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NOTE ON TAUTOLOGICAL CLASSES OF MODULI OF K3 SURFACES

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Dedicated to Michael Tsfasman on the occasion of his 50th birthday

ABSTRACT. In this note, we prove some cycle class relations on moduli of K3 surfaces.

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1. Introduction

This note deals with a few properties of tautological classes on moduli spaces of K3 surfaces. Let $\mathcal{M}_{2d}$ denote a moduli stack of K3 surfaces over an algebraically closed field with a polarization of degree $2d$ prime to the characteristic of the field. The Chern classes of the relative cotangent bundle $\Omega^1_{\mathcal{X}/\mathcal{M}}$ of the universal K3 surface $\mathcal{X}_{2d} = \mathcal{X}$ define classes t_1 and t_2 in the Chow groups $CH^i_{\mathbb{Q}}(\mathcal{X}_{2d})$ of the universal K3 surface over $\mathcal{M}_{2d}$. The class t_1 is the pull-back from $\mathcal{M}_{2d}$ of the first Chern class $v = c_1(V)$ of the Hodge line bundle $V = \pi_*(\Omega^2_{\mathcal{X}/\mathcal{M}})$. We use Grothendieck–Riemann–Roch to determine the push-forwards of the powers of t_2. These are powers of v. We then prove that $v^{18} = 0$ in the Chow group with rational coefficients of $\mathcal{M}_{2d}$. We show that this implies that a complete subvariety of $\mathcal{M}_{2d}$ has dimension at most 17 and that this bound is sharp. These results are in line with those for moduli of abelian varieties. There the top Chern class λ_g of the Hodge bundle vanishes in the Chow group with rational coefficients. The idea is that if the boundary of the Baily–Borel compactification has codimension r, then some tautological class of codimension r vanishes. Our result means that v^{18} is a torsion class. It would be very interesting to determine the order of this class as well as explicit representations of this class as a cycle on the boundary, cf. [EvdG04a], [EvdGb].

2. The Moduli Space $\mathcal{M}_{2d}$

Let k be an algebraically closed field. We consider the moduli space $\mathcal{M}_{2d}$ of polarized K3 surfaces over k with a primitive polarization of degree $2d$. This is a

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19-dimensional algebraic space. Over the complex numbers, we can describe it as an orbifold quotient $\Gamma_{2d}\backslash\Omega_{2d}$, where Ω_{2d} is a bounded symmetric domain and Γ_{2d} is the arithmetic subgroup of SO(3, 19) obtained as follows. Consider the lattice $U^3 \oplus E_8^2$, where U is the hyperbolic plane and E_8 is the usual rank 8 lattice. Let h be an element of this lattice with $\langle h,\, h\rangle = 2d$. Then $L_{2d} = h^{\perp} \cong U^2 \perp E_8^2 \perp \mathbb{Z}u$ with $\langle u,\, u\rangle = -2d$ is a lattice of signature (2, 19), and we put

$$\Omega_{2d} = \{[\omega] \in \mathbb{P}(L_{2d} \otimes \mathbb{C})\colon \langle \omega,\, \omega\rangle = 0,\ \ \langle \omega,\, \bar{\omega}\rangle > 0\}.$$

The group Γ_{2d} is the automorphism group of L_{2d}. It acts on Ω_{2d}, and the quotient (an orbifold) is the analytic space of $\mathcal{M}_{2d}$. It is well known by Baily–Borel that the sections of a sufficiently high power of V give an embedding of $\Gamma_{2d}\backslash\Omega_{2d}$ as a quasi-projective variety.

