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COVARIANTS OF BINARY SEXTICS AND MODULAR FORMS OF DEGREE 2 WITH CHARACTER

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ABSTRACT. We use covariants of binary sextics to describe the structure of modules of scalar-valued or vector-valued Siegel modular forms of degree 2 with character, over the ring of scalar-valued Siegel modular forms of even weight. For a modular form defined by a covariant we express the order of vanishing along the locus of products of elliptic curves in terms of the covariant.

1. Introduction

In [4] we describe a map from covariants of binary sextics to Siegel modular forms of degree 2. If V denotes the standard 2-dimensional representation of $\mathrm{GL}(2,\mathbb{C})$ with basis x_1, x_2 we consider the space $\mathrm{Sym}^6(V)$ of binary sextics. A general element $f \in \mathrm{Sym}^6(V)$ will be written as

$$f = \sum_{i=0}^{6} a_i \binom{6}{i} x_1^{6-i} x_2^i \,.$$

The group $\mathrm{GL}(2,\mathbb{C})$ acts on $\mathrm{Sym}^6(V)$. We denote by $\mathcal{C}$ the ring of covariants of binary sextics. A bihomogeneous covariant has a bidegree (a,b), meaning that it can be seen as a homogeneous expression of degree a in the coefficients a_i of f and as a form of degree b in x_1, x_2; such a covariant will be denoted by $C_{a,b}$. The map from covariants to Siegel modular forms defined in [4] is a map

$$\nu : \mathcal{C} \to M_{\chi_{10}} \,,$$

where M is the ring of vector-valued modular forms of degree 2 on $\Gamma_2 = \mathrm{Sp}(4,\mathbb{Z})$ and the subscript χ_{10} means that Igusa's cusp form χ_{10} of weight 10 is inverted. It sends the binary sextic f to the meromorphic vector-valued modular form $\chi_{6,8}/\chi_{10}$ of weight $(6,-2)$, where $\chi_{6,8}$ is the unique holomorphic modular form of weight $(6,8)$ (it is a cusp form). Using modular forms with character, we can also write this as $\chi_{6,3}/\chi_5$. This map provides us with a very effective method for constructing Siegel modular forms on Γ_2 with or without character. We used it in [4, 5] to construct modular forms.

Since the image of a covariant under ν may be meromorphic on $\mathcal{A}_2$, with possible poles along the locus $\mathcal{A}_{1,1}$ of abelian surfaces that are products of elliptic curves, it is important to have a method to determine the order of vanishing of modular forms obtained from covariants along this locus. In this paper we give such a method. In

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our earlier papers [4] and [5] we relied on restriction of the corresponding modular forms to the diagonal in the Siegel upper half space instead.

To exhibit the effectiveness of our method, we use it here to construct generators for certain modules of vector-valued Siegel modular forms of degree 2.

We denote by $M_{j,k}(\Gamma_2)$ (resp., $S_{j,k}(\Gamma_2)$) the vector space of Siegel modular forms (resp., of cusp forms) of weight (j,k) on Γ_2, that is, the weight corresponds to the irreducible representation $\mathrm{Sym}^j(\mathrm{St})\otimes\det^k(\mathrm{St})$ with St the standard representation of GL(2). The group Γ_2 admits a character ϵ of order 2 and χ_5, the square root of χ_{10}, is a modular form of weight 5 with this character. We refer to the last section for a way to calculate the character. We denote the space of modular forms (resp., of cusp forms) of weight (j,k) with character ϵ by $M_{j,k}(\Gamma_2,\epsilon)$ (resp., by $S_{j,k}(\Gamma_2,\epsilon)$).

Let $R=\bigoplus_{k\,\mathrm{even}} M_k(\Gamma_2)$ be the ring of scalar-valued Siegel modular forms of degree 2 of even weight. Igusa showed that it is a polynomial ring generated by E_4, E_6, χ_{10}, and χ_{12}.

We are interested in the structure of the R-modules

$$\mathcal{M}_j^{\mathrm{ev}}(\Gamma_2,\epsilon)=\bigoplus_{k\,\mathrm{even}} M_{j,k}(\Gamma_2,\epsilon) \quad \text{and} \quad \mathcal{M}_j^{\mathrm{odd}}(\Gamma_2,\epsilon)=\bigoplus_{k\,\mathrm{odd}} M_{j,k}(\Gamma_2,\epsilon)\,.$$

The structure of the analogous modules for modular forms without character

$$\mathcal{M}_j^{\mathrm{ev}}(\Gamma_2)=\bigoplus_{k\,\mathrm{even}} M_{j,k}(\Gamma_2) \quad \text{and} \quad \mathcal{M}_j^{\mathrm{odd}}(\Gamma_2)=\bigoplus_{k\,\mathrm{odd}} M_{j,k}(\Gamma_2)$$

is known for some values of j by work of Satoh, Ibukiyama, van Dorp, Kiyuna, and Takemori; see [8, 12, 15, 17, 19]. The next table summarizes the results.

j	2	4	6	8	10
even	Satoh [17]	Ibukiyama [12]	Ibukiyama [12]	Kiyuna [15]	Takemori [19]
odd	Ibukiyama [12]	Ibukiyama [12]	van Dorp [8]	Kiyuna [15]	Takemori [19]

The difficult part is the construction of the generators and the authors just mentioned used an array of methods to construct generators. For example, Satoh used generalized Rankin-Cohen brackets, Ibukiyama used theta series for even unimodular lattices and Rankin-Cohen brackets, van Dorp used differential operators, and so on. Here we produce the generators we need by a uniform method via the covariants of binary sextics. We treat the cases $j=0,2,4,6,8,10$ even and odd. In all these cases the module turns out to be a free R-module.

2. The ring of covariants of binary sextics

We recall some facts about the ring $\mathcal{C}$ of covariants of binary sextics. For a description of $\mathcal{C}$ we refer to [4, 5] and the classical literature mentioned there. The book of Grace and Young [11, p. 156] gives 26 generators for this ring. All these generators can be obtained as (repeated) so-called transvectants of the binary sextic f. The kth transvectant of two forms $g\in\mathrm{Sym}^m(V)$, $h\in\mathrm{Sym}^n(V)$ is defined as

$$(g,h)_k=\frac{(m-k)!(n-k)!}{m!\,n!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{\partial^k g}{\partial x_1^{k-j}\partial x_2^{j}}\frac{\partial^k h}{\partial x_1^{j}\partial x_2^{k-j}}$$

and the index k is usually omitted if $k=1$. If g is a covariant of bidegree (a,m) and h a covariant of bidegree (b,n), then $(g,h)_k$ is a covariant of bidegree $(a+b, m+n-2k)$ (cf. [3]). The following table summarizes the construction of the 26 generators.

1	$C_{1,6} = f$			
2	$C_{2,0} = (f,f)_6$	$C_{2,4} = (f,f)_4$	$C_{2,8} = (f,f)_2$	
3	$C_{3,2} = (f,C_{2,4})_4$	$C_{3,6} = (f,C_{2,4})_2$	$C_{3,8} = (f,C_{2,4})$	$C_{3,12} = (f,C_{2,8})$
4	$C_{4,0} = (C_{2,4},C_{2,4})_4$	$C_{4,4} = (f,C_{3,2})_2$	$C_{4,6} = (f,C_{3,2})$	$C_{4,10} = (C_{2,8},C_{2,4})$
5	$C_{5,2} = (C_{2,4},C_{3,2})_2$	$C_{5,4} = (C_{2,4},C_{3,2})$	$C_{5,8} = (C_{2,8},C_{3,2})$	
6	$C_{6,0} = (C_{3,2},C_{3,2})_2$	$C_{6,6}^{(1)} = (C_{3,6},C_{3,2})$	$C_{6,6}^{(2)} = (C_{3,8},C_{3,2})_2$	
7	$C_{7,2} = (f,C_{3,2}^2)_4$	$C_{7,4} = (f,C_{3,2}^2)_3$		
8	$C_{8,2} = (C_{2,4},C_{3,2}^2)_3$			
9	$C_{9,4} = (C_{3,8},C_{3,2}^2)_4$			
10	$C_{10,0} = (f,C_{3,2}^3)_6$	$C_{10,2} = (f,C_{3,2}^3)_5$		
12	$C_{12,2} = (C_{3,8},C_{3,2}^3)_6$			
15	$C_{15,0} = (C_{3,8},C_{3,2}^4)_8$			

3. Covariants and modular forms

The group Γ_2 acts on the Siegel upper half space $\mathfrak{H}_2$ and the orbifold quotient $\Gamma_2\backslash\mathfrak{H}_2$ can be identified with the moduli space $\mathcal{A}_2$ of principally polarized abelian surfaces. If $\mathcal{M}_2$ denotes the moduli space of complex smooth projective curves of genus 2 we have the Torelli map $\mathcal{M}_2 \hookrightarrow \mathcal{A}_2$. This is an embedding and the complement of the image is the locus $\mathcal{A}_{1,1}$ of products of elliptic curves. This is the image of the "diagonal"

$$\{\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} \in \mathfrak{H}_2 : \tau_{12} = 0\}$$

and also the zero locus of the cusp form χ_{10} that vanishes with order 2 there.

The moduli space $\mathcal{M}_2$ has another description as a stack quotient of the action of $\mathrm{GL}(2,\mathbb{C})$ on the space of binary sextics. We take the opportunity to correct an erroneous representation of this stack quotient in [4].

