarrow U\longrightarrow\tilde{U}\to 0,$$

with $\tilde{U}=\mathrm{R}_{F/\mathbb{Q}}(\mathbb{G}_a)$ and the projection $\mathrm{pr}:U\to\tilde{U}$ defined by

$$\begin{pmatrix} 1 & x & -\bar{y}\\ 0 & 1 & 0\\ 0 & y & 1\end{pmatrix}\mapsto y.$$

The subgroup W is given by $\{x\in F: x+\bar{x}=0\}$ and is the center of U. The action of U on B is by translations

$$(u,v)\mapsto(u+y,v+x-\bar{y}u).$$

For the group Γ the unipotent radical has $W=\sqrt{-3}\mathbb{Z}$ and $\tilde{U}=O_F$ (in fact, for given $y\in O_F$, take $x=\rho\mathrm{N}(y)+m\sqrt{-3})$, while the congruence subgroup $\Gamma_1[\sqrt{-3}]$ has the same W and $\tilde{U}=\sqrt{-3}O_F$.

The cusps of $\Gamma_1[\sqrt{-3}]\backslash B$ are singular points but can be resolved by elliptic curves $\mathbb{C}/\sqrt{-3}O_F$ (see [4], [6]).

§4. Fourier–Jacobi expansion of scalar-valued modular forms

A scalar-valued modular form on the congruence subgroup $\Gamma_1[\sqrt{-3}]$ of Γ is invariant under the translations in the unipotent radical $\Gamma_1[\sqrt{-3}] \cap U$ of the parabolic subgroup that fixes the cusp $(1:0:0)$; in particular, it is invariant under the translations $v \mapsto v + m\sqrt{-3}$ with $m \in \mathbb{Z}$ of W (see preceding section), thus giving rise to a Fourier–Jacobi expansion

$$f(u,v) = \sum_{n=0}^{\infty} f_n(u) w^n \quad \text{with } w = e^{2\pi v/\sqrt{3}}.$$

Here the function f_n defines a section of a line bundle $L^{\otimes n}$ on the elliptic curve $E = \mathbb{C}/\sqrt{-3}O_F$. More precisely, the function f_n satisfies for all $\xi \in \sqrt{-3}O_F$ the relation

$$f_n(u+\xi) = \exp\bigl(2\pi n(\bar{\xi}u - \rho\xi\bar{\xi})/\sqrt{3}\bigr) f_n(u).$$

Let L be the line bundle on the elliptic curve E corresponding to this factor of automorphy for $n=1$. It is the line bundle defined by the divisor class of degree 3 on E represented by $O_F/\sqrt{-3}O_F$ in $\mathbb{C}/\sqrt{-3}O_F$. We know that $\dim H^0(E, L^{\otimes n}) = 3n$ for $n \geq 1$. A generator $-\rho^2 \in \mu_6$ acts on the space of sections of L with eigenvalues $-\rho^2, 1, -1$. We choose a basis of eigenfunctions X, $Y+Z$, and $Y-Z$ of $\Gamma(E,L)$ for this action of μ_6, with X,Y,Z as in Finis [5, p. 157]. Then the sections X,Y,Z satisfy the equation $X^3 = \rho(Y^3 - Z^3)$. We have a standard basis of $H^0(E, L^n)$

$$\{X^a Y^b Z^c : 0 \leq a \leq 2, 0 \leq b \leq n-a, a+b+c=n\}.$$

The endomorphism ring O_F of E acts on the space $H^0(E,L^n)$ via the so-called *Shintani operators*

$$m_\alpha : H^0(E, L^n) \to H^0(E, L^{n\mathrm{N}(\alpha)}), \qquad s(z) \mapsto s(\alpha z).$$

There are also operators in the other direction:

$$t_\alpha : H^0(E, L^{n\mathrm{N}(\alpha)}) \to H^0(E, L^n),$$

given by

$$s(z) \mapsto \sum_c s\bigl(\alpha^{-1}(z+c)\bigr) e^{2\pi n(\rho\mathrm{N}(c) - \bar{c}z)/\sqrt{3}},$$

where c runs over a complete set of representatives for $\sqrt{-3}O_F/\alpha\sqrt{-3}O_F$. We refer to [5] and the literature given there.

Because the isotropy group of a cusp in S_4 is isomorphic to $S_3 = \{\sigma \in S_4 : \sigma(1) = 1\}$, we find an action of S_3 on the Fourier–Jacobi expansion of a Picard modular form. This action is given by $(X, Y, Z) \mapsto (-X, Z, Y)$ for $R_2 \sim (34)$ (Finis's notation in [5, p. 153]) and $(X, Y, Z) \mapsto (X, \rho Y, \rho^2 Z)$ for $R_3 \sim (234)$.

§5. The Fourier–Jacobi expansion for vector-valued modular forms

For a vector-valued modular form on $\Gamma_1[\sqrt{-3}]$, the invariance under the unipotent radical given in Section 3 implies that in the Fourier–Jacobi expansion the functions f_n satisfy the relation

$$f(u+\xi, v+\eta-\bar{\xi}u) = \mathrm{Sym}^j \begin{pmatrix} 1 & \bar{\xi} \\ 0 & 1 \end{pmatrix} f(u,v),$$

and this implies that

$$f_n(u+\xi) = \exp\big(2\pi n(\bar{\xi}u - \rho\bar{\xi}\xi)/\sqrt{3}\big)\, \mathrm{Sym}^j \begin{pmatrix} 1 & \bar{\xi} \\ 0 & 1 \end{pmatrix} f_n(u). \tag{4}$$

The functions $f_n = (f_n^{(1)}, \ldots, f_n^{(j+1)})$ represent sections of a vector bundle A_n of rank $j+1$ on the elliptic curve. The vector bundle A_n is a tensor product $L^{\otimes n} \otimes \mathrm{Sym}^j A$, with A given on $\mathbb{C}$ by the cocycle

$$\xi \mapsto \begin{pmatrix} 1 & \bar{\xi} \\ 0 & 1 \end{pmatrix}.$$

This vector bundle A is an indecomposable bundle, and hence A_n has a filtration with $j+1$ quotients isomorphic to $L^{\otimes n}$.

Proposition 5.1. *Let f be a vector-valued modular form of weight (j,k) and character* $\det^\ell$ *on $\Gamma[\sqrt{-3}]$. If f is not zero, then $j \equiv k \pmod 3$. Moreover, f is a cusp form if $\ell \not\equiv j \pmod 3$.*

Proof. The first statement follows by looking at the action of $\rho 1_3$. For the second, we write $f = \sum_n f_n w^n$. Equation (4) implies that in the constant vector f_0, all but the first coordinate are zero. If we apply $\mathrm{diag}(1,1,\rho)$, we find that $f_0(u) = f_0(\rho u) = \rho^\ell\, \mathrm{Sym}^j(\mathrm{diag}(\rho^2, 1)) f_0(u)$, implying that if $f_0 \neq 0$, we must have $\ell \equiv j \pmod 3$. ∎

§6. The Hecke rings

Finis [5] analyzed the Hecke rings for the arithmetic groups Γ and $\Gamma_1[\sqrt{-3}]$. These Hecke rings are the same outside 3 and are generated by operators $T(\nu)$, $T(\nu,\nu)$ for $\nu \in O_F$ with $\mathrm{N}(\nu)=p$, a prime congruent to 1 mod 3, and operators $T(p)$, $T(p,p)$ for primes $p \equiv 2 \pmod 3$ in O_F and for Γ operators $T(\sqrt{-3})$ and $T(\sqrt{-3},\sqrt{-3})$.

If $\Gamma g \Gamma$ is a double coset that can be written as a finite disjoint union of left cosets $\sum \Gamma g_i$, then the action of the corresponding operator T on modular forms in $M_{j,k}(\Gamma)$ is given by

$$Tf = \det(g)\eta(g)^{k-3} \sum_i f_{|j,k} g_i.$$

There is a Petersson scalar product for pairs (f,g) in $M_{j,k}(\Gamma)$ such that one of them is a cusp form. The Hecke operator $T(\nu)$ (resp., $T(\nu,\nu)$) with $\mathrm{N}(\nu)=p \equiv 1 \pmod 3$ is adjoint with $T(\bar{\nu})$ (resp., $T(\bar{\nu},\bar{\nu})$) and for $p \equiv 2 \pmod 3$ $T(p)$ (resp., $T(p,p)$) is self-adjoint. As a result, these Hecke operators are simultaneously diagonalizable, and the eigenvalues λ_ν of an eigenform are algebraic integers if $k \geq 3$ satisfying $\lambda_{\bar{\nu}} = \bar{\lambda}_\nu$.

For the congruence subgroup $\Gamma_1[\sqrt{-3}]$, we need Hecke operators $T(\nu)$ and $T(p)$ for primes congruent to 1 (mod 3). We will write T_ν for $T(\nu)$ and T_{-p} for $T(-p)$ for rational primes $p \equiv 2 \pmod 3$. For our calculations, we now need disjoint left coset decompositions of the double cosets representing the Hecke operators T_ν for ν with $\mathrm{N}(\nu)=p$, a prime $\equiv 1 \pmod 3$, and T_{-p} for the primes $p \equiv 2 \pmod 3$.

Lemma 6.1 ([5, Proposition 5]). *Write $\Gamma' = \Gamma_1[\sqrt{-3}]$. Then for $\nu \equiv 1 \pmod 3$, the operator T_ν is represented by*

$$\begin{aligned} T_\nu &= \Gamma' \begin{pmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & \nu \end{pmatrix} \Gamma' \\ &= \Gamma' \begin{pmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix} \oplus \bigoplus_{a,c} \Gamma' \begin{pmatrix} 1 & a & c \\ 0 & p & 0 \\ 0 & -\nu\bar{c} & \nu \end{pmatrix} \oplus \bigoplus_{b} \Gamma' \begin{pmatrix} \nu & A(b) & -\bar{b} \\ 0 & \nu & 0 \\ 0 & b & \bar{\nu} \end{pmatrix}, \end{aligned}$$

where (a,c) runs through the set of pairs $\{(\rho\mathrm{N}(c)+\sqrt{-3}n, c) : c \in O_F \bmod (\nu), n \in \mathbb{Z} \bmod p\}$ and b through $O_F \bmod (\nu)$, and the algebraic integer $A(b) \in O_F$ is uniquely determined $\bmod\, \nu$ by $\mathrm{Tr}(A(b)\bar{\nu}) = -\mathrm{N}(b)$.