3. GRR Applied to the Sheaf $\Omega^i_{\mathcal{X}/M}$

To determine the push-forward $\pi_*(t_2^a)$ under $\pi\colon \mathcal{X}_{2d} \to \mathcal{M} := \mathcal{M}_{2d}$, we apply Grothendieck–Riemann–Roch to the structure sheaf of the universal (polarized) K3 surface $\pi\colon \mathcal{X} \to \mathcal{M}$. We work in the Chow ring with rational coefficients. We have

$$\mathrm{ch}(\pi_!\mathcal{O}_{\mathcal{X}}) = \pi_*(\mathrm{ch}(\mathcal{O}_{\mathcal{X}})\,\mathrm{Td}^{\vee}(\Omega^1_{\mathcal{X}/\mathcal{M}})) = \pi_*(\mathrm{Td}^{\vee}(\Omega^1_{\mathcal{X}/\mathcal{M}})).$$

As to the left-hand side, we have $\pi_!\mathcal{O}_{\mathcal{X}} = 1 + V^{\vee}$, where $V = R^0\pi_*\Omega^2_{\mathcal{X}/\mathcal{M}}$ is the line bundle with fiber $H^0(X,\, \Omega^2_X)$ over $[X]$. We write v for the first Chern class of this bundle on $\mathcal{M}$. So the left-hand side is $1 + e^{-v}$. For the right-hand side, note that the determinant bundle of $\Omega^1_{\mathcal{X}/\mathcal{M}}$ is a line bundle that is trivial on each K3 surface that is a fiber of π. Therefore, this line bundle is a pull-back from $\mathcal{M}$, and we can identify it with $\pi^*(V)$. If we denote the Chern classes of $\Omega^1_{\mathcal{X}/\mathcal{M}}$ by $t_i = c_i(\Omega^1_{\mathcal{X}/\mathcal{M}})$, then the right-hand side has the form

$$\pi_*(1 - t_1/2 + (t_1^2 + t_2)/12 - t_1t_2/24 + \cdots).$$

Comparing the degree 0 terms gives $1+1 = 24/12$, since $c_1^2(X) = 0$ and $c_2(X) = 24$ for a K3 surface. For the terms of degree 1, we find $-v = \pi_*(-t_1t_2)/24 = -v \cdot (\pi_*(t_2)/24)$, and this is in agreement. The degree 2 terms yield $\pi_*(t_2^2) = 88\, t_1^2$. This is in agreement with the next term:

$$v^3/6 = \frac{1}{1440}\pi_*(3t_2^2t_1 - t_2t_1^3).$$

Continuing this way, we can determine $\pi_*(t_2^j)$ for all $j \geq 1$. More precisely, put $B(x) = x/(1 - e^{-x})$ and write γ_1 and γ_2 for the Chern roots of $\Omega^1_{\mathcal{X}/\mathcal{M}}$. Then

$$\mathrm{Td}^{\vee}(\Omega^1_{\mathcal{X}/\mathcal{M}}) = B(\gamma_1)B(\gamma_2) = \sum_{n,j\,:\ 0\leq 2j\leq n} c(n,\, j)(\gamma_1 + \gamma_2)^{n-2j}(\gamma_1\gamma_2)^j$$

with $t_1 = \gamma_1 + \gamma_2$ and $t_2 = \gamma_1\gamma_2$, and the Riemann–Roch identity says that if $\pi_*(t_2^{n+1}) = a_n v^{2n}$, then the a_n satisfy the relation

$$\sum_{j\geq 0} a_j\, c_{n,j} = \begin{cases} 1/(n-2)! & \text{for } n \equiv 0 \pmod 2,\ \ n > 2,\\ 2 & \text{for } n = 2.\end{cases}$$

Denoting $\pi_*(t_2^{n+1}) = a_n v^{2n}$, we find the following values for a_n.

n	$\pi_*(t_2^{n+1})/v^{2n}$
0	24
1	88
2	184
3	352
4	736
5	1295488/691
6	4292224/691
7	68418650624/2499347
8	17412311922527744/109638854849
9	22654813560476770158592/19144150084038739

Proposition 3.1. *Write $\pi_*(t_2^{n+1}) = a_n v^{2n}$ for $n \geq 0$. The generating function*

$$A(t) = \sum_{n=0}^{\infty} a_n\, t^n = 24 + 88\,t + 184\,t^2 + \cdots$$

is uniquely characterized by the property that the coefficient of t^{2n-1} in

$$\frac{2-t}{1-t} A\left(\frac{-t^2}{1-t}\right)$$

is equal to $4n/B_{2n}$ for every $n > 0$. Here B_m is the mth Bernoulli number.