Let V be a 2-dimensional vector space, say generated by x_1, x_2, and consider $\mathrm{Sym}^6(V)$, the space of binary sextics. The group $\mathrm{GL}(V)$ acts from the right; an element $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ sends $f(x_1,x_2)$ to $f(ax_1+bx_2, cx_1+dx_2)$. We twist the action by $\det^{-2}(V)$ and consider then

$$\mathcal{X} = \mathrm{Sym}^6(V) \otimes \det{}^{-2}(V)\,.$$

We let $\mathcal{X}^0 \subset \mathcal{X}$ be the open set of binary sextics with nonvanishing discriminant. An element f of $\mathcal{X}^0$ defines a nonsingular curve of genus 2 via the equation $y^2 = f(x)$. The action on the equation $y^2 = f(x)$ is now induced by

$$x \mapsto (ax+b)/(cx+d), \quad y \mapsto (ad-bc)\,y/(cx+d)^3\,.$$

Then $\eta\mathrm{id}_V$ acts on the binary sextics as η^2, so that only $\pm\mathrm{id}_V$ acts trivially. The action of $-\mathrm{id}_V$ on (x,y) is $(x,y) \mapsto (x,-y)$ and induces the hyperelliptic involution. So the stack quotient $[\mathcal{X}^0/\mathrm{GL}(V)]$ equals the stack $\mathcal{M}_2$. Let $\alpha : \mathcal{X}^0 \to \mathcal{M}_2$ be the quotient map.

The equation $y^2 = f(x)$ defines two differentials xdx/y and dx/y that form a basis of the space of regular differentials on the curve and the action of $\mathrm{GL}(V)$ is by the standard representation. Thus the pullback under α of the Hodge bundle $\mathbb{E}$ from $\mathcal{M}_2$ to $\mathcal{X}^0$ is the equivariant bundle defined by the standard representation $V \times \mathcal{X}^0$. The equivariant bundle $\mathrm{Sym}^6(V) \otimes \det^{-2}(V)$ has the diagonal section $f \mapsto (f,f)$. This diagonal section, the universal binary sextic, thus defines a meromorphic section $\chi_{6,-2}$ of $\mathrm{Sym}^6(\mathbb{E}) \otimes \det(\mathbb{E})^{-2}$. Since the construction extends to the locus of binary sextics with zeros of multiplicity at most 2, the section extends regularly over $\delta_0 \setminus \delta_1$. (Here, δ_0 corresponds to $\overline{\mathcal{A}}_2 \setminus \mathcal{A}_2$, the divisor at infinity, and δ_1 to the closure of $\mathcal{A}_{1,1}$.) With this construction, the pole order along δ_1 is not yet known, but after multiplication with a power of χ_{10} the section becomes regular.

In fact, it is not hard to see that $\chi_{6,-2}$ has a simple pole along δ_1. Using Taylor series expansions in the normal direction to $\mathfrak{H}_1 \times \mathfrak{H}_1$ with coordinate $t = 2\pi i \tau_{12}$ as in [4, §5] and coordinates c_i on Sym^j corresponding to the monomials $\binom{j}{i} x_1^{j-i} x_2^i$, we see that the coefficient of t^m in c_i in the expansion of a meromorphic section of $\mathrm{Sym}^j(\mathbb{E}) \otimes \det(\mathbb{E})^{\otimes k}$ that is holomorphic outside $\mathcal{A}_{1,1}$, is of the form $g \otimes h$, with g quasimodular of weight $j - i + k + m$ and h quasimodular of weight $i + k + m$. See the appendix where we prove that we get quasimodular forms. To get nonzero coefficients, the two weights and hence their sum $j + 2k + 2m$ must be nonnegative. For $\chi_{6,-2}$, we get $2 + 2m \geq 0$, hence $m \geq -1$, proving the claim. Multiplying $\chi_{6,-2}$ with χ_{10}, we obtain the holomorphic modular form $\chi_{6,8}$, unique up to a scalar; alternatively, $\chi_{6,-2}$ can be written as $\chi_{6,3}/\chi_5$; see [6] for $\chi_{6,3}$.

We can interpret modular forms as sections of vector bundles made out of $\mathbb{E}$ by Schur functors, like $\mathrm{Sym}^j(\mathbb{E}) \otimes \det(\mathbb{E})^{\otimes k}$. Since the pullback of the Hodge bundle is the equivariant bundle defined by V, the pullback of such a section can be interpreted as a covariant. Recall that the ring of covariants is the ring of invariants for the action of $\mathrm{SL}(V)$ on $V \oplus \mathrm{Sym}^6(V)$; see for example [18, p. 55]. Conversely, a (bihomogeneous) covariant corresponds to a meromorphic modular form, with poles at most along δ_1, hence to an element of $M_{\chi_{10}}$.

We thus get maps

$$M \to \mathcal{C} \xrightarrow{\nu} M_{\chi_{10}}$$

with $\mathcal{C}$ the ring of covariants of binary sextics and $M = \bigoplus_{j,k} M_{j,k}(\Gamma_2)$ and $M_{\chi_{10}}$ its localization at the multiplicative system generated by χ_{10}. For another perspective on the map ν, see [4, §6].

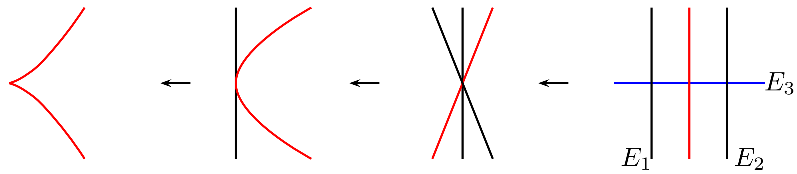
4. The order of vanishing

In this section we will describe a way to calculate the order of vanishing along the locus $\mathcal{A}_{1,1}$ of a modular form defined by a covariant. A covariant C has a bidegree (a, b): if we consider C as a form in the variables $a_0, \ldots, a_6$ and x_1, x_2, then it is of degree a in the a_i and degree b in x_1, x_2. The map $\nu : \mathcal{C} \to M_{\chi_{10}}$ associates to C a meromorphic modular form of weight $(b, a - b/2)$ on Γ_2. It has the property that $\chi_5^a\, \nu(C)$ is a holomorphic modular form on Γ_2, but with character if a is odd.

Recall that $\mathcal{M}_2$ is represented as the stack quotient $[\mathcal{X}^0/\mathrm{GL}(V)]$. The relation with the compactification of $\mathcal{M}_2$ is as follows.

In the (projectivized) space of binary sextics $\mathbb{P}(\mathcal{X})$ the discriminant defines a hypersurface Δ. This hypersurface has a codimension 1 singular locus, one component of which is the locus Δ' of binary sextics with three coinciding roots. So we are in codimension 2 in $\mathbb{P}(\mathcal{X})$ and we take a general plane Π in $\mathbb{P}(\mathcal{X})$ intersecting Δ transversally at a general point of Δ'.

In the plane Π the intersection with Δ gives rise to a curve with a cusp singularity corresponding to the intersection with Δ'; we assume this latter point is the origin of Π. In local coordinates u, v in the plane the discriminant is given by $u^2 = v^3$. One then blows up the plane at the origin three times. This is illustrated in the following picture (cf. the picture in [7, p. 80]).



Then one blows down the exceptional fibres E_1 and E_2. The image of E_3 corresponds in $\overline{\mathcal{M}}_2$ (resp., $\overline{\mathcal{A}}_2$) to the locus δ_1 (resp., $\overline{\mathcal{A}}_{1,1}$) of unions (resp., products) of elliptic curves.

If C is a covariant, then it defines a section of an equivariant vector bundle on $\mathcal{X}$ and we can pull this back to the blow-up. It then makes sense to speak of the order of this section along the divisor E_3.

If we consider in the last setting a vertical line that intersects the image of E_3 transversally at a general point, then this corresponds in the original plane with u, v coordinates to a curve $u^2 = c\,v^3$. We can calculate the order of vanishing along E_3 by calculating the order of the covariant on a general family corresponding to $u^2 = c\,v^3$.

The plane Π corresponds to a family of binary sextics of the form

$$g = (x^3 + vx + u)h$$

with h a general cubic polynomial in x. The substitution $u = c^2t^3$, $v = ct^2$ (with c general) gives a family corresponding to $u^2 = c\,v^3$ and the order in t of the covariant after substitution gives the order along E_3.

Theorem 1. *Let C be a covariant of binary sextics of degree a in the a_i and let $\chi_C = \nu(C)$ be the meromorphic modular form obtained by substituting $\chi_{6,-2}$. Then the order of χ_C along $\mathcal{A}_{1,1}$ is given by*

$$\mathrm{ord}_{\mathcal{A}_{1,1}}(\chi_C) = 2\,\mathrm{ord}_{E_3}(C) - a\,.$$

Proof. Since χ_C is obtained by substituting the components of $\chi_{6,-2}$ in C (cf. [4, §6]) and since $\chi_{6,-2}$ has a simple pole along δ_1, the order of χ_C along δ_1 (a.k.a. $\overline{\mathcal{A}}_{1,1}$) is at least $-a$. It can only be larger when C vanishes along E_3, the exceptional divisor of the third blow-up of $\mathcal{X}$. To work this out precisely, note first that the degree (resp., the order) of a product equals the sum of the degrees (resp., the orders) of the factors. Hence, after replacing C by its square if necessary, we may assume that a is even, equal to $2c$. Consider the invariant A of degree 2:

$$A = a_0a_6 - 6a_1a_5 + 15a_2a_4 - 10a_3^2$$

(proportional to $C_{2,0}$). Clearly, it doesn't vanish on E_3, and the associated scalar-valued meromorphic modular form χ_A of weight 2 has a pole of order 2 along δ_1. We can write C as $(C/A^c)\cdot A^c$ and χ_C as $\chi_{C/A^c}\cdot\chi_A^c$, where C/A^c is a meromorphic covariant and χ_{C/A^c} a meromorphic vector-valued modular form, regular along δ_1 but with possible poles along the zero locus of χ_A. The components of C/A^c are meromorphic functions on $\mathbb{P}(\mathcal{X})$ that descend to the components of χ_{C/A^c}. The (minimal) orders of vanishing along E_3, respectively, δ_1 are clearly closely related, but since E_3 in the picture above corresponds to the *coarse* moduli space $M_{1,1}$, not to the stack $\mathcal{M}_{1,1}$, the order of χ_{C/A^c} along δ_1 equals twice the order of C/A^c along E_3. □

5. Rings and modules of modular forms

Let $R = \bigoplus_{k \text{ even}} M_k(\Gamma_2)$ be the graded ring of scalar-valued Siegel modular forms of even weight on Γ_2. One knows that $R = \mathbb{C}[E_4, E_6, \chi_{10}, \chi_{12}]$ and so its Hilbert-Poincaré series equals $1/(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})$.