Moreover, for a prime $p \equiv 2 \pmod 3$, *the operator* T_{-p} *is represented by*

$$\begin{aligned} T_{-p} &= \Gamma' \begin{pmatrix} 1 & 0 & 0 \\ 0 & p^2 & 0 \\ 0 & 0 & -p \end{pmatrix} \Gamma' \\ &= \Gamma' \begin{pmatrix} p^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -p \end{pmatrix} \oplus \bigoplus_{a,c} \Gamma' \begin{pmatrix} 1 & a & c \\ 0 & p^2 & 0 \\ 0 & p\bar{c} & -p \end{pmatrix} \\ &\quad \oplus \bigoplus_{m} \Gamma' \begin{pmatrix} -p & \sqrt{-3}m & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}, \end{aligned}$$

where (a, c) *runs through the set* $\{(\rho \mathrm{N}(c) + \sqrt{-3}n, c) : c \in O_F \bmod (p), c \equiv 0 \pmod{\sqrt{-3}}, n \in \mathbb{Z} \bmod p^2\}$ *and* m *through* $(\mathbb{Z}/p\mathbb{Z})^*$.

Both Shintani and Finis showed how the Hecke operators act on the Fourier–Jacobi expansion in the scalar-valued case. We quote from Finis [5, Proposition 6]. Let $f = \sum_n f_n w^n$ be the Fourier–Jacobi expansion, and let $g = T_\nu f$ (resp., $g = T_{-p} f$) have Fourier–Jacobi expansion $g = \sum_n g_n w^n$; then we have for the case $T = T_\nu$ for g_n the expression

$$g_n = \nu p^{k-2} m_\nu(f_{n/p}) + \nu^{-1} t_{\bar{\nu}}(f_{np}) + \bar{\nu}^{k-2} \nu^{-1} t_\nu m_{\bar{\nu}}(f_n),$$

while for the case $T = T_{-p}$ we have the expression

$$g_n = (-p)^{2k-3} m_{-p}(f_{n/p^2}) + (-p)^{k-3} \big(p \mathbf{1}_{\mathbb{Z}}(n/p) - 1\big) f_n - t_{-p}(f_{np^2})/p,$$

with $\mathbf{1}_{\mathbb{Z}}$ the characteristic function of $\mathbb{Z}$ and $f_m = 0$ if $m \notin \mathbb{Z}$. Note that t_ν is defined in Section 4.

We now give a partial analogue for the vector-valued case. We write $f = \sum_n f_n w^n$ and $g = Tf = \sum_n g_n w^n$, where

$$f_n = \begin{pmatrix} f_n^{(1)} \\ \vdots \\ f_n^{(j+1)} \end{pmatrix} \qquad \text{and} \qquad g_n = \begin{pmatrix} g_n^{(1)} \\ \vdots \\ g_n^{(j+1)} \end{pmatrix}.$$

We give the action on the last coordinate.

LEMMA 6.2. *We have for $T = T_\nu$ with $\mathrm{N}(\nu) = p \equiv 1 \pmod 3$ a prime*

$$g_n^{(j+1)} = \nu p^{k-2}\left(p^j m_\nu f_{n/p}^{(j+1)} + p^{1-k} t_{\bar\nu} f_{np}^{(j+1)} + \nu^{j-k} t_\nu m_{\bar\nu} f_n^{(j+1)}\right),$$

where we put $f_{n/p}^{(j+1)} = 0$ if $n/p \notin \mathbb{Z}$.

For T_{-p} with p a prime $\equiv 2 \pmod 3$, we have for $g_n^{(j+1)}$ the expression

$$(-p)^{k+j-3}\big(p\mathbf{1}_{\mathbb{Z}}(n/p) - 1\big) f_n^{(j+1)} - p^{2j+2k-3} m_{-p}(f_{n/p^2}^{(j+1)}) - t_{-p}(f_{np^2}^{(j+1)})/p.$$

Proof. Since the left coset representatives g of T_ν and T_{-p} act by upper triangular factors of automorphy, we can express $g_n^{(j+1)}$ solely in terms of the last component of f_n. An explicit calculation gives the result. ∎

§7. The ring of scalar-valued Picard modular forms

We recall the structure of the rings of modular forms on $\Gamma_1[\sqrt{-3}]$, $\Gamma[\sqrt{-3}]$, and Γ as obtained by Feustel and Holzapfel (see also [5]). The ring $M(\Gamma[\sqrt{-3}])$ is polynomial ring

$$M\big(\Gamma[\sqrt{-3}]\big) = \mathbb{C}[\varphi_0, \varphi_1, \varphi_2],$$

with $\varphi_i \in M_3(\Gamma[\sqrt{-3}]) = M_{0,3}(\Gamma[\sqrt{-3}])$ given by their Fourier–Jacobi expansions; in fact, $\varphi_\nu = \vartheta_\nu^3$ for $\nu = 0,1,2$, with

$$\vartheta_\nu = \sum_{\xi \in O_F} \rho^{-\nu\,\mathrm{Tr}(\xi)} m_\xi(Y) w^{\mathrm{N}(\xi)}.$$

Here m_ξ is the endomorphism of $\bigoplus_n H^0(E, L^{\otimes n})$ defined in Section 4; we have

$$\varphi_0 = 1 + (9Y + 9Z)w + (27Y^2 + 54YZ + 27Z^2)w^2 + (36Y^3 + 81Y^2Z + 81YZ^2 + 36Z^3)w^3 + \cdots.$$

The expansions of φ_i are obtained by substituting $(\rho^i Y, \rho^{2i} Z)$ for (Y, Z), as follows from the definition of ϑ_ν.

Notation. Before we proceed, a word about our notation for representations of S_4. The irreducible representations of S_4 correspond to the partitions of 4 and are denoted by $s[4], s[3,1], s[2,2], s[2,1,1]$, and $s[1,1,1,1]$. They are of dimensions $1,3,2,3,1$. Here $s[4]$ is the trivial and $s[1,1,1,1]$

is the alternating representation. The representation $s[3,1]$ is given by the permutation representation on $\sum_{i=1}^{4} x_i = 0$ in $(x_1, \dots, x_4)$-space.

The group $\Gamma/\Gamma[\sqrt{-3}] \cong S_4 \times \mu_2$ acts on $M_3(\Gamma[\sqrt{-3}])$; the generator of μ_2 acts by -1 on this space, while the representation of S_4 is the irreducible representation $s[2,1,1]$. More precisely, define forms $x_1, \dots, x_4$ in $M_3(\Gamma[\sqrt{-3}])$ by

$$\varphi_0 + \varphi_1 + \varphi_2, \qquad -3\varphi_0 + \varphi_1 + \varphi_2, \qquad \varphi_0 - 3\varphi_1 + \varphi_2, \qquad \varphi_0 + \varphi_1 - 3\varphi_2.$$

In this way, we have generators $x_1, \dots, x_4$ with $\sum x_i = 0$, and $\sigma \in S_4$ acts by $x_i \mapsto \operatorname{sign}(\sigma) x_{\sigma(i)}$.

The ring $M(\Gamma_1[\sqrt{-3}])$ of modular forms on $\Gamma_1[\sqrt{-3}]$ is an extension of degree 3 of $M(\Gamma[\sqrt{-3}])$ by a modular form

$$\zeta \in S_6\big(\Gamma[\sqrt{-3}], \det\big)$$

satisfying a relation

$$\zeta^3 = \frac{-\rho}{\sqrt{-3}3^7} \varphi_0\varphi_1\varphi_2(\varphi_1 - \varphi_0)(\varphi_2 - \varphi_0)(\varphi_2 - \varphi_1). \tag{5}$$

In fact, ζ is given by its Fourier–Jacobi expansion

$$(1/6) \sum_{\xi \in O_F} \xi^5 m_\xi(X) w^{\mathrm{N}(\xi)}.$$

Concretely,

$$\begin{aligned}\zeta = Xw - 27XYZw^3 + (32XY^3 + 32XZ^3)w^4 \\ + (-211XY^6 + 136XY^3Z^3 - 211XZ^6)w^7 + \cdots .\end{aligned}$$

The action of S_4 on ζ is by the sign character.

Since -1_3 acts on $M_k(\Gamma_1[\sqrt{-3}])$ by $(-1)^k$, we find the decomposition under $\Gamma/\Gamma_1[\sqrt{-3}]$

$$M_k\big(\Gamma_1[\sqrt{-3}]\big) = M_k\big(\Gamma[\sqrt{-3}]\big) \oplus M_k\big(\Gamma[\sqrt{-3}], \det\big) \oplus M_k\big(\Gamma[\sqrt{-3}], \det^2\big)$$

with the recursions

$$M_k\big(\Gamma[\sqrt{-3}], \det\big) = M_{k-6}\big(\Gamma[\sqrt{-3}]\big)\zeta$$

and

$$M_k\big(\Gamma[\sqrt{-3}], \det^2\big) = M_{k-12}\big(\Gamma[\sqrt{-3}]\big)\zeta^2.$$

Moreover, we have for $\ell = 1, 2$ (see Proposition 5.1)

$$M_k\big(\Gamma[\sqrt{-3}], \det^\ell\big) = S_k\big(\Gamma[\sqrt{-3}], \det^\ell\big).$$

The ring $M(\Gamma)$ equals the ring of invariants $\mathbb{C}[\varphi_0, \varphi_1, \varphi_2]^{S_4 \times \mu_2}$ and is a polynomial ring generated by elements σ_2, σ_4, and σ_3^2 of weight 6, 12, and 18.

The ring $M(\Gamma_1)$ is the ring of invariants $\mathbb{C}[\varphi_0, \varphi_1, \varphi_2, \zeta]^{S_4}$ and is the quotient of the ring $\mathbb{C}[\sigma_2, \sigma_4, \sigma_3^2, \zeta\sigma_3, \zeta^2]$ by the ideal of relations implied by (5) and the notation (i.e., $(\zeta\sigma_3)^2 = \zeta^2\sigma_3^2$).

The Satake compactification $\Gamma[\sqrt{-3}]\backslash B^*$ of the ball quotient $\Gamma[\sqrt{-3}]\backslash B$ is isomorphic to $\mathbb{P}^2 = \operatorname{Proj}\mathbb{C}[\varphi_0, \varphi_1, \varphi_2]$ (see [6, Chapter I, Section 4]). Viewing $\mathbb{P}^2$ as the hyperplane $\sum_{i=1}^4 x_i = 0$ in $\mathbb{P}^3$, we have six lines (viz., $x_i = x_j$) that make up the divisor of ζ^3.