Although the numbers a_n are defined for all $n \geq 0$, they apparently have a geometric interpretation only for $n \leq 9$.

We now apply Grothendieck–Riemann–Roch to the sheaf $\Omega^1_{\mathcal{X}/\mathcal{M}}$, or, equivalently, to its dual $\Theta_{\mathcal{X}/\mathcal{M}}$. It says that

$$\mathrm{ch}(\pi_! \Theta_{\mathcal{X}/\mathcal{M}}) = \pi_*(\mathrm{ch}(\Theta_{\mathcal{X}/\mathcal{M}})\, \mathrm{Td}^\vee(\Omega^1_{\mathcal{X}/\mathcal{M}})).$$

Note that $\pi_! \Theta_{\mathcal{X}/\mathcal{M}} = R^1\pi_*\Theta_{\mathcal{X}/\mathcal{M}}$ on the left-hand side, since a K3 surface has no nonzero vector fields [RS81]. Since the push-forwards of powers of t_2 are powers of v and $t_1 = \pi^*(v)$, we see that $\mathrm{ch}(\pi_!\Theta_{\mathcal{X}/\mathcal{M}})$ is a polynomial in v. This can be determined by looking at cohomology once we show that the tautological ring of $\mathcal{M}_{2d}$ is $\mathbb{Q}[v]/v^{18}$.

Note that the fiber of $R^1\pi_*\Theta_{\mathcal{X}/\mathcal{M}}$ is $H^1(X, \Theta_X)$, the space of infinitesimal deformations of X. The tangent space to M at $[X]$ can be identified with the orthogonal complement of h, the hyperplane class in $H^1(X, \Omega^1_X) = H^1(X, \Theta_X)$. On the other hand, we know that Hodge theory gives the following description of this tangent space. Let

$$0 \subset F^2 \subset F^1 = (F^2)^\perp \subset H^2_{\mathrm{dR}}$$

be the Hodge filtration on $H^2_{\mathrm{dR}}(X)$, and let h be the hyperplane class that gives a section of $H^2_{\mathrm{dR}} \otimes O_{\mathcal{M}}$. Then the tangent space to $\mathcal{M}$ can be identified with $\mathrm{Hom}(F^2, (F^1 \cap h^\perp)/F^2)$. Using the cup product, we can identify $(F^2)^\vee$ with H^2_{dR}/F^1; i.e., in the Grothendieck group we have $[H^2_{\mathrm{dR}}] = V^\vee + V^\perp$, where we identify F^2 with V. Now consider the restriction to the orthogonal subbundle

$h^\perp$ of the hyperplane class h whose class in the Grothendieck group is $[H^2_{\mathrm{dR}}] - 1$. Therefore, we find $[\Theta_{\mathcal{M}}] = [(H^2_{\mathrm{dR}} - 1 - V - V^{-1}) \otimes V^{-1}]$.

Proposition 3.2. *The relation* $[\Theta_{\mathcal{M}}] = [H^2_{\mathrm{dR}} - 1] \otimes V^{-1} - 1 - V^{-2}$ *holds in the Grothendieck group of* $\mathcal{M}$.

In view of the Gauss–Manin connection on H^2_{dR}, we see that the Chern classes vanish in cohomology, the total Chern class of the bundle F^1 on $\mathcal{M}$ is $1/(1-v)$, and $\mathrm{ch}(\Theta_{\mathcal{M}}) = -1 + 21e^{-v} - e^{-2v}$. In particular, we find that $c_1(\Theta_{\mathcal{X}/\mathcal{M}}) = -19v$. We have already seen that the total Chern class of $R^1\pi_*(\Omega^1_{\mathcal{X}/\mathcal{M}})$ is $1/(1-v^2)$ and so its first Chern class vanishes. This is in agreement with the global duality $(R^1\pi_*\Theta_{\mathcal{X}/\mathcal{M}})^\vee \cong (R^1\pi_*\Omega^1_{\mathcal{X}/\mathcal{M}}) \otimes V$.