We denote by ϵ the unique nontrivial character of order 2 of Γ_2 (see Section 12 for a description of this character). Let $\Gamma_2[2]$ be the principal congruence subgroup of level 2 of Γ_2. The group $\mathrm{Sp}(4, \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathfrak{S}_6$. We fix an explicit isomorphism by identifying the symplectic lattice over $\mathbb{Z}/2\mathbb{Z}$ with the subspace $\{(a_1, \ldots, a_6) \in (\mathbb{Z}/2\mathbb{Z})^6 : \sum a_i = 0\}$ modulo the diagonally embedded $\mathbb{Z}/2\mathbb{Z}$ with form $\sum_i a_i b_i$ as in [1, Section 2]; it is given explicitly on generators of $\mathfrak{S}_6$ in [6, Section 3, (3.2)]. Thus $\mathfrak{S}_6$ acts on the space of modular forms $M_{j,k}(\Gamma_2[2])$ and the space $M_{j,k}(\Gamma_2, \epsilon)$ can be identified with the subspace of $M_{j,k}(\Gamma_2[2])$ on which $\mathfrak{S}_6$ acts via the alternating representation. Since -1_4 belongs to $\Gamma_2[2]$, we have $M_{j,k}(\Gamma_2, \epsilon) = (0)$ for j odd. In what follows, the integer j will always be even. The following result is in [13]; for the reader's convenience we give an alternative proof.

Lemma 2. *We have $M_{j,k}(\Gamma_2, \epsilon) = S_{j,k}(\Gamma_2, \epsilon)$ for $(j,k) \neq (0,0)$.*

Proof. In case $k = 0$ and $j \neq 0$ it is well known that $M_{j,0}(\Gamma_2, \epsilon) = (0)$; see [9, Satz1]. The Siegel operator Φ_2 maps $M_{j,k}(\Gamma_2[2])$ to $S_{j+k}(\Gamma_1[2])$ which is (0) if k is odd and j is even. Since $M_{j,k}(\Gamma_2, \epsilon) \subseteq M_{j,k}(\Gamma_2[2])$ we find $M_{j,k}(\Gamma_2, \epsilon) = S_{j,k}(\Gamma_2, \epsilon)$ for k odd. For $k \geq 2$ even, the Eisenstein part $E_{j,k}(\Gamma_2[2])$ of $M_{j,k}(\Gamma_2[2])$, that is, the orthogonal complement of $S_{j,k}(\Gamma_2[2])$, was described in [6, Section 13] as an $\mathfrak{S}_6$-representation. From the description there we see that the isotypical component $s[1^6]$ never occurs in $E_{j,k}(\Gamma_2[2])$; the result follows since $S_{j,k}(\Gamma_2, \epsilon) = S_{j,k}(\Gamma_2[2])^{s[1^6]}$. (Note that there is a misprint in the expression in [6, Prop. 13.1]: Sym^k should be read as $\mathrm{Sym}^{(j+k)/2}$.) □

The preceding lemma allows us to study cusp forms only. The dimensions of the spaces $S_{j,k}(\Gamma_2, \epsilon)$ are known by work of Tsushima (private communication) as completed by Bergström (see [2]) and independently by [13, Thm. 6.2 and the tables on p. 203 for $k \geq 5$]. The next table gives the Hilbert-Poincaré series of $\mathcal{M}_j^{\mathrm{odd}}(\Gamma_2, \epsilon)$ and $\mathcal{M}_j^{\mathrm{ev}}(\Gamma_2, \epsilon)$ as R-modules. We give only the numerators since in all cases we have

$$\sum_{k \equiv_2 0 \, (\text{or } 1)} \dim S_{j,k}(\Gamma_2, \epsilon)\, t^k = \frac{N_j}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})},$$

with N_j a polynomial in t.

j	$k \bmod 2$	$N_j(t)$
0	1	t^5
	0	t^{30}
2	1	$t^9+t^{11}+t^{17}$
	0	$t^{16}+t^{22}+t^{24}$
4	1	$t^9+t^{11}+t^{13}+t^{15}+t^{17}$
	0	$t^{14}+t^{16}+t^{18}+t^{20}+t^{22}$
6	1	$t^3+t^5+t^{11}+t^{13}+t^{17}+t^{19}+t^{21}$
	0	$t^8+t^{10}+t^{12}+t^{16}+t^{18}+t^{24}+t^{26}$
8	1	$t^5+t^7+2\,t^9+t^{11}+t^{13}+t^{15}+t^{17}+t^{23}$
	0	$t^4+t^{10}+t^{12}+t^{14}+t^{16}+2\,t^{18}+t^{20}+t^{22}$
10	1	$t^5+t^7+2\,t^9+2t^{11}+2t^{13}+2t^{15}+t^{17}$
	0	$t^8+2\,t^{10}+2\,t^{12}+2t^{14}+2t^{16}+t^{18}+t^{20}$
12	1	$t^3+2\,t^5+t^7+2\,t^9+3\,t^{11}+2\,t^{13}+t^{17}+t^{19}-t^{23}+t^{27}$
	0	$t^2+t^4+t^6+t^8+t^{10}+t^{12}+t^{14}+2\,t^{16}+2\,t^{18}+t^{20}+t^{22}+t^{24}-t^{28}$

For $j \in \{0,2,4,6,8,10\}$ and both for k odd and even the shape of the polynomials N_j is as follows:

$$N_j(t) = a_{k_{j,1}}\, t^{k_{j,1}} + \ldots + a_{k_{j,n}}\, t^{k_{j,n}} \quad \text{with} \quad n, a_{k_{j,i}} \in \mathbb{Z}_{>0} \quad \text{and} \quad \sum_{i=1}^{n} a_{k_{j,i}} = j+1.$$

This suggests that the R-modules $\mathcal{M}_j^{\mathrm{ev}}(\Gamma,\epsilon)$ and $\mathcal{M}_j^{\mathrm{odd}}(\Gamma,\epsilon)$ are generated by $j+1$ cusp forms with $a_{j,k_{j,i}}$ generators of weight $(j,k_{j,i})$. As the table shows this does not hold for $j=12$.

Therefore the strategy of the proof for the structure of the modules will be to show first that there is no cusp form of weight (j,k) for $k < k_{j,1}$ for $j \in \{0,2,4,6,8,10\}$. In the cases at hand this follows from the above formula and the results in [5]. Then we will construct $j+1$ cusp forms and check that their wedge product is not identically 0. In fact in all cases we find that the wedge product of the $j+1$ forms is a nonzero multiple of a product of powers of χ_5 and χ_{30}. This proves that the submodule they generate has the same Hilbert-Poincaré series as the whole module, hence that we found the whole module. We will give the covariants that define the generators explicitly in a number of cases, but in view of their size we refer for the other cases to [2] where we will make these available.

6. The scalar-valued cases

In this section we deal with the modules of scalar-valued modular forms with character. In this case the weight (j,k) is of the form $(0,k)$ and we simply indicate it by k.

The diagonal element $\gamma_1 = \mathrm{diag}(1,-1,1,-1) \in \Gamma_2$ defines an involution fixing the coordinates τ_{11} and τ_{22} and replacing τ_{12} by $-\tau_{12}$. Its fixed point set is the locus defined by $\tau_{12}=0$. This defines the Humbert surface $H_1 = \mathcal{A}_{1,1}$ parametrizing products of elliptic curves in $\mathcal{A}_2$. There is another involution ι_2 given by $\gamma_2 = (a,b;c,d)$ with $b=c=0$ and $a=d=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ which interchanges τ_{11} and τ_{22}, but fixes τ_{12}. The fixed point set of ι_2 is the locus $\tau_{11}=\tau_{22}$ and defines the Humbert surface H_4 in $\mathcal{A}_2$; see [10]. One checks that the action on modular forms is as follows:

$$(1) \qquad \gamma_1 : f \mapsto (-1)^k f, \quad \gamma_2 : f \mapsto (-1)^{k+1} f \qquad \text{for } f \in M_k(\Gamma_2,\epsilon).$$

Note $\epsilon(\gamma_2) = -1$. It follows that $f \in M_k(\Gamma_2,\epsilon)$ vanishes on H_1 for k odd and on H_4 for k even.

We have two modular forms χ_5 and χ_{30} of weight 5 and weight 30 whose zero loci in $\mathcal{A}_2$ equals H_1 and H_4. We recall their construction.

The cusp form $\chi_5 \in S_5(\Gamma_2, \epsilon)$ is defined in terms of theta functions. For $(\tau, z) \in \mathfrak{H} \times \mathbb{C}$ and (μ_1, μ_2), (ν_1, ν_2) in $\mathbb{Z}^2$ we have the standard theta series with characteristics

$$\vartheta_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}(\tau, z) = \sum_{n=(n_1,n_2)\in\mathbb{Z}^2} e^{i\pi(n+\mu/2)(\tau\,(n+\mu/2)^t+2(z+\nu/2))}.$$

By letting μ and ν be vectors consisting of zeros and ones with $\mu^t\nu \equiv 0 \,(\mathrm{mod}\, 2)$ and setting $z = 0$ we obtain ten so-called theta constants and their product defines a cusp form of weight 5 on Γ_2 with character ϵ:

$$\chi_5 = -\frac{1}{64} \prod \vartheta_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}.$$

Its Fourier expansion starts with

$$\chi_5(\tau) = (u - 1/u)XY + \ldots,$$

where $X = e^{\pi i \tau_1}$, $Y = e^{\pi i \tau_2}$, and $u = e^{\pi i \tau_{12}}$. We note that $\chi_5^2 = \chi_{10}$ and the vanishing locus of χ_{10} in $\mathcal{A}_2$ is $2H_1$.