The cusps are the four points (in the φ_i-coordinates)

$$c_1 = (1:1:1), \qquad c_2 = (1:0:0), \qquad c_3 = (0:1:0), \qquad c_4 = (0:0:1).$$

The surface $\Gamma_1[\sqrt{-3}]\backslash B^*$ is a degree 3 cover of $\Gamma[\sqrt{-3}]\backslash B^*$ branched along the union of the six lines $x_i - x_j = 0$ with $1 \leq i < j \leq 4$. This surface has three singular points (order 3 quotient singularities) corresponding to the three intersections of these lines outside the four cusps, namely, $p_{14,23} = (1:1:0)$, $p_{13,24} = (1:0:1)$, and $p_{12,34} = (0:1:1)$. In $p_{ij,kl}$ we have $x_i = x_j$ and $x_k = x_l$.

REMARK 7.1. The eigenvalues of the action of T_ν on φ_i for ν with $\nu\bar{\nu} = p \equiv 1 \pmod 3$ are $(p+1)\nu + \bar{\nu}^2$, and for T_{-p} these are $-1 - p^3$ (see (9a) and (9b) below).

§8. Expansion of Picard modular forms along a modular curve

Picard modular surfaces contain many modular curves. In the following we need only one curve, namely, the one that in the moduli space interpretation corresponds to the degree 3 covers of $\mathbb{P}^1$ that are hyperelliptic curves of genus 3. This curve consists of six irreducible components and is defined as follows.

Let $\mathcal{H} \to B$ be the embedding of the upper half-plane in B given by $\tau \mapsto (0, \sqrt{-3}\tau)$. The corresponding embedding of algebraic groups $\mathrm{GL}_2 \to G$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & \sqrt{-3}b & 0 \\ c/\sqrt{-3} & d & 0 \\ 0 & 0 & ad-bc \end{pmatrix}.$$

The image defines an algebraic curve in the Satake compactification of $\Gamma_1[\sqrt{-3}]\backslash B$ and $\Gamma[\sqrt{-3}]\backslash B$, and it passes through two cusps. Using the action of S_4, we get six curves C_{ij} with $1 \le i < j \le 4$ on $\Gamma_1[\sqrt{-3}]\backslash B^*$ and six image curves on $\Gamma[\sqrt{-3}]\backslash B^*$. On the latter surface, these curves are given by $x_i = x_j$, as the next lemma shows.

LEMMA 8.1. *The stabilizer in* $\Gamma = G^0(O_F)$ *of the modular curve* $C = C_{34}$ *given by* $u = 0$ *equals*

$$\left\{ g = \begin{pmatrix} a & \sqrt{-3}b & 0 \\ c/\sqrt{-3} & d & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(3), \varepsilon \in O_F^* \right\}.$$

If f is a scalar-valued modular form of weight $(0,k)$ and character $\det^\ell$, then we can develop f in a Taylor expansion along the curve $u = 0$:

$$f(u, \sqrt{-3}\tau) = \sum_{n=0}^{\infty} f_n(\tau) u^n. \tag{6}$$

The functional equation of f implies the following proposition.

PROPOSITION 8.2. *The coefficients* f_n *of* f *in (6) are modular forms of weight* $k+n$ *on* $\Gamma_1(3)$ *and cusp forms for* $n > 0$. *Moreover,* $f_n = 0$ *unless* $n \equiv \ell \pmod 3$.

EXAMPLE 8.3. Writing $\varphi_i = \sum \varphi_{i,n} u^n$, we have

$$\begin{aligned} \varphi_0 &= \varphi_{0,0} + \varphi_{0,6} u^6 + O(u^{12}), \\ \varphi_1 &= \varphi_{1,0} + \varphi_{1,3} u^3 + O(u^6), \\ \varphi_2 &= \varphi_{1,0} - \varphi_{1,3} u^3 + O(u^6), \end{aligned}$$

with

$$\begin{aligned} \varphi_{0,0} &= 1 + 18q + 108q^2 + 234q^3 + 234q^4 + O(q^5), \\ \varphi_{1,0} = \varphi_{2,0} &= \sqrt{-3}\big(1 - 9q + 27q^2 - 9q^3 - 117q^4 + O(q^5)\big). \end{aligned}$$

For the modular form ζ of weight 6, we have

$$\zeta(u, \sqrt{-3}\tau) = \zeta_1 u + \zeta_7 u^7 + O(u^{13}),$$

with $\zeta_1 \in S_7(\Gamma_1(3))$ and $\zeta_7 \in S_{13}(\Gamma_1(3))$.

Similarly, we can develop vector-valued modular forms along the curve C. We write such a modular form $F \in M_{j,k}(\Gamma[\sqrt{-3}], \det^{\ell})$ as

$$F(u, \sqrt{-3}\tau) = \sum_{n=0}^{\infty} \begin{pmatrix} F_n^{(1)} \\ \vdots \\ F_n^{(j+1)} \end{pmatrix} u^n.$$

Proposition 8.4. *The first component $F_n^{(1)}$ is a modular form of weight $k+n$ on $\Gamma_1(3)$ and a cusp form if $n>0$. Moreover, $F_n^{(m)}$ vanishes unless $n+(j+1-m) \equiv \ell \pmod 3$. The function $F_0^{(m)}$ is a modular form of weight $k+m-1$ on $\Gamma_1(3)$, while for $n>0$ the function $F_n^{(m)}$ is a quasi-modular form of weight $k+m+n-1$ on $\Gamma_1(3)$.*

Proof. We refer to [8, Section 1] for the definition of a quasi-modular form. The proof of the first statement follows from writing out the transformation behavior. The second statement follows by applying $\mathrm{diag}(1,1,\rho)$. □

§9. Rankin–Cohen brackets

We now construct vector-valued modular forms by a variant of the Rankin–Cohen brackets. Recall that we have the relation

$$j_2(g,u,v)^{-1} = j_1(g,u,v)\big(J(g,u,v)\big)^t$$

between the automorphy factors and the Jacobian of the group action on the ball B. This implies that, for a differentiable function $f: B \to \mathbb{C}$ with gradient

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial u} \\ \frac{\partial f}{\partial v} \end{pmatrix},$$

we get, using coordinates $b=(u,v)$ on B and writing $g=(g_{ij}) \in G$,

$$\nabla\Big(\frac{f(g\cdot b)}{j_1(g,b)^k}\Big) = -k\frac{f(g\cdot b)}{j_1(g,b)^{k+1}}\begin{pmatrix} g_{23} \\ g_{21}\end{pmatrix} + \frac{1}{j_1(g,b)^{k+1}} j_2(g,b)^{-1}\nabla f(g\cdot b).$$

We can get rid of the first term on the right-hand side by using a bracket.

Definition 9.1. For $k,l \in \mathbb{Z}_{\geq 1}$ and $f,h: B \to \mathbb{C}$ differentiable functions, we put

$$[f,h]_{k,l}(b) = \frac{1}{l} f(b)\nabla h(b) - \frac{1}{k} h(b)\nabla f(b).$$

A straightforward computation leads to the following proposition.

PROPOSITION 9.2. *For every unitary similitude g and functions $f,h: B \to \mathbb{C}$, we have*

$$[f|_k g, h|_l g]_{k,l}(b) = j_1(g,b)^{-k-l-1} j_2(g,b)^{-1}[f,h]_{k,l}(g.b)$$
$$= \big([f,h]_{k,l}|_{1,k+l+1} g\big)(b).$$

Let Γ' be a finite index subgroup of the Picard modular group, and let χ_1, χ_2 be finite order characters. Then we define the bracket for $f \in M_k(\Gamma', \chi_1)$ and $h \in M_l(\Gamma', \chi_2)$ by

$$[f,h] := [f,h]_{k,l}.$$

COROLLARY 9.3. *Let $f \in M_k(\Gamma', \chi_1)$ and $h \in M_l(\Gamma', \chi_2)$ be scalar-valued Picard modular forms. Then $[f,h]$ is a vector-valued modular cusp form in $S_{1,k+l+1}(\Gamma', \chi_1 \cdot \chi_2)$.*

Proof. In view of Proposition 9.2, the only thing to check is that we obtain a cusp form, that is, that the Fourier–Jacobi expansions at the different cusps of the group have no constant terms. This is immediate because differentiation kills constant terms in the Fourier–Jacobi expansions. ∎

§10. Modules of vector-valued Picard modular forms

We denote the vector space of Picard modular forms of weight (j,k) on the group $\Gamma_1[\sqrt{-3}]$ (resp., $\Gamma[\sqrt{-3}]$) by $M_{j,k}(\Gamma_1[\sqrt{-3}])$ (resp., by $M_{j,k}(\Gamma[\sqrt{-3}])$). Note that -1_3 acts by $(-1)^k \operatorname{Sym}^j(-1_2)$; we thus can decompose $M_{j,k}(\Gamma_1[\sqrt{-3}])$ as

$$M_{j,k}\big(\Gamma_1[\sqrt{-3}]\big) = M_{j,k}\big(\Gamma[\sqrt{-3}]\big) \oplus M_{j,k}\big(\Gamma[\sqrt{-3}], \det\big) \oplus \big(\Gamma[\sqrt{-3}], \det^2\big).$$

The corresponding spaces of cusp forms are denoted by $S_{j,k}$. We thus have modules

$$\mathcal{M}_j = \mathcal{M}_j^0 \oplus \mathcal{M}_j^1 \oplus \mathcal{M}_j^2$$

with $\mathcal{M}_j^\ell = \bigoplus_k M_{j,k}(\Gamma[\sqrt{-3}], \det^\ell)$ and for the cusp forms

$$\Sigma_j = \bigoplus_k S_{j,3k+j}\big(\Gamma_1[\sqrt{-3}]\big) = \Sigma_j^0 \oplus \Sigma_j^1 \oplus \Sigma_j^2.$$

Note that by Proposition 5.1 we have $\mathcal{M}_j^\ell = \Sigma_j^\ell$ if $\ell \not\equiv j \pmod 3$. These are modules over $M = \mathcal{M}_0^0 = \bigoplus_r M_{3r}(\Gamma[\sqrt{-3}]) = \mathbb{C}[\varphi_0, \varphi_1, \varphi_2]$.

By using the Hirzebruch–Riemann–Roch theorem, one can show that for $j+3k>4$ we have

$$\dim M_{j,j+3k}\bigl(\Gamma_1[\sqrt{-3}]\bigr) = \frac{3(k-1)(j+1)(j+k)}{2} + \frac{j(j+1)(j+2)}{3} + c,$$

where c is a constant depending only on congruences for j. We have $c = 4,2,2,4$ for $j=0,1,2,3$. We refer to [2] for this. In fact, we have by the holomorphic Lefschetz formula for $j \equiv 2 \pmod 3$ the more precise formula

$$\dim M_{j,j+3k}\bigl(\Gamma[\sqrt{-3}], \det^{\ell}\bigr) = \frac{j+1}{2}k^2 + \frac{j^2-1}{2}k + c',$$

while for $j \not\equiv 2 \pmod 3$ we have

$$\dim M_{j,j+3k}\bigl(\Gamma[\sqrt{-3}], \det^{\ell}\bigr) = \frac{j+1}{2}k^2 + \Bigl(\frac{j^2-1}{2} + \varepsilon\Bigr)k + c'',$$

with ε given by

$$\varepsilon = \begin{cases} 2, & j \equiv \ell \pmod 3, \ell \neq 2, \\ 0, & j \not\equiv \ell \pmod 3, \ell \neq 2, \\ -2, & \ell = 2 \end{cases}$$

and with c' and c'' not depending on k. In fact, given such formulas, our cohomological calculations in [2] determine the constants c, c', and c'' for small j.