4. Vanishing of Tautological Classes in Characteristic Zero

Let $\mathrm{II}_{3,19}$ be the unique even unimodular lattice of signature (3, 19), and let S be some Lorentzian sublattice of $\mathrm{II}_{3,19}$, say, of signature $(1, m)$. Recall that an S-K3 surface X is a K3 surface with a fixed primitive embedding of S in the Picard group such that the image of S contains a semi-ample class, i.e., a class D such that $D^2 > 0$ and $D \cdot C \geq 0$ for all curves C on the K3 surface X, cf. [BKPSB98]. The period space Y of marked S-K3 surfaces is an orbifold that is a quotient of an Hermitian symmetric domain of dimension $19 - m$ by an arithmetic subgroup of $\mathrm{Aut}(S^\perp)$.

Theorem 4.1. *For* $m \leq 16$*, the cycle class* v^{18-m} *vanishes in the Chow group* $CH^{18-m}_{\mathbb{Q}}(Y)$ *with rational coefficients.*

Proof. By imposing a level structure, we can replace our period space by a finite cover and assume that we are working with a fine moduli space.

The proof is by descending induction on m. For $m = 16$, the period domain can be identified with the Siegel upper half-space $\mathcal{H}_2$ and the orbifold Y can be viewed as a moduli space of abelian surfaces. It thus carries a natural vector bundle, the Hodge bundle $\pi_*(\Omega^1_{\mathcal{X}/Y})$ with Chern classes λ_1 and λ_2. One shows that $\lambda_1 = v$ by comparing the factors of automorphy or by noticing that $H^0(X, \Omega^2_X) \cong \bigwedge^2 H^0(\Omega^1_X)$ for an abelian surface. Furthermore, it is known that λ_1^2 vanishes by [vdG99, Prop. 2.2]. We conclude that v^2 vanishes.

The induction step is now provided by Theorem 1.2 in [BKPSB98]. There exists a modular form Φ of weight $k \geq 12$ whose zero-divisor is of the form $\sum m_i W_i$ with $m_i \in \mathbb{Z}_{>0}$ and with orbifolds W_i that are images in Y of quotients $\Gamma_{L_t}\backslash\Omega_{L_t}$ under finite maps. Here Ω_{L_t} is an Hermitian symmetric domain of dimension one less than the original domain Ω, and the quotient parametrizes a family of S'-K3 surfaces with $S' \supset S$ of signature $(1, m+1)$. We know by induction that the class v^{17-m} vanishes on each of the orbifolds $\Gamma_{L_t}\backslash\Omega_{L_t}$. The zero-divisor of Φ represents the class $k\,v$. We thus find that a nonzero multiple of $v^{18-m} = v \cdot v^{17-m}$ vanishes. □

In characteristic 0, we can use the existence of the Satake compactification whose boundary is 1-dimensional to conclude that intersecting twice with a sufficiently general hyperplane yields a complete 17-dimensional subvariety of $\mathcal{M}$. Since the class v is ample by Baily–Borel, this shows that $v^{17} \neq 0$.

Corollary 4.2. *The tautological ring of* $\mathcal{M}_{2d}$ *is* $\mathbb{Q}[v]/(v^{18})$.

Corollary 4.3. *The maximal dimension of a complete subvariety of* $\mathcal{M}_{2d}$ *is* 17.

In positive characteristic, the locus of K3 surfaces with height ≥ 3 defines a complete subvariety of dimension 17, cf. [vdGK00].

If $\mathcal{M}^*$ is the Baily–Borel compactification of $\mathcal{M}$, then the "boundary" is a 1-dimensional cycle. In the Chow group $CH^{18}_{\mathbb{Q}}(\mathcal{M}^*)$, the class v^{18} is represented by a 1-cycle with support on the boundary.

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