In order to construct χ_{30} we consider the invariant $C_{15,0}$, given in the table in Section 2. By the procedure of [4] it provides a meromorphic cusp form of weight 15 on Γ_2. One checks using Theorem 1 that the order of this form along $\mathcal{A}_{1,1}$ is -3. So we obtain a holomorphic modular form by multiplying by χ_5^3 and we set

$$\chi_{30} = 2^{-11} 3^{11} \cdot 5^{11} \cdot 11 \cdot 13\, \nu(C_{15,0})\chi_5^3\,;$$

it is a cusp form in $S_{30}(\Gamma_2, \epsilon)$ whose Fourier expansion starts with

$$\chi_{30}(\tau) = (u + 1/u)X^3Y^5 - (u + 1/u)X^5Y^3 + \ldots.$$

The following result is due to Igusa, see [14, pp. 402-404].

Theorem 3. *We have* $\mathcal{M}_0^{\mathrm{odd}}(\Gamma_2, \epsilon) = R\,\chi_5$ *and* $\mathcal{M}_0^{\mathrm{ev}}(\Gamma_2, \epsilon) = R\,\chi_{30}$.

Proof. Clearly $\mathcal{M}_0^{\mathrm{odd}}(\Gamma_2, \epsilon)$ contains $R\,\chi_5$ and $\mathcal{M}_0^{\mathrm{ev}}(\Gamma_2, \epsilon)$ contains $R\,\chi_{30}$. The generating function for the dimensions shows that χ_5 (resp., χ_{30}) generates. □

Remark 4. We know the cycle classes of the closures of H_1 and H_4 in the compactified moduli space $\tilde{\mathcal{A}}_2$. In the divisor class group with rational coefficients of $\tilde{\mathcal{A}}_2$ we have

$$5\lambda_1 = [\overline{H}_1] + [D], \quad 30\lambda_1 = [\overline{H}_4] + [D]$$

with D the divisor at infinity of $\tilde{\mathcal{A}}_2$, and λ_1 the first Chern class of the determinant of the Hodge bundle; see [10, Thm. 2.6]. From this it follows that the vanishing locus of χ_{30} in $\mathcal{A}_2$ is H_4. Then (1) implies that for k odd (resp., k even) any $f \in M_k(\Gamma_2, \epsilon)$ is divisible by χ_5 (resp., by χ_{30}). This implies the theorem as well.

For later identifications (for example in the proof of Theorem 11) we need the restriction of $\chi_{6,3}$ to the Humbert surface H_4. This surface can be given by $\tau_{11} = \tau_{22}$, or equivalently by $\tau_{12} = 1/2$. Let χ denote the Dirichlet character modulo 4 defined by the Kronecker symbol $\left(\frac{-4}{\cdot}\right)$. The space $S_3^{\mathrm{new}}(\Gamma_0(16), \chi)$ is generated by $\eta^6(2\tau)$. The space $S_5^{\mathrm{new}}(\Gamma_0(16), \chi)$ has dimension 2 and a basis of eigenforms g', g'' with Fourier expansions

$$q - 8\sqrt{-3}\,q^3 + 18\,q^5 - 16\sqrt{-3}\,q^7 - 111\,q^9 + \ldots$$

and similarly $S_7^{\rm new}(\Gamma_0(16),\chi)$ has dimension 2 and a basis of eigenforms f', f'' with Fourier expansions

$$q - 16\sqrt{-3}\, q^3 - 150\, q^5 - 352\sqrt{-3}\, q^7 - 39\, q^9 + \dots .$$

Lemma 5. *The restriction of* $\chi_{6,3}$ *to* H_4 *is given by*

$$\chi_{6,3}\begin{pmatrix} \tau_1 & 1/2 \\ 1/2 & \tau_2 \end{pmatrix} = 2\,i \begin{bmatrix} 16\,\eta^{18}(2\,\tau_1)\otimes\eta^6(2\,\tau_2) \\ 0 \\ F_1(\tau_1)\otimes F_2(\tau_2) \\ 0 \\ F_2(\tau_1)\otimes F_1(\tau_2) \\ 0 \\ 16\,\eta^{6}(2\,\tau_1)\otimes\eta^{18}(2\,\tau_2) \end{bmatrix},$$

where

$$F_1 = \frac{3+\sqrt{-3}}{6}\, f' + \frac{3-\sqrt{-3}}{6}\, f'' \quad \textit{and} \quad F_2 = \frac{3+\sqrt{-3}}{6}\, g' + \frac{3-\sqrt{-3}}{6}\, g''\, .$$

7. The case $j = 2$

We start with the case k odd.

Theorem 6. *The R-module* $\mathcal{M}_2^{\rm odd}(\Gamma_2,\epsilon)$ *is free with three generators of weight* $(2,9)$, $(2,11)$, *and* $(2,17)$.

Proof. We recall that the numerator N_2 of the Hilbert-Poincaré series is $t^9+t^{11}+t^{17}$. We construct the three generators by considering the covariants

$$\begin{aligned}
\xi_1 &= 4\, C_{2,0}C_{3,2} - 15\, C_{5,2},\\
\xi_2 &= 32\, C_{2,0}^2 C_{3,2} + 135\, C_{2,0}C_{5,2} - 300\, C_{3,2}C_{4,0} - 15750\, C_{7,2},\\
\xi_3 &= C_{3,2}.
\end{aligned}$$

These three covariants define meromorphic modular forms vanishing with order -1, -1, -3 along $\mathcal{A}_{1,1}$ (by Theorem 1), so we obtain holomorphic modular forms

$$F_{2,9} = -\frac{3375}{4}\nu(\xi_1)\chi_5, \quad F_{2,11} = -\frac{10125}{8}\nu(\xi_2)\chi_5, \quad F_{2,17} = \frac{1125}{2}\nu(\xi_3)\chi_5^3$$

of weights $(2,9)$, $(2,11)$, and $(2,17)$ and their Fourier expansions start as

$$F_{2,9} = \begin{pmatrix} u-1/u \\ u+1/u \\ u-1/u \end{pmatrix} XY + \dots \qquad F_{2,11} = \begin{pmatrix} u-1/u \\ u+1/u \\ u-1/u \end{pmatrix} XY + \dots$$

and

$$F_{2,17} = \begin{pmatrix} u^3+9\,u-9\,u^{-1}-u^{-3} \\ u^3+71\,u+71\,u^{-1}+u^{-3} \\ u^3+9\,u-9\,u^{-1}-u^{-3} \end{pmatrix} X^3Y^3 + \dots .$$

To prove the theorem we have to show that these three generators satisfy

$$F_{2,9}\wedge F_{2,11}\wedge F_{2,17} \neq 0.$$

Note that $\det({\rm Sym}^j(\mathbb{E})) = \det(\mathbb{E})^{j(j+1)/2}$, so this is a form in $S_{40}(\Gamma_2,\epsilon)$. The Fourier expansion of $F_{2,9}\wedge F_{2,11}\wedge F_{2,17}$ starts with

$$86400\,((-u^3+u+u^{-1}-u^{-3})Y^7X^5 + (u^3-u-u^{-1}+u^{-3})Y^5X^7 + \dots)$$

and this shows the result. □

Remark 7. The space $S_{40}(\Gamma_2,\epsilon)$ is 2-dimensional, generated by $\chi_5^2\chi_{30}$ and $E_4E_6\chi_{30}$. We check that $F_{2,9}\wedge F_{2,11}\wedge F_{2,17} = -86400\,\chi_5^2\chi_{30}$.

The case k even is similar.

Theorem 8. *The R-module $\mathcal{M}_2^{\rm ev}(\Gamma_2,\epsilon)$ is free with generators of weight $(2,16)$, $(2,22)$, and $(2,24)$.*

Proof. We use the covariants

$$\xi_1 = 1211\, C_{2,0}^2 C_{8,2} - 8910\, C_{2,0}C_{10,2} - 5250\, C_{4,0}C_{8,2} + 277200\, C_{12,2},$$
$$\xi_2 = C_{8,2}, \quad \xi_3 = 7\, C_{2,0}C_{8,2} - 110\, C_{10,2},$$

and set

$$F_{2,16} = \frac{34171875}{2048}\nu(\xi_1)\chi_5 = \begin{pmatrix} 0 \\ 2(u-1/u) \\ u+1/u \end{pmatrix} XY^3 + \begin{pmatrix} -(u+1/u) \\ -2(u-1/u) \\ 0 \end{pmatrix} X^3Y + \ldots$$
$$F_{2,22} = \frac{26578125}{8}\nu(\xi_2)\chi_5^3 = \begin{pmatrix} u+1/u \\ 0 \\ -(u+1/u) \end{pmatrix} X^3Y^3 + \ldots$$
$$F_{2,24} = -\frac{102515625}{16}\nu(\xi_3)\chi_5^3 = \begin{pmatrix} u+1/u \\ 0 \\ -(u+1/u) \end{pmatrix} X^3Y^3 + \ldots.$$

By the criterion these are holomorphic modular forms of weight $(2,16)$, $(2,22)$, and $(2,24)$. The Fourier expansion of $F_{2,16}\wedge F_{2,22}\wedge F_{2,24}$ starts with

$$F_{2,16}\wedge F_{2,22}\wedge F_{2,24} = -2880\,(u^3+u-u^{-1}-u^{-3})\,X^7Y^{11} + \ldots$$

and in fact equals $-2880\,\chi_5\,\chi_{30}^2$. This finishes the proof in view of the Hilbert-Poincaré series. □

8. The case $j=4$.

Theorem 9. *The R-module $\mathcal{M}_4^{\rm odd}(\Gamma_2,\epsilon)$ is free with generators of weight $(4,9)$, $(4,11)$, $(4,13)$, $(4,15)$, and $(4,17)$.*

Proof. We use the covariants

$$\begin{aligned}
\xi_1 &= 49\, C_{2,0}^2C_{2,4} + 45\, C_{2,0}C_{4,4} - 375\, C_{2,4}C_{4,0} - 225\, C_{3,2}^2,\\
\xi_2 &= 772\, C_{2,0}^3C_{2,4} - 1260\, C_{2,0}^2C_{4,4} - 4875\, C_{2,0}C_{2,4}C_{4,0} - 900\, C_{2,0}C_{3,2}^2\\
&\quad - 5625\, C_{2,4}C_{6,0} + 13500\, C_{3,2}C_{5,2} + 6750\, C_{4,0}C_{4,4}\\
\xi_3 &= 64\, C_{2,0}^4C_{2,4} - 1200\, C_{2,0}^2C_{2,4}C_{4,0} - 3600\, C_{2,0}^2C_{3,2}^2 + 27000\, C_{2,0}C_{3,2}C_{5,2}\\
&\quad + 5625\, C_{2,4}C_{4,0}^2 - 50625\, C_{5,2}^2,\\
\xi_4 &= C_{2,4}, \qquad \xi_5 = 3\, C_{2,0}C_{2,4} - 5\, C_{4,4}\,.
\end{aligned}$$