§11. Examples of vector-valued Picard modular forms

11.1. Forms in $S_{1,7}$

As a first example, we consider the forms

$$\Phi_0 = -\frac{[\varphi_1,\varphi_2]}{6\pi\sqrt{-1}}, \qquad \Phi_1 = -\frac{[\varphi_2,\varphi_0]}{6\pi\sqrt{-1}}, \qquad \Phi_2 = -\frac{[\varphi_0,\varphi_1]}{6\pi\sqrt{-1}}.$$

By Corollary 9.3 these forms belong to $S_{1,7}(\Gamma[\sqrt{-3}])$, and they are linearly independent, as one sees by calculating the Fourier–Jacobi expansions (see below). Since the φ_i generate the S_4-representation $s[2,1,1]$, the Φ_i generate the S_4-representation $\wedge^2 s[2,1,1] = s[2,1,1]$. To make the action of S_4 more transparent, we define

$$X_1 = \Phi_0 + \Phi_1 + \Phi_2, \qquad X_2 = -\Phi_0, \qquad X_3 = -\Phi_1, \qquad X_4 = -\Phi_2$$

and observe that $\sum_{i=1}^{4} X_i = 0$ and that the action of $\sigma \in S_4$ is by $X_i \mapsto \operatorname{sgn}(\sigma) X_{\sigma(i)}$.

We find the Fourier–Jacobi expansions

$$\begin{pmatrix} \Phi_0^{(1)} \\ \Phi_0^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2\pi}((Y'-Z')w - 3(2YY'+3Y'Z-3YZ'-2ZZ')w^2+\cdots) \\ (Y-Z)w + (-6Y^2+6Z^2)w^2+\cdots \end{pmatrix}.$$

Here the primes refer to the derivative with respect to u. The corresponding expansions for Φ_i are obtained from this one by substituting $(\rho^i Y, \rho^{2i} Z)$ for (Y,Z).

We determine the expansion of the Φ_i along the curve C_{34} given by $\{(0,\sqrt{-3}\tau) : \tau \in \mathcal{H}\} \subset B$. We find that

$$\Phi_0(u,\sqrt{-3}v) = \begin{pmatrix} g_2u^2 + g_8u^8 + O(u^{14}) \\ g_3u^3 + O(u^9) \end{pmatrix},$$

with $g_2 = q - 15q^2 + O(q^3) \in S_9(\Gamma_1(3))$, $g_8 \in S_{15}(\Gamma_1(3))$, and g_3 a quasi-modular form of weight 11 on $\Gamma_1(3)$. Moreover,

$$\Phi_1(u,\sqrt{-3}v) = \begin{pmatrix} h_2u^2 + O(u^5) \\ h_0 + h_3u^3 + O(u^6) \end{pmatrix}$$

and

$$\Phi_2(u,\sqrt{-3}v) = \begin{pmatrix} h_2u^2 + O(u^5) \\ -h_0 + h_3u^3 + O(u^6) \end{pmatrix},$$

with $h_0 \in S_8^{\mathrm{new}}(\Gamma_0(3))$, $h_2 = q + 12q^2 + O(q^3) \in S_9(\Gamma_1(3))$, and h_3 quasi-modular of weight 11 for $\Gamma_1(3)$.

By relation (1) the form $\Phi_1 \wedge \Phi_2$ is a scalar-valued modular form of weight 15 and character $\det^2$. Using the Fourier–Jacobi expansion, we see that

$$\Phi_1 \wedge \Phi_2 = 2\pi\sqrt{-1}(Y'Z - Z'Y)w^2 + \cdots$$

and that, up to a factor $1/Z^2$, the coefficient of the first term is the derivative of Y/Z, which is not constant. Since $\Phi_1 \wedge \Phi_2 \in S_{15}(\Gamma[\sqrt{-3}], \det^2)$, it is divisible by ζ^2, in fact, of the form $f\zeta^2$ with f of weight 3; using the action of S_4, we see that there is a nonzero constant $c \in \mathbb{C}$ such that

$$\Phi_1 \wedge \Phi_2 = c\zeta^2\varphi_0. \tag{7}$$

We now draw an important conclusion about the vanishing locus of the forms Φ_i (or X_i).

COROLLARY 11.1. *The forms X_i with $i=1,2,3,4$ do not vanish outside the union of the modular curves C_{ij}. More precisely, the vanishing locus of X_i consists of the three curves C_{jk}, C_{jl}, and C_{kl} passing through cusp c_i.*

Proof. From the expansions given above, we deduce that Φ_0 vanishes on three of the six C_{ij}. On the other three C_{ij}, the first component vanishes, while the second component is a nonzero modular form of weight 8 on $\Gamma_0(3)$. Since it vanishes on the intersections of the C_{ij}, we see that there cannot be more zeros in view of the formula for the number of zeros of a modular form on $\Gamma_0(3)$. ∎

11.2. Forms in $S_{1,7}(\Gamma[\sqrt{-3}],\det)$

Recall that according to [4, Theorem 5] the form ζ satisfies the identity $\zeta = c_\zeta \prod_{0\le i\le 5} \vartheta_i \in S_6(\Gamma[\sqrt{-3}],\det)$, with $c_\zeta\in\mathbb{C}^*$ and $\varphi_k=\vartheta_k^3$ for $0\le k\le 2$. We form the bracket with one of the forms φ_k with $k=0,1,2$:

$$[\zeta,\varphi_k] = c_\zeta\Big[\prod_{0\le i\le 5}\vartheta_i,\vartheta_k^3\Big] = \frac{c_\zeta}{3}\Big(\prod_{0\le i\le 5}\vartheta_i\Big)\nabla\vartheta_k^3 - \frac{1}{6}\vartheta_k^3\Big(\nabla\Big(\prod_{0\le i\le 5}\vartheta_i\Big)\Big)$$
$$= c_\zeta\vartheta_k^3\Big(\Big(\prod_{\substack{0\le i\le 5\\ i\ne k}}\vartheta_i\Big)\nabla\vartheta_k - \frac{1}{6}\nabla\Big(\prod_{0\le i\le 5}\vartheta_i\Big)\Big).$$

So we can divide by $\vartheta_k^3=\varphi_k$ to obtain $[\zeta,\varphi_k]/\varphi_k\in S_{1,7}(\Gamma[\sqrt{-3}],\det)$. More generally, we put

$$\gamma_{ij} = \frac{1}{x_i-x_j}[\zeta,x_i-x_j] \quad \big(\{i,j,k,l\}=\{1,2,3,4\}\big)$$

and thus obtain six elements in $S_{1,7}(\Gamma[\sqrt{-3}],\det)$ satisfying the relation

$$\sum_{1\le i<j\le 4}\gamma_{ij}=0.$$

These γ_{ij} generate a 5-dimensional space which decomposes as $s[2,2]\oplus s[2,1,1]$ as an S_4-representation. The $s[2,1,1]$-space is generated by the four elements $a_i=\gamma_{jk}+\gamma_{jl}+\gamma_{kl}$ with $\sum a_i=0$, while the $s[2,2]$-space is generated by the three $b_{ij,kl}=\gamma_{ij}+\gamma_{kl}$ satisfying $\sum b_{ij,kl}=0$.

The Fourier–Jacobi expansion of the second component of $6\gamma_{12}$ is

$$-Xw+18X(Y+Z)w^2-27X(2Y^2+YZ+2Z^2)w^3$$
$$+88X(Y^3+Z^3)w^4-18X(11Y^4-Y^3Z-YZ^3+11Z^4)w^5+O(w^6),$$

and the expansion of $\gamma_{13}^{(2)}$ (resp., $\gamma_{14}^{(2)}$) is obtained by substituting $(\rho Y, \rho^2 Z)$ (resp., $(\rho^2 Y, \rho Z)$) for (Y, Z); the expansion for $6\gamma_{34}^{(2)}$ is

$$Xw - 6X(Y+Z)w^2 - 9X(2Y^2 - 3YZ + 2Z^2)w^3 + O(w^4),$$

and then $\gamma_{23}^{(2)}$ and $\gamma_{24}^{(2)}$ are obtained by substituting $(\rho^2 Y, \rho Z)$ (resp., $(\rho Y, \rho^2 Z)$).

The relation between the γ_{ij} and the Φ_i is as follows.

LEMMA 11.2. *We have*

$$\frac{\zeta}{\varphi_1\varphi_2}\Phi_0 = -\frac{16\zeta X_2}{(x_3 - x_1)(x_4 - x_1)} = \frac{1}{3\sqrt{-3}}(\gamma_{13} - \gamma_{14}).$$

Proof. The proof is just a computation. □

11.3. Forms in $S_{1,7}(\Gamma[\sqrt{-3}], \det^2)$ and $S_{1,10}(\Gamma[\sqrt{-3}], \det^2)$

We start by defining a form Ψ_1 in $S_{1,7}(\Gamma[\sqrt{-3}], \det^2)$. The form Ψ_1 is defined as the quotient of the projection of $\varphi_0\varphi_1\Phi_2$ to the $s[1,1,1,1]$-space in $S_{1,13}(\Gamma[\sqrt{-3}])$ divided by ζ:

$$\Psi_1 = \frac{\varphi_0(\varphi_1 - \varphi_0)\Phi_0 - \varphi_2(\varphi_2 - \varphi_1)\Phi_2}{\zeta}.$$

This form behaves in the right way; the only thing to check is that it is holomorphic and a cusp form. Since the divisor of zeta consists of the six curves C_{ij}, we have to check holomorphicity along these curves. But Ψ_1 is S_4-invariant; hence, it suffices to check this along one of the C_{ij}. This can be read off from the Taylor expansion. The form Ψ_1 vanishes at the cusp ∞.

The Fourier–Jacobi expansion of the second component of Ψ_1 is up to a nonzero factor

$$X^2w^2 - 24X^2YZw^4 + 34X^2(Y^3 + Z^3)w^5 - 81X^2Y^2Z^2w^6 + \cdots.$$

The form Ψ_2 in $S_{1,10}(\Gamma[\sqrt{-3}], \det^2)$ is defined as F/ζ, with F given by

$$\varphi_0(\varphi_0 - \varphi_1)(\varphi_0 + \varphi_1 - 3\varphi_2)\Phi_0 - \varphi_2(\varphi_1 - \varphi_2)(\varphi_1 + \varphi_2 - 3\varphi_0)\Phi_2,$$

and the second component of Ψ_2 has Fourier–Jacobi expansion (up to a nonzero factor)

$$X^2w^2 - 6X^2YZw^4 + 70X^2(Y^3 + Z^3)w^5 - 405X^2Y^2Z^2w^6 + \cdots.$$

The group S_4 acts on Ψ_2 by the sign character.

We finish by calculating some wedge products. The form $\Psi_1 \wedge \Psi_2$ is an S_4-anti-invariant scalar-valued modular form of weight 18 with trivial character; in fact,

$$\Psi_1 \wedge \Psi_2 = 2^2 3^7 (\rho - 1) c \zeta^3, \tag{8}$$

with $c \in \mathbb{C}^*$ given in (7). One can also calculate the wedge of Ψ_1 with the space $S_{1,7}(\Gamma[\sqrt{-3}], \det)$:

$$\Psi_1 \wedge \gamma_{1j} = -\frac{c}{6\sqrt{-3}} \zeta^2 (\varphi_0 + \varphi_1 + \varphi_2 - 2\varphi_{j+1}) \quad \text{for } j = 2, 3, 4.$$

This shows, for example, that wedging by Ψ_1 annihilates the $s[2,2]$-subspace of $S_{1,7}(\Gamma[\sqrt{-3}], \det)$.

§12. Low-weight Eisenstein series

In this section we construct Eisenstein series of low weight for our Picard modular group $\Gamma[\sqrt{-3}]$. Note that by Proposition 5.1 the weight of a nontrivial Eisenstein series in $M_{j,k}(\Gamma[\sqrt{-3}], \det^{\ell})$ satisfies $j \equiv \ell \pmod 3$. Eisenstein series exist if $j + k > 4$ (see, e.g., [17]).

Proposition 12.1. *For $j + k > 4$ and $j \equiv \ell \pmod 3$, the space of Eisenstein series in $M_{j,k}(\Gamma[\sqrt{-3}], \det^{\ell})$ has dimension 4, and as an S_4-representation, it is of the form $(s[4] \oplus s[3,1]) \otimes s[1,1,1,1]^{\otimes k}$.*

Proof. Since the group S_4 permutes the cusps, it follows that the representation is either $s[4] \oplus s[3,1]$ or $s[2,1,1] \oplus s[1,1,1,1]$. Let now $E \in M_{j,k}(\Gamma[\sqrt{-3}], \det^{\ell})$ be an invariant or anti-invariant element in the space of Eisenstein series under the action of S_4. Then the matrix $\mathrm{diag}(-1,-1,1) = R_2$ corresponds to the transposition (34) and acts on an Eisenstein series E by

$$\mathrm{diag}\big((-1)^k, (-1)^{k+1}, \ldots, (-1)^{k+j}\big).$$

From the transformation rule (4), it follows that the constant term is a vector $(c^{(1)}, \ldots, c^{(j+1)})^t$ with zero entries $c^{(m)}$ on places $m > 1$ (see the proof of Proposition 5.1). Therefore, the action on E is by $(-1)^k$, and this proves the proposition. ∎

In general, the eigenvalue for T_ν with $\mathrm{N}(\nu) = p \equiv 1 \pmod 3$ of an Eisenstein series of weight (j,k) is

$$(p^{k-2} + 1)\nu^{j+1} + \bar{\nu}^{j+k-1}, \tag{9a}$$

and for T_{-p} with p a prime $\equiv 2 \pmod 3$,

$$(-1)^{j+1}\big(p^{2k+j-3}+p^{j+1}+(-1)^k(p-1)p^{k+j-3}\big). \tag{9b}$$

Now we look at the remaining cases with $j+k\le 4$.

PROPOSITION 12.2. *The nonzero Eisenstein spaces for $j+k\le 4$ are given in the following table as representations of S_4:*

(j,k,ℓ)	$(0,0,0)$	$(0,3,0)$	$(1,1,1)$	$(2,2,2)$
rep	$s[4]$	$s[2,1,1]$	$s[1,1,1,1]$	$s[4]$

Proof. The cases with $(j,k)=(0,0)$ and $(0,3)$ are well known (see Section 7). The space $M_{1,1}(\Gamma[\sqrt{-3}],\det)$ is generated by the form

$$E_{1,1}=\Psi_1/\zeta,$$

with Ψ_1 the S_4-invariant form that generates $S_{1,7}(\Gamma[\sqrt{-3}],\det^2)$. In fact, after multiplication by ζ, our Eisenstein series yields a cusp form of weight $(1,7)$ and character $\det^2$. Since in view of (8) the form Ψ_1 does not vanish outside the curves C_{ij}, it suffices to check the divisibility along the curve $C=C_{34}$. This form does not vanish at every cusp. Similarly, the space $M_{1,4}(\Gamma[\sqrt{-3}],\det)$ is generated by $\varphi_i E_{1,1}$ and by the invariant form Ψ_2/ζ. Finally, the space $M_{2,2}(\Gamma[\sqrt{-3}],\det^2)$ is generated by a form $K_2=\mathrm{Sym}^2(E_{1,1})$. ∎

REMARK 12.3. The eigenvalues of $E_{1,1}$ for the Hecke operators are given by formulas (9a) and (9b). Note that these eigenvalues are not integral.

§13. The structure of the module $\mathcal{M}_1$ of vector-valued modular forms

We will determine the structure of the M-module $\mathcal{M}_1$ of Picard modular forms, and we construct generators for the $\mathcal{M}_1^\ell$ and Σ_1^ℓ. In order to see that these generators exhaust Σ_1, we need the dimension formula

$$\dim S_{1,3k+1}\big(\Gamma_1[\sqrt{-3}]\big)=3k^2-3 \quad \text{for } k\ge 1 \tag{10}$$

given in Section 10. In fact, we have the more precise formula for $k\ge 1$:

$$\dim S_{1,3k+1}\big(\Gamma[\sqrt{-3}],\det^\ell\big)=\begin{cases} k^2-1, & \ell=0,\\ (k+1)^2-4, & \ell=1,\\ (k-1)^2, & \ell=2,\end{cases} \tag{11}$$

but assuming (10), it will follow from our proof. In fact, we will show the existence of a submodule of Σ_1 whose graded part of degree k has dimension $3k^2-3$. Then (10) shows that this exhausts all of Σ_1; hence, we have exhausted Σ_1^i for $i=0,1,2$ as well. Thus, (10) will imply (11).

13.1. The module $\mathcal{M}_1 = \Sigma_1^0$

We give a presentation of this module.

THEOREM 13.1. *There exist cusp forms Φ_i for $i=0,1,2$ of weight $(1,7)$ spanning an irreducible S_4-representation of type $s[2,1,1]$ that generate Σ_1^0 as M-modules, and the module of relations is generated over M by the S_4-invariant relation*

$$\sum_{i=0}^{2} \varphi_i \Phi_i = 0.$$

In particular, we see that as an S_4-representation we have, for $k \geq 1$,

$$S_{1,3k+7}\big(\Gamma[\sqrt{-3}]\big) = s[2,1,1] \otimes \mathrm{Sym}^k\big(s[2,1,1]\big) - \mathrm{Sym}^{k-1}\big(s[2,1,1]\big).$$

Table 1* gives the multiplicities of the irreducible representations of S_4 in $S_{1,1+3k}(\Gamma[\sqrt{-3}])$.

Proof. The forms Φ_i defined in Section 11 belong to $S_{1,7}(\Gamma[\sqrt{-3}])$ and are linearly independent. Because $M_3(\Gamma[\sqrt{-3}])$ is the S_4-representation $s[2,1,1]$, the Φ_i generate the irreducible representation $s[2,1,1] = \wedge^2 s[2,1,1]$. We consider the kernel of the map of M-modules $M \otimes \langle \Phi_0, \Phi_1, \Phi_2 \rangle \to \Sigma_1^0$ given by

Table 1: The space $S_{1,1+3k}(\Gamma[\sqrt{-3}])$ as an S_4-representation.

k	$s[4]$	$s[3,1]$	$s[2,2]$	$s[2,1,1]$	$s[1,1,1,1]$
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	0	$\underline{1}$	0
3	0	$\underline{1}$	$\underline{1}$	$\underline{1}$	0
4	0	2	1	2	1
5	1	3	2	3	1

*An underline indicates a form whose eigenvalues are given in Tables 1–6; italics, lifts; boldface, Eisenstein series.

$f \otimes \Phi_i \mapsto f\Phi_i$. Take an irreducible S_4-representation in the kernel. Suppose that it has dimension at least 2. Then we have two independent relations $\sum_{i=0}^{2} f_i\Phi_i = 0$ and $\sum_{i=0}^{2} g_i\Phi_i = 0$. By suitably subtracting and multiplying with elements of M and using the action of S_4, we obtain a nontrivial relation $h_1\Phi_1 + h_2\Phi_2 = 0$, which implies that $\Phi_1 \wedge \Phi_2$ is zero, contradicting relation (6). Hence, any nontrivial relation corresponds to a 1-dimensional subspace. But this implies that the relation is essentially unique: if we had two such relations that are independent over M, we would obtain by the same argument $\Phi_1 \wedge \Phi_2 = 0$. One can then check that (for $k \geq 1$) the dimension of $\dim S_{1,3k+1}(\Gamma[\sqrt{-3}])$ indeed equals $3\dim M_{0,3k-6}(\Gamma[\sqrt{-3}]) - \dim M_{0,3k-9}(\Gamma[\sqrt{-3}]) = k^2 - 1$, and the result follows. ∎

13.2. The modules $\mathcal{M}_1^1$ and Σ_1^1

THEOREM 13.2. *The M-module $\mathcal{M}_1^1$ is generated by an S_4-anti-invariant Eisenstein series of weight* $(1,1)$ *and an S_4-invariant Eisenstein series of weight* $(1,4)$.

The results are given in Table 2 for the irreducible representations contained in $M_{1,1+3k}(\Gamma[\sqrt{-3}], \det)$.