The Fourier expansions of

$$F_{4,9} = -\frac{675}{4}\nu(\xi_1)\chi_5, \qquad F_{4,11} = \frac{2025}{8}\nu(\xi_2)\chi_5, \quad \text{and} \quad F_{4,13} = -\frac{30375}{8}\nu(\xi_3)\chi_5$$

start all three as

$$\begin{pmatrix} u-1/u \\ 2(u+1/u) \\ 3(u-1/u) \\ 2(u+1/u) \\ u-1/u \end{pmatrix} XY + \ldots.$$

The other two modular forms we need are

$$F_{4,15} = \frac{75}{2}\nu(\xi_4)\chi_5^3 = \begin{pmatrix} u^3-3u+3/u-1/u^3 \\ 2(u^3-u-1/u+1/u^3) \\ 3(u^3+5u-5/u-1/u^3) \\ 2(u^3-u-1/u+1/u^3) \\ u^3-3u+3/u-1/u^3 \end{pmatrix} X^3Y^3 + \dots,$$

$$F_{4,17} = -\frac{675}{2}\nu(\xi_5)\chi_5^3 = \begin{pmatrix} u^3+9u-9/u-1/u^3 \\ 2(u^3-u-1/u+1/u^3) \\ 3(u^3-3u+3/u-1/u^3) \\ 2(u^3-u-1/u+1/u^3) \\ u^3+9u-9/u-1/u^3 \end{pmatrix} X^3Y^3 + \dots.$$

The Fourier expansion of $F_{4,9} \wedge F_{4,11} \wedge F_{4,13} \wedge F_{4,15} \wedge F_{4,17}$ starts with

$$-2866544640\,(u^5 - u^3 - 2u + 2/u + 1/u^3 - 1/u^5)X^9Y^{13} + \dots$$

and by a calculation we get

$$F_{4,9} \wedge F_{4,11} \wedge F_{4,13} \wedge F_{4,15} \wedge F_{4,17} = -2866544640\,\chi_5^3\chi_{30}^2\,. \qquad \square$$

Theorem 10. *The R-module* $\mathcal{M}_4^{\mathrm{ev}}(\Gamma_2,\epsilon)$ *is free with generators of weight* $(4,14)$, $(4,16)$, $(4,18)$, $(4,20)$, *and* $(4,22)$.

Proof. For weight $(4,14)$ we consider the covariant ξ_1 given as

$$89\,C_{2,0}^3C_{5,4} + 12390\,C_{2,0}^2C_{7,4} - 750\,C_{2,0}C_{4,0}C_{5,4} \\ - 63000\,(C_{2,0}C_{9,4} + C_{3,2}C_{8,2} + C_{4,0}C_{7,4})$$

and set $F_{4,14} = -(151875/1024)\nu(\xi_1)\chi_5$. This is holomorphic and its Fourier expansion starts with

$$F_{4,14}(\tau) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2(u-1/u) \\ (u+1/u) \end{pmatrix} XY^3 - \begin{pmatrix} (u+1/u) \\ 2(u-1/u) \\ 0 \\ 0 \\ 0 \end{pmatrix} X^3Y + \dots.$$

For weight $(4,16)$ we consider the covariant ξ_2 given as

$$11176\,C_{2,0}^4C_{5,4} - 82320\,C_{2,0}^3C_{7,4} + 9576000\,C_{2,0}^2C_{9,4} - 15750\,C_{2,0}C_{3,2}C_{8,2} \\ - 220500\,C_{2,0}C_{4,0}C_{7,4} - 176625\,C_{2,0}C_{5,4}C_{6,0} \\ - 414000\,C_{4,0}^2C_{5,4} + 43213500\,C_{3,2}C_{10,2} \\ - 47250000\,C_{4,0}C_{9,4} + 20506500\,C_{5,2}C_{8,2} - 9308250\,C_{6,0}C_{7,4}$$

and set $F_{4,16} = (151875/4096)\nu(\xi_2)\chi_5$; it is holomorphic and its Fourier expansion starts with

$$F_{4,16}(\tau) = \begin{pmatrix} 0 \\ 2(u+1/u) \\ 3(u+1/u) \\ (u-1/u) \\ 0 \end{pmatrix} XY^3 + \dots.$$

We get a form $F_{4,18}$ of weight $(4,18)$ by putting $F_{4,18} = (16875/8)\nu(C_{5,4})\chi_5^3$; it is holomorphic and its Fourier expansion starts with

$$F_{4,18}(\tau) = \begin{pmatrix} 3(u+1/u) \\ 2(u-1/u) \\ 0 \\ -2(u-1/u) \\ -3(u+1/u) \end{pmatrix} X^3Y^3 + \dots.$$

For weight $(4,20)$ we consider the covariant $\xi_4 = C_{2,0}C_{5,4} + 70C_{7,4}$ and put $F_{4,20} = (151875/32)\nu(\xi_4)\chi_5^3$ with Fourier expansion

$$F_{4,20}(\tau) = \begin{pmatrix} 0 \\ (u-1/u) \\ 0 \\ -(u-1/u) \\ 0 \end{pmatrix} X^3Y^3 + \dots .$$

Finally, the covariant $\xi_5 = C_{2,0}^2 C_{5,4} - 10\, C_{2,0}C_{7,4} + 1000\, C_{9,4}$ yields the form $F_{4,22} = (3189375/32)\nu(\xi_5)\chi_5^3$ with Fourier expansion

$$F_{4,22}(\tau) = \begin{pmatrix} (u+1/u) \\ 2(u-1/u) \\ 0 \\ -2(u-1/u) \\ -(u+1/u) \end{pmatrix} X^3Y^3 + \dots .$$

The Fourier expansion of $F_{4,14} \wedge F_{4,16} \wedge F_{4,18} \wedge F_{4,20} \wedge F_{4,22}$ starts with

$$-20736\,(u^5 + u^3 - 2u - 2/u + 1/u^3 + 1/u^5)X^{11}Y^{17} + \dots$$

and in fact we checked that it equals $-20736\,\chi_5^2\chi_{30}^3$. □

9. The case $j = 6$

Theorem 11. *The R-module* $\mathcal{M}_6^{\mathrm{odd}}(\Gamma_2, \epsilon)$ *is free with generators of weight* $(6,3)$, $(6,5)$, $(6,11)$, $(6,13)$, $(6,17)$, $(6,19)$, *and* $(6,21)$.

Proof. We use the covariants

$$\begin{aligned}
\xi_1 =& C_{1,6}, \qquad \xi_2 = 8\,C_{1,6}C_{2,0} - 75\,C_{3,6},\\
\xi_3 =& 125\,C_{1,6}C_{2,0}^2C_{4,0} + 249\,C_{1,6}C_{2,0}C_{6,0} - 840\,C_{1,6}C_{4,0}^2 - 189\,C_{2,0}C_{2,4}C_{5,2}\\
&- 1008\,C_{2,0}C_{3,2}C_{4,4} - 72\,C_{2,0}C_{3,6}C_{4,0} + 630\,C_{3,2}^3 + 132300\,C_{2,4}C_{7,2}\\
&+ 2430\,C_{3,6}C_{6,0} - 1890\,C_{4,4}C_{5,2},\\
\xi_4 =& 768\,C_{1,6}C_{2,0}^5 + 768\,C_{2,0}^4C_{3,6} - 487520\,C_{1,6}C_{2,0}^2C_{6,0} - 36075\,C_{2,0}^2C_{2,4}C_{5,2}\\
&+ 33600\,C_{2,0}^2C_{3,2}C_{4,4} - 52500\,C_{2,0}C_{3,2}^3 - 11061300\,C_{1,6}C_{4,0}C_{6,0}\\
&- 314861750\,C_{2,0}C_{2,4}C_{7,2} - 112500\,C_{2,0}C_{3,6}C_{6,0} + 8956675\,C_{2,0}C_{4,4}C_{5,2}\\
&+ 17767100\,C_{2,4}C_{3,2}C_{6,0} + 230625\,C_{2,4}C_{4,0}C_{5,2} - 39779100\,C_{3,2}^2C_{5,2}\\
&+ 17834600\,C_{3,2}C_{4,0}C_{4,4} + 9482503800\,C_{1,6}C_{10,0} - 932772750\,C_{4,4}C_{7,2},\\
\xi_5 =& 8\,C_{1,6}C_{2,0}^2 - 125\,C_{2,4}C_{3,2},\\
\xi_6 =& 128\,C_{1,6}C_{2,0}^3 + 6600\,C_{2,0}^2C_{3,6} + 6750\,C_{2,4}C_{5,2}\\
&- 9000\,C_{3,2}C_{4,4} - 52875\,\mathit{Cov}_{3,6}C_{4,0},\\
\xi_7 =& -837\,C_{1,6}C_{2,0}^2C_{4,0} + 415\,C_{1,6}C_{2,0}C_{6,0}\\
&+ 9450\,C_{2,0}C_{2,4}C_{5,2} + 6075\,C_{2,0}C_{3,6}C_{4,0}\\
&+ 3150\,C_{3,2}^3 - 1543500\,C_{2,4}C_{7,2} - 17475\,C_{3,6}C_{6,0} + 14175\,C_{4,4}C_{5,2}\,.
\end{aligned}$$