Proof. Let $E_{1,1} = \Psi_1/\zeta$ and $E_{1,4} = \Psi_2/\zeta$ be the two Eisenstein series. Since $\Psi_1 \wedge \Psi_2$ does not vanish, we cannot have relations of the form $\alpha E_{1,1} + \beta E_{1,4} = 0$ with α and β scalar-valued modular forms. Therefore, these forms generate a submodule with graded piece of degree $1 + 3k$ of dimension equal to $\dim M^0_{3k} + \dim M^0_{3k-3} = (k+1)^2$. ∎

Table 2: The space $M_{1,1+3k}(\Gamma[\sqrt{-3}], \det)$ as an S_4-representation.

k	$s[4]$	$s[3,1]$	$s[2,2]$	$s[2,1,1]$	$s[1,1,1,1]$
0	0	0	0	0	**1**
1	**1**	**1**	0	0	0
2	0	0	*1*	$\underline{1}$ + **1**	**1**
3	**1**+*1*	**1** + 2	*1*	$\underline{1}$	0
4	0	2	2	4	3
5	3	6	3	3	0

From the structure of the module $\mathcal{M}_1^1$, one can deduce the structure of the M-module Σ_1^1. Recall that we constructed forms a_i and $b_{ij,kl}$ in Section 11.2. We summarize our results.

THEOREM 13.3. *There exist three modular forms A_1, A_2, A_3 spanning the $s[2,1,1]$-space of $S_{1,7}(\Gamma[\sqrt{-3}], \det)$ and two modular forms B_1, B_2 spanning the $s[2,2]$-space of $S_{1,7}(\Gamma[\sqrt{-3}], \det)$ that generate Σ_1^1 over M, and the module of relations is generated over M by three relations spanning a representation of type $s[2,1,1]$.*

13.3. The module $\mathcal{M}_1^2 = \Sigma_1^2$

THEOREM 13.4. *There exist an S_4-invariant modular cusp form Ψ_1 in $S_{1,7}(\Gamma[\sqrt{-3}], \det^2)$ and an S_4-anti-invariant form Ψ_2 in $S_{1,10}(\Gamma[\sqrt{-3}], \det^2)$ that generate Σ_1^2 as a free M-module: for $k \geq 1$, we have*

$$S_{1,3k+4}\big(\Gamma[\sqrt{-3}], \det^2\big) = M_{3k-3}\Psi_1 \oplus M_{3k-6}\Psi_2.$$

The results are given in Table 3 for the irreducible representations contained in $S_{1,1+3k}(\Gamma[\sqrt{-3}], \det^2)$.

Proof. Note that by (8) we have $\Psi_1 \wedge \Psi_2 \neq 0$, so there are no relations between Ψ_1 and Ψ_2. This together with the dimension formula proves our claim. ∎

REMARK 13.5. The form Ψ_1 is a lift of

$$f_\pm = \sum a(n)q^n = q \pm 6\sqrt{10}q^2 + 232q^4 + 260q^7 + \cdots$$

in $S_8(\Gamma_0(9))$ with $f_+ + f_-$ lying in the so-called *plus space*; Ψ_1 has eigenvalues $\lambda_\nu = a(p) + \nu^2\bar{\nu}^5$ for ν with $\mathrm{N}(\nu) = p \equiv 1 \pmod 3$.

Table 3: The space $S_{1,1+3k}(\Gamma[\sqrt{-3}], \det^2)$ as an S_4-representation.

k	$s[4]$	$s[3,1]$	$s[2,2]$	$s[2,1,1]$	$s[1,1,1,1]$
0	0	0	0	0	0
1	0	0	0	0	0
2	*1*	0	0	0	0
3	0	0	0	$\underline{1}$	$\underline{1}$
4	1	2	1	0	0
5	0	1	1	3	2
6	3	4	2	2	0

§14. The structure of $\mathcal{M}_2$

14.1. The module $\mathcal{M}_2^0 = \Sigma_2^0$

We begin by constructing some modular forms.

LEMMA 14.1. *The form* $\mathrm{Sym}^2(\Phi_0)/\varphi_1\varphi_2(\varphi_2-\varphi_1)$ *lies in* $S_{2,5}(\Gamma[\sqrt{-3}])$.

Proof. Since $[\varphi_1,\varphi_2]=(\vartheta_1\vartheta_2)^2[\vartheta_1,\vartheta_2]$, we have

$$-36\pi^2\mathrm{Sym}^2(\Phi_0)=(\vartheta_1\vartheta_2)^4\mathrm{Sym}^2\big([\vartheta_1,\vartheta_2]\big),$$

from which it is obvious that we can divide by $\varphi_1\varphi_2$. The stabilizer in S_3 of the space spanned by $\Phi_0\in S_{1,7}(\Gamma[\sqrt{-3}])$ is isomorphic to S_3, and the orbit of φ_1 is $\{\varphi_1,-\varphi_2,\varphi_2-\varphi_1\}$; we thus see that we can divide by $\varphi_2-\varphi_1$ and obtain a modular form in $M_{2,5}(\Gamma[\sqrt{-3}])$.

Using the explicit formulas of the functions ϑ_1 and ϑ_2, we see that the Fourier–Jacobi expansion of the form

$$-36\pi^2\mathrm{Sym}^2(\Phi_0)/\varphi_2\varphi_1(\varphi_2-\varphi_1)=\vartheta_2\vartheta_1\mathrm{Sym}^2\big([\vartheta_1,\vartheta_2]\big)/(\varphi_2-\varphi_1)$$

at the cusp $(1:0:0)$ has no constant term, and using the action of the group S_4 on the cusps, we see the same at the other cusps. ∎

We now put

$$D_0=9\sqrt{-3}\,\frac{\mathrm{Sym}^2(\Phi_0)}{\varphi_1\varphi_2(\varphi_1-\varphi_2)}\in S_{2,5}\big(\Gamma[\sqrt{-3}]\big)$$

with Fourier–Jacobi expansion of the last component of D_0

$$(Y-Z)w+9(Y^3-3Y^2Z+3YZ^2-Z^3)w^3$$
$$+8(Y^4-7Y^3Z+7YZ^3-Z^4)w^4+\cdots,$$

and by the action of R_3 we get forms D_1 and D_2 whose Fourier–Jacobi expansion is obtained by substituting $(\rho^iY,\rho^{2i}Z)$ $(i=1,2)$ for (Y,Z). These are linearly independent and generate an S_4-representation of type $s[3,1]$.

We have $\Phi_0\wedge\Phi_1=c\zeta^2\varphi_2$, $\Phi_0\wedge\Phi_2=-c\zeta^2\varphi_1$, and $\Phi_1\wedge\Phi_2=c\zeta^2\varphi_0$, and from this we get

$$\mathrm{Sym}^2(\Phi_0)\wedge\mathrm{Sym}^2(\Phi_1)\wedge\mathrm{Sym}^2(\Phi_2)=-c^3\zeta^6\varphi_0\varphi_1\varphi_2.$$

We conclude the following.

LEMMA 14.2. *We have the identity*

$$D_0 \wedge D_1 \wedge D_2 = -\rho c^3 \zeta^3.$$

We now have the structure of Σ_2^0, as follows.

THEOREM 14.3. *The M-module Σ_2^0 is generated by the three modular forms D_0, D_1, and D_3 of weight $(2,5)$ that generate an S_4-representation of type $s[3,1]$; for $k \geq 0$ we have*

$$S_{2,3k+5}\big(\Gamma[\sqrt{-3}]\big) = M_{3k}\big(\Gamma[\sqrt{-3}]\big) \otimes \langle D_0, D_1, D_2 \rangle.$$

Proof. Suppose that there is a relation $\sum_{i=1}^{3} f_i D_i$ with f_i modular forms of some weight. Then the wedge $D_0 \wedge D_1 \wedge D_2$ would vanish identically, and this contradicts Lemma 14.2. The dimension formula from [2] that we need is

$$\dim S_{2,3k+2}\big(\Gamma[\sqrt{-3}]\big) = 3k(k+1)/2 \quad \text{for } k \geq 1.$$

The dimension argument is the same as in the beginning of the preceding section. It suffices to have the formula for $\dim S_{2,3k+2}(\Gamma_1[\sqrt{-3}])$. This finishes the proof. ∎

We finish with Table 4 for the irreducible representations contained in $S_{2,2+3k}(\Gamma[\sqrt{-3}])$.

14.2. The module $\mathcal{M}_2^1 = \Sigma_2^1$

We begin by constructing the modular forms in $S_{2,5}(\Gamma[\sqrt{-3}], \det)$ by considering

$$D_0' = \frac{\zeta(D_1 + D_2)}{\varphi_0(\varphi_1 - \varphi_2)}$$

and observing that these are holomorphic using the expansion along the C_{ij}. The action of S_4 thus gives rise to forms D_i' for $i = 0, 1, 2$ in $S_{2,5}(\Gamma[\sqrt{-3}], \det)$. The D_i' generate a representation of type $s[3,1]$.

Table 4: The space $S_{2,2+3k}(\Gamma[\sqrt{-3}])$ as an S_4-representation.

k	$s[4]$	$s[3,1]$	$s[2,2]$	$s[2,1,1]$	$s[1,1,1,1]$
0	0	0	0	0	0
1	0	$\underline{1}$	0	0	0
2	0	1	1	1	1
3	1	3	1	2	0

THEOREM 14.4. *The M-module Σ_2^1 is generated by the three modular forms D_0', D_1', and D_2' of weight $(2,5)$ that generate an S_4-representation of type $s[3,1]$; for $k \geq 0$ we have*

$$S_{2,3k+5}\big(\Gamma[\sqrt{-3}],\det\big) = M_{3k}\big(\Gamma[\sqrt{-3}]\big) \otimes \langle D_0', D_1', D_2'\rangle.$$

Proof. The wedge $D_0' \wedge D_1' \wedge D_2'$ is a nonzero multiple of ζ^3, and this shows that there can be no relations between these generators. The dimension formula now implies the result. ∎

The eigenvalues of Hecke eigenforms in $S_{2,5}(\Gamma[\sqrt{-3}],\det)$ are the F-conjugates of the corresponding ones in $S_{2,5}(\Gamma[\sqrt{-3}])$, as follows from cohomological arguments (see [2]). By Theorems 14.3 and 14.4, the spaces $S_{2,2+3k}(\Gamma[\sqrt{-3}])$ and $S_{2,2+3k}(\Gamma[\sqrt{-3}],\det)$ are isomorphic as S_4-representations.

14.3. The module $\mathcal{M}_2^2$

THEOREM 14.5. *The M-module $\mathcal{M}_2^2$ is freely generated by an S_4-invariant form K_2 of weight $(2,2)$, an S_4-anti-invariant form K_5 of weight $(2,5)$, and an S_4-invariant form K_8 of weight $(2,8)$:*

$$M_{2,2+3k}\big(\Gamma[\sqrt{-3}],\det^2\big) = M_{3k}K_2 \oplus M_{3k-3}K_5 \oplus M_{3k-6}K_8.$$

Table 5 gives the irreducible representations in $M_{2,2+3k}(\Gamma[\sqrt{-3}],\det^2)$ for a few values of k.

Proof. The form $K_2 = \mathrm{Sym}^2(E_{1,1})$; it has an alternative description as $(\varphi_0 D_0 + \varphi_1 D_1 + \varphi_2 D_2)/\zeta$. The form K_5 is the form

$$\big(\varphi_0(\varphi_0 - \varphi_1 - \varphi_2)D_0 + \varphi_1(-\varphi_0 + \varphi_1 - \varphi_2)D_1 + \varphi_2(-\varphi_0 - \varphi_1 + \varphi_2)D_2\big)/\zeta,$$

Table 5: The space $M_{2,2+3k}(\Gamma[\sqrt{-3}],\det^2)$ as an S_4-representation.

k	$s[4]$	$s[3,1]$	$s[2,2]$	$s[2,1,1]$	$s[1,1,1,1]$
0	**1**	0	0	0	0
1	0	0	0	**1**	**1**
2	**1**+*1*	**1**+$\underline{1}$	*1*	0	0
3	0	1	1	4	2
4	4	5	3	2	0

which is S_4-anti-invariant, while $K_8 = \mathrm{Sym}^2(E_{1,4})$. The wedge $K_2 \wedge K_5 \wedge K_8$ is the product of $D_0 \wedge D_1 \wedge D_2$ with a nonzero rational function and hence, by Lemma 14.2, does not vanish. This implies that there can be no relations over M between these generators. Therefore, these elements generate an M-module with graded piece of dimension $(3k^2+3k+2)/2$. By the dimension formula (see Section 10), these generators thus exhaust the whole module $\mathcal{M}_2^2$. ∎

§15. The structure of $\mathcal{M}_3^0$

In this section we give the structure of the M-modules $\mathcal{M}_3^0$ and Σ_3^0. We start by constructing Eisenstein series in weight $(3,3)$. For this we define

$$E_i = \frac{\mathrm{Sym}^3(\Phi_i)}{(\varphi_{i+1}\varphi_{i+2}(\varphi_{i+1}-\varphi_{i+2}))^2}$$

for $i = 0,1,2$ (taken $\bmod 3$) and

$$E_3 = \frac{\mathrm{Sym}^3(\Phi_0+\Phi_1+\Phi_2)}{((\varphi_0-\varphi_1)(\varphi_0-\varphi_2)(\varphi_1-\varphi_2))^2}.$$

LEMMA 15.1. *The forms E_0, E_1, E_2, and E_3 are modular forms of weight $(3,3)$ on $\Gamma[\sqrt{-3}]$, and at each of the four cusps exactly one of these is nonzero. They generate an S_4-representation $s[2,1,1] \oplus s[1,1,1,1]$, and we have*

$$E_0 \wedge E_1 \wedge E_2 \wedge E_3 = c_2\zeta^3 \quad \text{with } c_2 \in \mathbb{C}^*.$$

Proof. Just as above in the proof of Lemma 14.1, we see that $\mathrm{Sym}^3(F_0)$ is divisible by $\varphi_1^2\varphi_2^2$, and then using the action of S_4, we check that it is also divisible by $(\varphi_1-\varphi_2)^2$. The form $E_0+E_1+E_2-E_3$ is anti-invariant under S_4, and for $i=0,1,2$ the forms $4E_i + E_3 - (E_0+E_1-E_2)$ generate an S_4-representation $s[2,1,1]$. One calculates the Fourier–Jacobi expansion of these series. For example, one finds for the fourth component of E_0 the expansion

$$-3^{-5}\big((Y-Z)w + 6(Y^2-Z^2)w^2 + 9(3Y^3+Y^2Z-YZ^2-3Z^3)w^3+\cdots\big)$$

and the ones of E_i $(i=1,2)$ are obtained by substituting $(\rho^iY, \rho^{2i}Z)$ for (Y,Z), while the fourth component of E_3 has an expansion

$$-3^{-3}\big((Y^3-Z^3)w^3 - 18(Y^4Z-YZ^4)w^5+\cdots\big).$$

One also checks that the coefficient of w in the Fourier–Jacobi expansion of E_0 is

$$-3^{-5}\begin{pmatrix}\kappa(Y'-Z')^3/(Y-Z)^2\\ \kappa(Y'-Z')^2/(Y-Z)\\ \kappa(Y'-Z')\\ (Y-Z)\end{pmatrix}$$

but that E_3 does not vanish at the first cusp. From these facts, the proof can be deduced. ∎

The E_i are Hecke eigenforms with eigenvalues as given in formulas (9a) and (9b).

PROPOSITION 15.2. *The M-module $\mathcal{M}_3^0$ is freely generated by the Eisenstein series E_i $i=0,1,2,3$.*

Table 6 gives the irreducible representations in $M_{3,3+3k}(\Gamma[\sqrt{-3}])$ for a few values of k.

Proof. By Lemma 15.1, there can be no relations between the E_i with coefficients in the vector space M_k. Therefore, these Eisenstein series generate an M-module with a graded part of weight $3k+3$ a vector space $M_{3,3k+3}$ of dimension equal to $4\dim M_{3k}$. The dimension formula says that the dimension of $M_{3,3k+3}$ equals $2k^2+6k+4$. This shows that the E_i generate the whole module. ∎

It is easy to check that the forms

$$G_0=\varphi_2E_1-\varphi_1E_2+(\varphi_2-\varphi_1)E_3,$$
$$G_1=-\varphi_2E_0+\varphi_0E_2+(\varphi_0-\varphi_2)E_3,$$
$$G_2=\varphi_1E_0-\varphi_0E_1+(\varphi_1-\varphi_0)E_3$$

Table 6: The space $M_{3,3+3k}(\Gamma[\sqrt{-3}])$ as an S_4-representation.

k	$s[4]$	$s[3,1]$	$s[2,2]$	$s[2,1,1]$	$s[1,1,1,1]$
-1	0	0	0	0	0
0	0	0	0	**1**	**1**
1	**1**	$\mathbf{1}+\underline{1}$	1	$\underline{1}$	0
2	0	2	2	4	2
3	3	6	3	4	1

generate an $s[2,1,1]$-representation in $S_{3,6}(\Gamma[\sqrt{-3}])$. Similarly, the forms

$$H_1 = \varphi_1 E_0 + (\varphi_0 - \varphi_2)E_1 - \varphi_1 E_2 + (\varphi_0 - \varphi_2)E_3,$$
$$H_2 = \varphi_2 E_0 - \varphi_2 E_1 + (\varphi_0 - \varphi_1)E_2 + (\varphi_0 - \varphi_1)E_3$$

generate a representation of type $s[2,2]$ in $S_{3,6}(\Gamma[\sqrt{-3}])$. Finally, the orbit of the form

$$J_0 = 3\varphi_1 E_0 + (\varphi_0 + \varphi_2)E_1 + 3\varphi_1 E_2 + (\varphi_0 - 2\varphi_1 + \varphi_2)E_3$$

generates a representation of type $s[3,1]$. Writing the forms in terms of multiples of E_i with $0 \le i \le 3$ with coefficients from $M_6(\Gamma[\sqrt{-3}])$, it is obvious that we have the invariant relation

(R4) $$\varphi_0 G_0 + \varphi_1 G_1 + \varphi_2 G_2 = 0.$$

We also have an $s[3,1]$-space of relations generated (under the S_4-action) by the following relation among the elements $K_{02} = \varphi_0 E_2$, $K_{12} = \varphi_1 E_2$, $K_{23} = (\varphi_1 - \varphi_0)E_3$, and $K_{13} = (\varphi_2 - \varphi_0)E_3$ of $S_{3,6}(\Gamma[\sqrt{-3}])$:

(R5) $$\varphi_1 K_{02} - \varphi_0 K_{12} - (\varphi_2 - \varphi_0)K_{23} + (\varphi_1 - \varphi_0)K_{13} = 0,$$

which generates a space of relations of type $s[3,1]$.

THEOREM 15.3. *The M-module Σ_3^0 is generated by the S_4-orbits of the forms G_0, H_1, and J_0. The relations are generated over M by the S_4-orbits of the relations (R4) and (R5).*

Proof. We know that the dimension of $M_{3,3k+3} = 2k^2 + 6k + 4$ for $k \ge 0$; hence, $\dim S_{3,3k+3} = 2k^2 + 6k$. Any element of $S_{3,3k+3}$ can be written as a linear combination $\sum_{i=0}^{3} \epsilon_i E_i$, with $\epsilon_i \in M_{3k}$ a modular form that vanishes in the cusp where E_i does not vanish. It is easy to see that these are generated by the G_i, H_i, and the orbit of J_0. ∎

§16. Eigenvalues of Hecke operators

In this section we give a number of eigenvalues for vector-valued modular forms. A number of modular forms we encountered are lifts from $U(1)$ or GL(2). In general, there are Kudla lifts (see [9], [10])

$$S_{b+2}\big(\Gamma_1(3^?)\big) \to S_{a,b+3}\big(\Gamma[\sqrt{-3}], \det^{\ell}\big)$$

with $\ell \equiv a \pmod 3$. Another type of lift is given in [14]:

$$S_{a+b+3}\big(\Gamma_1(3)\big) \to S_{a,b+3}\big(\Gamma[\sqrt{-3}], \det^2\big).$$

In the first case, the eigenvalues for T_ν are $a(p)\nu^{a+1} + \bar{\nu}^{a+b+2}$, and in the second case they are of the form $a(p) + \nu^{a+1}\bar{\nu}^{b+1}$. In all tables below, the cusp form is not a lift and corresponds to a 3-dimensional Galois representation.

16.1. Eigenforms in Σ_1^0

EXAMPLE 16.1. We consider the eigenforms $\Phi_i \in S_{1,7}(\Gamma[\sqrt{-3}])$ with representation $s[2,1,1]$. We give the eigenvalues in Table 7.

EXAMPLE 16.2. The space $S_{1,10}(\Gamma[\sqrt{-3}])$ has a decomposition $s[3,1] \oplus s[2,2] \oplus s[2,1,1]$ as an S_4-representation. In the $s[3,1]$-space and $s[2,2]$-space, there are eigenforms with eigenvalues as given in Table 8.

In the $s[2,1,1]$-part we find an eigenform with eigenvalues as given in Table 9.

16.2. Eigenforms in Σ_1^1

EXAMPLE 16.3. The space $S_{1,7}(\Gamma[\sqrt{-3}], \det)$ equals $s[2,2] \oplus s[2,1,1]$ as a representation of S_4. An eigenform F in the $s[2,1,1]$-space is a lift, so for a prime $p \equiv 1 \pmod 3$ and $\nu \in O_F$ with $\nu \equiv 1 \pmod 3$ of norm p, the eigen-

Table 7: Hecke eigenvalues of Φ_i.