We consider the following cusp forms:

$$F_{6,3} = \nu(\xi_1)\chi_5, \;\; F_{6,5} = -15\,\nu(\xi_2)\chi_5, \;\; F_{6,11} = \frac{253125}{8}\nu(\xi_3)\chi_5,$$

$$F_{6,13} = \frac{2278125}{16}\nu(\xi_4)\chi_5,$$

and

$$F_{6,17} = -\frac{675}{4}\nu(\xi_5)\chi_5^3, \quad F_{6,19} = -\frac{675}{2}\nu(\xi_6)\chi_5^3, \quad F_{6,21} = -\frac{151875}{4}\nu(\xi_7)\chi_5^3 .$$

Then

$$W_{110} = F_{6,3} \wedge F_{6,5} \wedge F_{6,11} \wedge F_{6,13} \wedge F_{6,17} \wedge F_{6,19} \wedge F_{6,21}$$

is a cusp form in $S_{0,110}(\Gamma_2, \epsilon)$ and its Fourier expansion starts with

$$2^{30} \cdot 3^5 \cdot 5^8 \cdot 7^3 \, (u^7 - u^5 - 3u^3 + 3u + 3/u - 3/u^3 - 1/u^5 + 1/u^7) \, X^{13}Y^{17} + \ldots .$$

The order of vanishing of W_{110} along H_1 is 4 while along H_4 it is 3, so W_{110} is a multiple of $\chi_5^4\chi_{30}^3$ and a calculation at the level of covariants yields $W_{110} = 2^{30} \cdot 3^5 \cdot 5^8 \cdot 7^3 \, \chi_5^4\chi_{30}^3$. □

Theorem 12. *The R-module $\mathcal{M}_6^{\mathrm{ev}}(\Gamma_2, \epsilon)$ is free with generators of weight* $(6, 8)$, $(6, 10)$, $(6, 12)$, $(6, 16)$, $(6, 18)$, $(6, 24)$, *and* $(6, 26)$.

Proof. We use the covariants

$$\xi_1 = 16\, C_{2,0}C_{4,6} + 75\, C_{6,6}^{(1)} - 60\, C_{6,6}^{(2)}, \quad \xi_4 = C_{4,6}, \quad \xi_5 = 4\, C_{2,0}C_{4,6} - 15\, C_{6,6}^{(1)},$$

$$\xi_2 = -128\, C_{2,0}^2C_{4,6} + 75\, C_{2,0}C_{6,6}^{(1)} - 540\, C_{2,0}C_{6,6}^{(2)} - 1500\, C_{3,2}C_{5,4} + 1800\, C_{4,0}C_{4,6},$$

$$\begin{aligned}\xi_3 = {} & 64\, C_{2,0}^3C_{4,6} - 3975\, C_{2,0}^2C_{6,6}^{(1)} + 1740\, C_{2,0}^2C_{6,6}^{(2)} \\ & - 189000\, C_{2,4}C_{8,2} + 63000\, C_{3,2}C_{7,4} \\ & + 40500\, C_{4,0}C_{6,6}^{(1)} - 18000\, C_{4,0}C_{6,6}^{(2)} + 4500\, C_{5,2}C_{5,4},\end{aligned}$$

$$\begin{aligned}\xi_6 = {} & -17472\, C_{2,0}C_{2,4}C_{8,2} + 31360\, C_{2,0}C_{3,2}C_{7,4} \\ & - 513\, C_{2,0}C_{4,0}C_{6,6}^{(1)} + 180\, C_{2,0}C_{4,0}C_{6,6}^{(2)} \\ & - 64\, C_{2,0}C_{4,6}C_{6,0} + 342\, C_{2,0}C_{5,2}C_{5,4} + 39600\, C_{2,4}C_{10,2} - 126000\, C_{3,2}C_{9,4} \\ & - 16800\, C_{4,4}C_{8,2} - 60900\, C_{5,2}C_{7,4} + 600\, C_{6,0}C_{6,6}^{(1)},\end{aligned}$$

$$\begin{aligned}\xi_7 = {} & 1024\, C_{2,0}^5C_{4,6} - 257152000\, C_{2,0}^2C_{3,2}C_{7,4} + 5375048250\, C_{2,0}C_{2,4}C_{10,2} \\ & - 1808283750\, C_{2,0}C_{3,2}C_{9,4} + 785335250\, C_{2,0}C_{4,4}C_{8,2} + 1144763375\, C_{2,0}C_{5,2}C_{7,4} \\ & + 673186500\, C_{2,4}C_{4,0}C_{8,2} + 656687500\, C_{3,2}^2C_{8,2} - 938905625\, C_{3,2}C_{4,0}C_{7,4} \\ & + 3150000\, C_{4,0}^2C_{6,6}^{(2)} + 17435250\, C_{4,0}C_{5,2}C_{5,4} - 378064302000\, C_{2,4}C_{12,2} \\ & - 532125000\, C_{4,4}C_{10,2} - 415800000\, C_{4,6}C_{10,0} + 37292797500\, C_{5,2}C_{9,4} \\ & - 250254270000\, C_{7,2}C_{7,4} .\end{aligned}$$

We consider the following cusp forms:

$$F_{6,8} = \frac{10125}{8}\nu(\xi_1)\chi_5, \quad F_{6,10} = -\frac{30375}{16}\nu(\xi_2)\chi_5, \quad F_{6,12} = \frac{455625}{64}\nu(\xi_3)\chi_5,$$

$$F_{6,16} = -3375\, \nu(\xi_4)\chi_5^3, \quad F_{6,18} = -50625\, \nu(\xi_5)\chi_5^3 \quad F_{6,24} = -\frac{170859375}{32}\nu(\xi_6)\chi_5^3,$$

$$F_{6,26} = -\frac{20503125}{16}\nu(\xi_7)\chi_5^3 .$$

Then

$$W_{135} = F_{6,8} \wedge F_{6,10} \wedge F_{6,12} \wedge F_{6,16} \wedge F_{6,18} \wedge F_{6,24} \wedge F_{6,26}$$

is a cusp form in $S_{135}(\Gamma_2, \epsilon)$ and its Fourier expansion starts with

$$-2^{32} \cdot 3^8 \cdot 5^8 \cdot 7^2 \cdot 13 \cdot 23\, (u^7 + u^5 - 3u^3 - 3u + 3/u + 3/u^3 - 1/u^5 - 1/u^7) X^{15}Y^{23} + \ldots .$$

A calculation shows that the order of vanishing of W_{135} along H_1 is 3, while along H_4 it is 4, so W_{135} is a multiple of $\chi_5^3\chi_{30}^4$ and a calculation at the level of covariants tells us

$$W_{135} = -2^{32} \cdot 3^8 \cdot 5^8 \cdot 7^2 \cdot 13 \cdot 23\, \chi_5^3\chi_{30}^4 \,. \qquad \square$$

10. The case $j = 8$

Theorem 13. *The R-module $\mathcal{M}_8^{\mathrm{odd}}(\Gamma_2,\epsilon)$ is free with generators of weight* $(8,5)$, $(8,7)$, $(8,9)$, $(8,9)$, $(8,11)$, $(8,13)$, $(8,15)$, $(8,17)$, *and* $(8,23)$.

Proof. We use the covariants

$$\begin{aligned}
\xi_1 =& 160\, C_{1,6}C_{3,2} - 208\, C_{2,0}C_{2,8} + 250\, C_{2,4}^2,\\
\xi_2 =& 60\, C_{1,6}C_{2,0}C_{3,2} + 16\, C_{2,0}^2C_{2,8} - 225\, C_{1,6}C_{5,2} - 150\, C_{2,8}C_{4,0},\\
\xi_3^{(1)} =& 4032\, C_{2,0}^3C_{2,8} + 55800\, C_{1,6}C_{2,0}C_{5,2} - 25000\, C_{1,6}C_{3,2}C_{4,0} - 46125\, C_{2,0}C_{2,4}C_{4,4},\\
& - 159500\, C_{2,0}C_{3,2}C_{3,6} + 17377500\, C_{1,6}C_{7,2} + 90750\, C_{2,8}C_{6,0} + 675000\, C_{3,6}C_{5,2} - 384375\, C_{4,4}^2\\
\xi_3^{(2)} =& 112\, C_{1,6}C_{2,0}^2C_{3,2} - 60\, C_{1,6}C_{2,0}C_{5,2} - 150\, C_{1,6}C_{3,2}C_{4,0} - 135\, C_{2,0}C_{2,4}C_{4,4} - 1440\, C_{2,0}C_{3,2}C_{3,6}\\
& + 31500\, C_{1,6}C_{7,2} + 450\, C_{2,8}C_{6,0} + 5625\, C_{3,6}C_{5,2} - 1125\, C_{4,4}^2,\\
\xi_4 =& 1792\, C_{2,0}^4C_{2,8} + 28750\, C_{1,6}C_{2,0}^2C_{5,2} - 3685500\, C_{1,6}C_{2,0}C_{7,2} - 139200\, C_{1,6}C_{3,2}C_{6,0}\\
& - 229650\, C_{1,6}C_{4,0}C_{5,2} - 93600\, C_{2,0}C_{2,8}C_{6,0} - 183150\, C_{2,0}C_{3,6}C_{5,2} + 166725\, C_{2,4}^2C_{6,0}\\
& - 40500\, C_{2,4}C_{3,2}C_{5,2} - 16875\, C_{2,4}C_{4,0}C_{4,4} - 72450\, C_{2,8}C_{4,0}^2 + 317700\, C_{3,2}^2C_{4,4}\\
& + 256500\, C_{3,2}C_{3,6}C_{4,0} + 38650500\, C_{3,6}C_{7,2} + 246600\, C_{5,4}^2,\\
\xi_5 =& 807424\, C_{2,0}^5C_{2,8} - 6707400000\, C_{1,6}C_{2,0}^2C_{7,2} - 1888920000\, C_{1,6}C_{2,0}C_{3,2}C_{6,0}\\
& - 785694375\, C_{1,6}C_{2,0}C_{4,0}C_{5,2} - 278572500\, C_{1,6}C_{3,2}C_{4,0}^2 - 120600000\, C_{2,0}^2C_{4,4}^2\\
& - 42918750\, C_{2,0}C_{2,8}C_{4,0}^2 + 5193090000\, C_{2,0}C_{3,2}^2C_{4,4} - 271446918750\, C_{1,6}C_{4,0}C_{7,2}\\
& - 5117321250\, C_{1,6}C_{5,2}C_{6,0} + 338190300000\, C_{2,0}C_{3,6}C_{7,2} + 1145700000\, C_{2,0}C_{5,4}^2\\
& + 62962200000\, C_{2,4}C_{3,2}C_{7,2} - 450720000\, C_{2,4}C_{4,4}C_{6,0} - 1831612500\, C_{2,4}C_{5,2}^2\\
& + 4053206250\, C_{2,8}C_{4,0}C_{6,0} - 12202200000\, C_{3,2}C_{3,6}C_{6,0} + 20030895000\, C_{3,2}C_{4,4}C_{5,2}\\
& + 6489787500\, C_{3,6}C_{4,0}C_{5,2} - 8640074520000\, C_{2,8}C_{10,0} - 245226240000\, C_{4,6}C_{8,2}\\
& + 170775360000\, C_{5,4}C_{7,4},\\
\xi_6 =& 8\, C_{2,0}C_{2,8} - 25\, C_{2,4}^2, \qquad \xi_7 = 48\, C_{2,0}^2C_{2,8} - 475\, C_{1,6}C_{5,2} + 625\, C_{3,2}C_{3,6},\\
\xi_8 =& 2588867072\, C_{2,0}^5C_{2,8} - 2215180800000\, C_{1,6}C_{2,0}^2C_{7,2} + 13431825000\, C_{1,6}C_{2,0}C_{4,0}C_{5,2}\\
& - 97632787500\, C_{2,0}C_{2,8}C_{4,0}^2 - 125273250000\, C_{2,0}C_{3,2}^2C_{4,4} + 1345443750000\, C_{1,6}C_{4,0}C_{7,2}\\
& + 7597800000000\, C_{2,0}C_{3,6}C_{7,2} + 95399876250000\, C_{2,4}C_{3,2}C_{7,2} - 968719500000\, C_{2,4}C_{4,4}C_{6,0}\\
& - 248030859375\, C_{2,4}C_{5,2}^2 - 178311712500\, C_{2,8}C_{4,0}C_{6,0} + 1077259500000\, C_{3,2}C_{4,4}C_{5,2}\\
& - 143877610800000\, C_{2,8}C_{10,0} - 5470416000000\, C_{4,6}C_{8,2} - 25300674000000\, C_{5,4}C_{7,4}\,.
\end{aligned}$$