α	p	$\lambda_\alpha(\Phi_i)$
$1+3\rho$	7	$759+261\rho$
$1-3\rho$	13	$-4137+1683\rho$
$-2+3\rho$	19	$24042+14733\rho$
$1+6\rho$	31	$-145401-241830\rho$
$4-3\rho$	37	$12900-114849\rho$
$1-6\rho$	43	$246567-8946\rho$
$4+9\rho$	61	$1048836-173205\rho$
$-2-9\rho$	67	$-1539510-1246887\rho$
$1+9\rho$	73	$-1563729+1261143\rho$
$7-3\rho$	79	$9921297+3294171\rho$
$-8+3\rho$	97	$5678616-3870891\rho$
-2	2	72
-5	5	89622

Table 8: Hecke eigenvalues on the $s[3,1]$- and $s[2,2]$-part of $S_{1,10}(\Gamma[\sqrt{-3}])$.

p	$s[3,1]$	$s[2,2]$
7	$-13515 + 3573\rho$	$15159 + 10863\rho$
13	$-321963 - 290475\rho$	$-95001 + 288351\rho$
19	$2154864 + 1895139\rho$	$-2977296 - 681147\rho$
31	$-4371693 - 1547568\rho$	$-24682119 - 22711896\rho$
37	$-13227720 - 83952837\rho$	$-76866504 + 46681047\rho$
43	$108861123 - 48030912\rho$	$32373957 + 31482576\rho$
61	$1122962232 + 554059467\rho$	$465758040 + 641801907\rho$
67	$878127888 + 1196423595\rho$	$-211962336 + 187424901\rho$
73	$-1637757627 - 2807114427\rho$	$3493044975 + 565725087\rho$
79	$-504410811 - 607778811\rho$	$-3018458193 - 2809124073\rho$
97	$-23598528 - 7910853813\rho$	$-6587510640 - 8420338791\rho$
-2	36	1008
-5	13464990	-11930940

Table 9: Hecke eigenvalues on the $s[2,1,1]$-part of $S_{1,10}(\Gamma[\sqrt{-3}])$.

α	p	$s[2,1,1]$
$1+3\rho$	7	$26985 + 20097\rho$
$1-3\rho$	13	$31521 + 13761\rho$
$-2+3\rho$	19	$1806888 + 842463\rho$
$1+6\rho$	31	$15911679 + 12552264\rho$
$4-3\rho$	37	$81911640 + 71598267\rho$
$1-6\rho$	43	$47737551 - 26870472\rho$
$4+9\rho$	61	$524111736 + 375028731\rho$
$-2-9\rho$	67	$489305208 - 4044033\rho$
$1+9\rho$	73	$-513904983 + 1961971497\rho$
$7-3\rho$	79	$-203501319 - 3483886959\rho$
$-8+3\rho$	97	$4237830912 - 1749247641\rho$
-2	2	-1548
-5	5	-1356390

Table 10: Hecke eigenvalues on the $s[2,2]$-part of $S_{1,7}(\Gamma[\sqrt{-3}], \det)$.

α	p	$\lambda_\alpha(F)$
$1+3\rho$	7	$-294+855\rho$
$1-3\rho$	13	$-2220-1017\rho$
$-2+3\rho$	19	$5817-5841\rho$
$1+6\rho$	31	$-23847-38466\rho$
$4-3\rho$	37	$152301-21375\rho$
$1-6\rho$	43	$-188403-18558\rho$
-2	2	180
-5	5	82764

Table 11: Hecke eigenvalues on the $s[2,1,1]$-part of $S_{1,10}(\Gamma[\sqrt{-3}], \det)$.

α	p	$\lambda_\alpha(F_1)$
$1+3\rho$	7	$-19320-7497\rho$
$1-3\rho$	13	$74208-298521\rho$
$-2+3\rho$	19	$877737+798561\rho$
$1+6\rho$	31	$10127631+22554360\rho$
$4-3\rho$	37	$-80206539-23638131\rho$
$1-6\rho$	43	$-113882937+15496200\rho$
-2	2	-36
-5	5	1289610

value λ_ν satisfies $\lambda_\nu = a_p\nu^2 + \bar{\nu}^7$, with a_p the eigenvalue of $(\eta(3\tau)\eta(\tau))^6 \in S_6(\Gamma_0(3))$. An eigenform in the $s[2,2]$-space has eigenvalues as given in Table 10.

EXAMPLE 16.4. The space $S_{1,10}(\Gamma[\sqrt{-3}], \det)$ decomposes as $s[4] \oplus 2s[3,1] \oplus s[2,2] \oplus s[2,1,1]$. The S_4-invariant eigenform is a lift of a form $g \in S_9(\Gamma_1(9))$, with $g = q + 45q^3 - 284q^4 + 1512q^6 + \cdots$. The eigenform in the $s[2,2]$-space is a lift of a form $g \in S_9(\Gamma_0(9), \psi)$ with q-expansion $q + 238q^4 + 1652q^7 - 4194q^{10} + \cdots$. In the $s[2,1,1]$-part we find an eigenform with eigenvalues as in Table 11.

Table 12: Hecke eigenvalues on $S_{1,10}(\Gamma[\sqrt{-3}], \det^2)$.

ν	p	$\lambda_\nu(\Psi_2)$
$1+3\rho$	7	$-6549 - 17352\rho$
$1-3\rho$	13	$223599 + 133992\rho$
$-2+3\rho$	19	$-492621 - 1294560\rho$
$1+6\rho$	31	$3832419 + 8618040\rho$
$4-3\rho$	37	$56905563 + 49705992\rho$
$1-6\rho$	43	$16590459 + 186818112\rho$
-2	2	-684
-5	5	6541650

Table 13: Hecke eigenvalues on $S_{2,5}(\Gamma[\sqrt{-3}])$ and $S_{2,8}(\Gamma[\sqrt{-3}])$.

ν	p	Weight $(2,5)$, irrep $s[3,1]$	Weight $(2,8)$, irrep $s[2,1,1]$
$1+3\rho$	7	$-105 - 297\rho$	$-3039 - 765\rho$
$1-3\rho$	13	$1137 + 945\rho$	$97707 + 110007\rho$
$-2+3\rho$	19	$-1536 - 891\rho$	$-268962 - 412137\rho$
$1+6\rho$	31	$20577 - 1728\rho$	$4182969 + 2591334\rho$
$4-3\rho$	37	$19200 - 37017\rho$	$-61836 + 6730299\rho$
$1-6\rho$	43	$-113667 - 127872\rho$	$13604205 - 6584742\rho$
$4+9\rho$	61	$-354048 - 242433\rho$	$-67731468 - 1452033\rho$
$-2-9\rho$	67	$271488 + 194805\rho$	$-45800610 + 117273771\rho$
$1+9\rho$	73	$-268107 - 235467\rho$	$-373673625 - 459690417\rho$
$7-3\rho$	79	$114159 + 449199\rho$	$235630047 + 382294197\rho$
$-8+3\rho$	97	$-60288 - 554013\rho$	$-95419824162 - 26086979421\rho$
-2	2	-72	-288
-5	5	-810	1629990

16.3. Eigenforms in Σ_1^2

EXAMPLE 16.5. The space $S_{1,7}(\Gamma[\sqrt{-3}], \det^2)$ is 1-dimensional and S_4-invariant. The generator Ψ_1 is a lift of an element in $S_8(\Gamma_0(9))$. The space $S_{1,10}(\Gamma(\sqrt{-3}), \det^2)$ decomposes as $s[1,1,1,1] \oplus s[2,1,1]$, where the component $s[1,1,1,1]$ is generated by the form Ψ_2 and the component $s[2,1,1]$ is generated by the forms $\varphi_i \Psi_1$. All forms in $S_{1,10}(\Gamma(\sqrt{-3}), \det^2)$ have the same eigenvalues given in Table 12.

Table 14: Hecke eigenvalues on $S_{2,8}(\Gamma[\sqrt{-3}], \det^2)$.

ν	p	Weight (2,8), irrep $s[3,1]$
$1+3\rho$	7	$-2175-1602\rho$
$1-3\rho$	13	$-58947-169740\rho$
$-2+3\rho$	19	$737949-220734\rho$
$1+6\rho$	31	$-90267+2362374\rho$
$4-3\rho$	37	$-8035881-17655156\rho$
$1-6\rho$	43	$6838329-67590\rho$
-2	2	792
-5	5	-408510

Table 15: Hecke eigenvalues on $S_{3,6}(\Gamma[\sqrt{-3}])$.

α	p	$s[2,1,1]$	$s[3,1]$
$1+3\rho$	7	$3189-459\rho$	$273+2457\rho$
$1-3\rho$	13	$-3543+8721\rho$	$-35619-46683\rho$
$-2+3\rho$	19	$29784+118179\rho$	$152256-30537\rho$
$1+6\rho$	31	$-29949-203904\rho$	$-167001-547992\rho$
$4-3\rho$	37	$355296+8667\rho$	$1545024+1338363\rho$
$1-6\rho$	43	$-66741-241272\rho$	$2292303+207792\rho$
$4+9\rho$	61	$-99168+3835107\rho$	$3969904-118989\rho$
$-2-9\rho$	67	$-5321544+13554459\rho$	$11562096+21366423\rho$
$1+9\rho$	73	$-58317351-62040087\rho$	$9680853+35351397\rho$
$7-3\rho$	79	$-44663451-34446411\rho$	$1196481+1434969\rho$
$-8+3\rho$	97	$-52988496+9813663\rho$	$2112240+65593827\rho$
-2	2	-36	-36
-5	5	-563670	-117522

16.4. Eigenforms in Σ_2^0

EXAMPLE 16.6. Table 13 gives the Hecke eigenvalues for the D_i of weight $(2,5)$ and of the forms in the $s[2,1,1]$-part of $S_{2,8}(\Gamma[\sqrt{-3}])$.

16.5. Eigenforms in Σ_2^2

EXAMPLE 16.7. The space $S_{2,8}(\Gamma[\sqrt{-3}], \det^2)$ splits as $s[4] \oplus s[3,1] \oplus s[2,2]$. The S_4-invariant form is a lift of $U(1)$ with eigenvalues $\nu^9 + \nu^3\bar{\nu}^6 + \bar{\nu}^9$.

The $s[2,2]$-forms are lifts from $S_7(\Gamma_1(3))$. Table 14 gives the eigenvalues for the $s[3,1]$-part of $S_{2,8}(\Gamma[\sqrt{-3}], \det^2)$.

16.6. Eigenforms in Σ_3^0

EXAMPLE 16.8. Table 15 gives the eigenvalues for the $s[2,1,1]$-part and the $s[3,1]$-part of $S_{3,6}(\Gamma[\sqrt{-3}])$.

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