We consider the following cusp forms:

$$F_{8,5} = \frac{135}{8}\nu(\xi_1)\chi_5, \quad F_{8,7} = -\frac{405}{4}\nu(\xi_2)\chi_5,$$

$$F_{8,9}^{(1)} = \frac{675}{16}\nu(\xi_3^{(1)})\chi_5, \quad F_{8,9}^{(2)} = \frac{10125}{4}\nu(\xi_3^{(2)})\chi_5,$$

$$F_{8,11} = \frac{18225}{16}\nu(\xi_4)\chi_5 \quad F_{8,13} = \frac{54675}{16}\nu(\xi_5)\chi_5, \quad F_{8,15} = -\frac{675}{4}\nu(\xi_6)\chi_5^3,$$

$$F_{8,17} = \frac{2025}{2}\nu(\xi_7)\chi_5^3, \quad F_{8,23} = -\frac{382725}{32}\nu(\xi_8)\chi_5^3\,.$$

The Fourier expansion of

$$W_{145} = F_{8,5} \wedge F_{8,7} \wedge F_{8,9}^{(1)} \wedge F_{8,9}^{(2)} \wedge F_{8,11} \wedge F_{8,13} \wedge F_{8,15} \wedge F_{8,17} \wedge F_{8,23}$$

starts with

$$c\,(u^9-u^7-4u^5+4u^3+6u-6/u-4/u^3+4/u^5+1/u^7-1/u^9)X^{17}Y^{25}+\dots$$

with $c=-2^{17}\cdot 3^{10}\cdot 5^3\cdot 7\cdot 59\cdot 67\cdot 103\cdot 429$. The order of vanishing of W_{145} along H_1 is 5, while along H_4 it is 4, so W_{145} is a multiple of $\chi_5^5\chi_{30}^4$ and a computation at the level of covariants gives

$$W_{145}=-2^{17}\cdot 3^{10}\cdot 5^3\cdot 7\cdot 59\cdot 67\cdot 103\cdot 429\,\chi_5^5\chi_{30}^4\,. \qquad \square$$

Theorem 14. *The R-module $\mathcal{M}_8^{\rm ev}(\Gamma_2,\epsilon)$ is free with generators of weight $(8,4)$, $(8,10)$, $(8,12)$, $(8,14)$, $(8,16)$,$(8,18)$, $(8,18)$, $(8,20)$, and $(8,22)$.*

Proof. We use the following covariants:

$$\xi_1=C_{3,8},\qquad \xi_5=C_{5,8},$$

$$\begin{aligned}\xi_2 &= 8\,C_{2,0}^3C_{3,8}-360\,C_{2,0}^2C_{5,8}-600\,C_{2,0}C_{3,2}C_{4,6}+28000\,C_{1,6}C_{8,2}-1875\,C_{3,2}C_{6,6}^{(1)}\\ &\quad+1500\,C_{3,2}C_{6,6}^{(2)}+3000\,C_{4,0}C_{5,8},\\ \xi_3 &= 64\,C_{2,0}^3C_{5,8}+960\,C_{2,0}^2C_{3,2}C_{4,6}-26880\,C_{1,6}C_{2,0}C_{8,2}-32760\,C_{2,0}C_{2,4}C_{7,4}\\ &\quad-600\,C_{2,0}C_{4,0}C_{5,8}+405\,C_{3,8}C_{4,0}^2-974160\,C_{1,6}C_{10,2}+705600\,C_{2,4}C_{9,4}+267120\,C_{3,6}C_{8,2}\\ &\quad-471240\,C_{4,4}C_{7,4}+3263400\,C_{4,6}C_{7,2}-44280\,C_{5,2}C_{6,6}^{(1)}+41760\,C_{5,8}C_{6,0},\\ \xi_4 &= -450785280\,C_{1,6}C_{2,0}C_{10,2}-209672400\,C_{1,6}C_{4,0}C_{8,2}-107933000\,C_{2,0}C_{2,4}C_{9,4}\\ &\quad+322793520\,C_{2,0}C_{3,6}C_{8,2}-93936640\,C_{2,0}C_{4,4}C_{7,4}+708825600\,C_{2,0}C_{4,6}C_{7,2}\\ &\quad+27870759840\,C_{1,6}C_{12,2}-6460961760\,C_{3,6}C_{10,2}-10179070440\,C_{3,8}C_{10,0}-6501163200\,C_{4,4}C_{9,4}\\ &\quad+2887120425\,C_{7,2}C_{6,6}^{(1)}+4910108700\,C_{7,2}C_{6,6}^{(2)}-19333170\,C_{2,0}C_{5,2}C_{6,6}^{(1)}+6700200\,C_{2,0}C_{5,2}C_{6,6}^{(2)}\\ &\quad+8466560\,C_{2,0}C_{5,8}C_{6,0}+104073340\,C_{2,4}C_{3,2}C_{8,2}+42245700\,C_{2,4}C_{4,0}C_{7,4}\\ &\quad+26659470\,C_{2,4}C_{5,4}C_{6,0}-21600\,C_{4,0}^2C_{5,8}+1024\,C_{2,0}^3C_{3,2}C_{4,6}+1024\,C_{2,0}^5C_{3,8},\end{aligned}$$

$$\xi_6^{(1)}=8\,C_{2,0}C_{5,8}+25\,C_{2,4}C_{5,4}+30\,C_{3,2}C_{4,6},\qquad \xi_6^{(2)}=C_{2,0}^2C_{3,8}-5\,C_{2,0}C_{5,8}-25\,C_{3,2}C_{4,6},$$

$$\begin{aligned}\xi_7 &= 128\,C_{2,0}^3C_{3,8}+158200\,C_{1,6}C_{8,2}+214200\,C_{2,4}C_{7,4}-88275\,C_{3,2}C_{6,6}^{(1)}+33900\,C_{3,2}C_{6,6}^{(2)}\\ &\quad+39900\,C_{4,0}C_{5,8},\\ \xi_8 &= 768\,C_{2,0}^4C_{3,8}+2800000\,C_{1,6}C_{2,0}C_{8,2}-2782500\,C_{2,0}C_{2,4}C_{7,4}-11979000\,C_{1,6}C_{10,2}\\ &\quad+66990000\,C_{2,4}C_{9,4}-27636000\,C_{3,6}C_{8,2}+30838500\,C_{4,4}C_{7,4}-117232500\,C_{4,6}C_{7,2}\\ &\quad+880875\,C_{5,2}C_{6,6}^{(1)}-1039500\,C_{5,2}C_{6,6}^{(2)}-1342500\,C_{5,8}C_{6,0}.\end{aligned}$$

We consider the following cusp forms:

$$F_{8,4}=-225\,\nu(\xi_1)\chi_5,\quad F_{8,10}=-\frac{6075}{512}\nu(\xi_2)\chi_5,\quad F_{8,12}=-\frac{6834375}{4}\nu(\xi_3)\chi_5,$$

$$F_{8,14}=\frac{102515625}{256}\nu(\xi_4)\chi_5,\quad F_{8,16}=50625\nu(\xi_5)\chi_5^3\quad F_{8,18}^{(1)}=\frac{151875}{4}\nu(\xi_6^{(1)})\chi_5^3,$$

$$F_{8,18}^{(2)}=-\frac{6075}{16}\nu(\xi_6^{(2)})\chi_5^3,\quad F_{8,20}=\frac{151875}{32}\nu(\xi_7)\chi_5^3,\quad F_{8,22}=-\frac{1366875}{16}\nu(\xi_8)\chi_5^3\,.$$

Then

$$W_{170}=F_{8,4}\wedge F_{8,10}\wedge F_{8,12}\wedge F_{8,14}\wedge F_{8,16}\wedge F_{8,18}^{(1)}\wedge F_{8,18}^{(2)}\wedge F_{8,20}\wedge F_{8,22}$$

is a cusp form in $S_{170}(\Gamma_2,\epsilon)$ and its Fourier expansion starts with

$$2^{36}\cdot 3^{13}\cdot 5^8\cdot 7^3\cdot 19\,(u^9+u^7-4u^5-4u^3+6u+6/u-4/u^3-4/u^5+1/u^7+1/u^9)X^{19}Y^{29}+\dots.$$

One can check that the order of vanishing of W_{170} along H_1 is 4 while along H_4 it is 5, so W_{170} is a multiple of $\chi_5^4\chi_{30}^5$. A calculation with the covariants shows

$$W_{170}=2^{36}\cdot 3^{13}\cdot 5^8\cdot 7^3\cdot 19\,\chi_5^4\chi_{30}^5\,. \qquad \square$$

11. The case $j = 10$

Theorem 15. *The R-module* $\mathcal{M}_{10}^{\rm odd}(\Gamma_2,\epsilon)$ *is free with generators of weight* $(10,5)$, $(10,7)$, $(10,9)$, $(10,9)$, $(10,11)$, $(10,11)$, $(10,13)$, $(10,13)$, $(8,15)$, $(10,15)$, *and* $(10,17)$.

Theorem 16. *The R-module* $\mathcal{M}_{10}^{\rm ev}(\Gamma_2,\epsilon)$ *is free with generators of weight* $(10,8)$, $(10,10)$, $(10,10)$, $(10,12)$, $(10,12)$, $(10,14)$,$(10,14)$, $(10,16)$, $(10,16)$, $(10,18)$, *and* $(10,20)$.

The proofs in both cases are similar to the cases above. The covariants used are quite big and we refer to [2] for these.

12. The character ϵ of Γ_2

Maass showed in [16] that the abelianization of Γ_2 is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. So Γ_2 has one nontrivial character ϵ and it is of order 2. It can be described as the composition

$$\mathrm{Sp}(4,\mathbb{Z}) \xrightarrow{\bmod 2} \mathrm{Sp}(4,\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cong} \mathfrak{S}_6 \xrightarrow{\rm sign} \{\pm 1\}\,.$$

The following rules may help in easily determining the value $\epsilon(\gamma)$. If

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then one has

$$\epsilon(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \epsilon(\begin{pmatrix} c & d \\ a & b \end{pmatrix}) = \epsilon(\begin{pmatrix} b & a \\ d & c \end{pmatrix}) = \epsilon(\begin{pmatrix} d & c \\ b & a \end{pmatrix})$$

as one sees by applying $J = (0, 1_g; -1_g, 0)$ on the left and/or on the right.

If γ satisfies

$$\det(a) \equiv \det(b) \equiv \det(c) \equiv \det(d) \equiv 0 \,(\mathrm{mod}\, 2),$$

then we have $\epsilon(\gamma) = -\epsilon(\gamma_0)$ with γ_0 obtained from γ by replacing the first row by minus the third row and the third row by the first row. For this matrix γ_0 at least one of $\det(a_0)$, $\det(b_0)$, $\det(c_0)$, $\det(d_0)$ is not zero modulo 2.

Using this we arrive at the case where γ has the property that $\det(c) \not\equiv 0(\mathrm{mod}\ 2)$.

Proposition 17. *For* $\gamma = (a,b;c,d) \in \Gamma_2$ *with* $\det(c) \not\equiv 0(\mathrm{mod}\, 2)$ *we have* $\epsilon(\gamma) = (-1)^\rho$ *with* ρ *given by*

$$\begin{aligned} a_1c_1 + a_2c_1 + a_2c_2 + a_3c_3 + a_4c_3 + a_4c_4 + c_1c_2 + c_2c_3 + c_3c_4 \\ + c_1d_4 + c_2d_3 + c_2d_4 + c_3d_2 + c_4d_1 + c_4d_2 \end{aligned}$$

where the 2×2 *matrices are written as* $\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right)$.

The proof is omitted.

13. Appendix on quasimodularity

We prove here that the Taylor expansion of a Siegel modular form of degree 2 along the diagonal $\mathfrak{H}_1^2$ yields quasimodular forms. A reference for quasimodular forms is [20, Section 5]. We write $QM_k(\Gamma_1)$ for the space of quasimodular forms of weight k on Γ_1. We will write an element τ of $\mathfrak{H}_2$ as $(\tau_1, z; z, \tau_2)$ and develop a modular form $F \in M_{j,k}(\Gamma_2)$ as a Taylor series in z, the normal coordinate of the diagonal.

Proposition 18. *Let $F \in M_{j,k}(\Gamma_2)$ and write $F = (F_0, F_1, \ldots, F_j)^t$. Then the restriction $F_l|_{\mathfrak{H}_1\times\mathfrak{H}_1}$ lies in $M_{j+k-l}(\Gamma_1) \otimes M_{k+l}(\Gamma_1)$ and for $n \geqslant 1$, we have*

$$\frac{\partial^n F_l}{\partial z^n}|_{\mathfrak{H}_1\times\mathfrak{H}_1} \in QM_{j+k-l+n}(\Gamma_1) \otimes QM_{k+l+n}(\Gamma_1).$$

Proof. The boundedness requirements for quasimodular forms are easily verified. Using the element of Γ_2 that maps $(\tau_1, z; z, \tau_2)$ to $(\tau_2, z; z, \tau_1)$ and which swaps the coordinates of F from bottom to top up to a sign $(-1)^k$, one sees that it suffices to prove

$$\frac{\partial^n F_l}{\partial z^n}(\left(\begin{smallmatrix} \gamma\tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)) = (c\tau_1+d)^{k+j-l+n} \sum_{s=0}^{n} f_s(\left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)) \left(\frac{c}{c\tau_1+d}\right)^s$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$ where the f_s are holomorphic and depend on n; see [20, p. 58]. We embed Γ_1 into Γ_2 via

$$\gamma \mapsto \tilde{\gamma} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with action} \quad \tau \mapsto \begin{pmatrix} \gamma\tau_1 & z/(c\tau_1+d) \\ z/(c\tau_1+d) & \tau_2 - cz^2/(c\tau_1+d) \end{pmatrix}.$$

The modularity of F gives $F(\tilde{\gamma}\tau) = (c\tau_1+d)^k \mathrm{Sym}^j(\left(\begin{smallmatrix} c\tau_1+d & cz \\ 0 & 1 \end{smallmatrix}\right))F(\tau)$ and a direct computation gives for $l = 0, \ldots, j$

$$F_l(\tilde{\gamma}\tau) = (c\tau_1+d)^{k+j-l} \sum_{m=0}^{j-l} (c\tau_1+d)^{-m} \binom{l+m}{l} c^m z^m F_{l+m}(\tau). \tag{2}$$

Setting $z = 0$ proves that $F_l(\tilde{\gamma}\tau) = (c\tau_1+d)^{k+j-l} F_l(\tau)$, hence the first statement and the (quasi)modularity for $n = 0$. We prove the rest by induction on n. We assume that the proposition is true for $a < n$, i.e.,

$$\frac{\partial^a F_l}{\partial z^a}(\left(\begin{smallmatrix} \gamma\tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)) = (c\tau_1+d)^{k+j-l+a} \sum_{s=0}^{a} f_s(\left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)) \left(\frac{c}{c\tau_1+d}\right)^s.$$

We differentiate n times both sides of the equation (2) with respect to z and evaluate at $z = 0$, and get

$$\frac{\partial^n F_l}{\partial z^n}(\left(\begin{smallmatrix} \gamma\tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)) \frac{1}{(c\tau_1+d)^n} + \sum_{\substack{2i+r=n \\ r\neq n}} \frac{\partial^i}{\partial \tau_2^i}\left(\frac{\partial^r F_l}{\partial z^r}(\left(\begin{smallmatrix} \gamma\tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right))\right) \frac{(-1)^i n!}{r!\, i!} \frac{c^i}{(c\tau_1+d)^{i+r}}$$

$$= (c\tau_1+d)^{k+j-l} \left(\sum_{m=0}^{j-l} \left(\frac{c}{c\tau_1+d}\right)^m \binom{l+m}{l} \frac{n!}{(n-m)!} \frac{\partial^{n-m} F_{l+m}}{\partial z^{n-m}}(\left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right))\right).$$

By using the induction hypothesis, we arrive at

$$\begin{aligned} &(c\tau_1+d)^{-(k+j-l+n)} \frac{\partial^n F_l}{\partial z^n}(\left(\begin{smallmatrix} \gamma\tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)) \\ &= \sum_{m=0}^{j-l} \left(\frac{c}{c\tau_1+d}\right)^m \binom{l+m}{l} \frac{n!}{(n-m)!} \frac{\partial^{n-m} F_{l+m}}{\partial z^{n-m}}(\left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)) \\ &\quad + \sum_{\substack{2i+r=n \\ r\neq n \\ 0\leqslant s\leqslant r}} \frac{\partial^i f_s}{\partial \tau_2^i}(\left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)) \frac{(-1)^{i+1} n!}{r!\, i!} \frac{c^{i+r+s}}{(c\tau_1+d)^{i+r+s}} \end{aligned}$$

and this shows the proposition. □

Using this proposition we can deduce that $\chi_{6,-2}$ has a Taylor expansion along $\mathfrak{H}_1^2$ with quasimodular coefficients. Indeed, suppose that a is a nonnegative integer such that $\chi_{10}^a \chi_{6,-2}$ is holomorphic. We then apply the proposition to χ_{10}^a and $\chi_{10}^a \chi_{6,-2}$ and get Taylor expansions $\sum_{\mu \geq 2a} a_\mu t^\mu$ and $\sum_{\nu \geq \nu_0} c_\nu t^\nu$ with quasimodular a_μ and c_ν. Writing the Taylor expansion of $\chi_{6,-2}$ as $\sum_\lambda b_\lambda t^\lambda$ with $c_\nu = \sum_{\mu+\lambda=\nu} a_\mu b_\lambda$ we see by induction that the b_λ are tensor products of quasimodular forms.

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