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# Modular forms via invariant theory



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## Abstract

We discuss two simple but useful observations that allow the construction of modular forms from given ones using invariant theory. The first one deals with elliptic modular forms and their derivatives, and generalizes the Rankin-Cohen bracket, while the second one deals with vector-valued modular forms of genus greater than 1.

## 1 Introduction

In this paper we present two observations on the use of invariant theory in the theory of modular forms.

The first observation is that the invariant theory of binary forms of degree  $r$  can be applied to an elliptic modular form to produce in a very simple and direct way new modular forms that are expressions in the first  $r$  derivatives of the elliptic modular form. For example, the  $r$ th transvectant of a binary form of degree  $r$  produces a Rankin-Cohen bracket. But one can use all invariants and there are many more invariants than just the transvectants that give rise to the Rankin-Cohen brackets. The novelty is to associate a binary form to a modular form and a natural number and to use invariant theory of this binary form. It extends to multi-invariants of several binary forms and then associates a modular form to a tuple of elliptic modular forms and their derivatives.

The second observation deals with vector-valued modular forms. We can view a modular form as a section of an automorphic vector bundle on a moduli space or on an arithmetic quotient of a bounded symmetric domain. Such a bundle corresponds to a factor of automorphy, or equivalently, it is obtained by applying a Schur functor to the Hodge bundle. In turn, this is given by a representation of a group, usually a general linear group, and the weight of the modular form refers to this representation. For example, for a Siegel modular form of degree  $g$  the corresponding factor of automorphy is given by an irreducible representation of  $GL(g)$ .

The second observation of this paper is now that we can apply invariant theory for this  $GL(g)$ -representation to construct new modular forms from a given one.

For example, for Siegel modular forms of degree 2 the factors of automorphy are indexed by the irreducible representations of  $GL(2)$  and here we can apply the invariant theory of binary forms to create new modular forms from a given one.

When we apply this observation to the invariant theory of binary sextics and a non-zero Siegel modular cusp form  $\chi_{6,8}$  of degree 2 and weight (6, 8), we recover the method of [2] to construct all degree two Siegel modular forms on  $\mathrm{Sp}(4, \mathbb{Z})$  from just the form  $\chi_{6,8}$ .

Similarly, for degree 3 we apply it to the invariant theory of ternary quartics and recover the method of the paper [3] that allows one to construct all Siegel modular forms on  $\mathrm{Sp}(6, \mathbb{Z})$  from one cusp form  $\chi_{4,0,8}$  of weight (4, 0, 8).

The second observation applies equally well to all kinds of arithmetic groups, to modular forms on other bounded symmetric domains, or to Teichmüller forms on the moduli spaces of curves.

We illustrate this method by examples involving Siegel modular forms, Teichmüller forms and Picard modular forms.

We finish the paper by giving an example of a generalization of the first observation to vector-valued modular forms of more variables.

### 2 The first observation

Here we start with an elliptic modular form  $f$  of weight  $k$  on some congruence subgroup  $\Gamma$  of  $\mathrm{SL}(2, \mathbb{Z})$ . The space of modular forms of weight  $k$  on  $\Gamma$  is denoted by  $M_k(\Gamma)$  and the subspace of cusp forms by  $S_k(\Gamma)$ . We write  $\tau$  for the variable in the upper half plane  $\mathfrak{H}$ .

We consider the derivatives  $f^{(n)} = d^n f / d\tau^n$  for  $n = 0, \dots, r$  of  $f$  and using these derivatives we shall associate a modular form to each invariant of binary forms of degree  $r$ .

Let  $V = \langle x_1, x_2 \rangle$  be the vector space generated by  $x_1$  and  $x_2$ . The group  $\mathrm{GL}(V) = \mathrm{GL}(2)$  acts on  $V$ . Recall that an invariant of a binary form  $\sum_{i=0}^r a_i \binom{r}{i} x_1^{r-i} x_2^i \in \mathrm{Sym}^r(V)$  of degree  $r$  is a (homogeneous) polynomial in the coefficients  $a_0, \dots, a_r$  invariant under  $\mathrm{SL}(V)$ . By its degree we mean the degree in the  $a_i$ . An invariant has order  $n$  if for all its monomials the sum of the indices of the  $a_i$  is equal to  $n$ ; equivalently, if it changes under  $\mathrm{GL}(V)$  by the  $n$ th power of the determinant. We note that for an invariant of degree  $d$  and order  $n$  we have  $dr = 2n$ . Such an invariant corresponds to a  $\mathrm{GL}(V)$ -equivariant embedding  $\det(V)^{\otimes dr} \hookrightarrow \mathrm{Sym}^d(\mathrm{Sym}^r(V))$ . We will write  $V_r$  for  $\mathrm{Sym}^r(V)$ . The set of invariants  $\mathcal{I}(V_r)$  forms in a natural way a ring graded by the degree:  $\mathcal{I}(V_r) = \bigoplus_d \mathcal{I}_d(V_r)$ .

**Theorem 2.1** *Suppose that  $I$  is an invariant of degree  $d$  and order  $n$  of a binary form  $\sum_{i=0}^r a_i \binom{r}{i} x_1^{r-i} x_2^i$  of degree  $r$  and let  $f$  be an elliptic modular form of weight  $k$  on a congruence subgroup  $\Gamma$  of  $\mathrm{SL}(2, \mathbb{Z})$ . Then by the substitution*

$$a_i \mapsto i! \binom{k+r-1}{i} f^{(r-i)}$$

for  $i = 0, \dots, r$  in  $I$ , we obtain a map  $\Psi : \mathcal{I}_d(V_r) \times M_k(\Gamma) \rightarrow M_{dk+2n}(\Gamma)$ . For fixed  $f$ , the map  $I \mapsto \Psi(I, f)$  defines a homomorphism  $\mathcal{I}(V_r) \rightarrow R(\Gamma)$  with  $R(\Gamma)$  the ring of modular forms on  $\Gamma$ .

*Proof* In the proof we will write  $\delta$  for the degree of  $I$ . To  $f \in M_k(\Gamma)$  we associate the vector-valued function

$$F = (F_0, \dots, F_r)^t : \mathfrak{H} \rightarrow \mathbb{C}^{r+1},$$

where  $F_i(\tau) = i! \binom{k+r-1}{i} f^{(r-i)}(\tau)$  for  $i = 0, \dots, r$ . By repeatedly differentiating with respect to  $\tau$  the functional equation  $f((a\tau + b)/(c\tau + d)) = (c\tau + d)^k f(\tau)$  we get

$$F\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \text{Sym}^r \begin{pmatrix} (c\tau + d)^2 & c(c\tau + d) \\ 0 & 1 \end{pmatrix} F(\tau) \tag{*}$$

for any  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ . Thus by the substitution  $a_i \mapsto F_i(\tau)$  we obtain a binary form of degree  $r$  and  $\gamma$  defines (apart from a factor  $(c\tau + d)^k$ ) an action on it by

$$A = \begin{pmatrix} (c\tau + d)^2 & c(c\tau + d) \\ 0 & 1 \end{pmatrix} \in \text{GL}(2, \mathbb{C}).$$

Since  $I$  is an invariant of order  $n$  it follows that  $\Psi(I, f)$  changes by  $\det(A)^n = (c\tau + d)^{2n}$  under this action. As  $I$  is of degree  $\delta$  we get under the substitution  $\tau \mapsto (a\tau + b)/(c\tau + d)$  in  $F$  also a factor  $(c\tau + d)^{\delta k}$  in  $\Psi(I, f)$  as equation (\*) shows. Together we get that  $\Psi(I, f)$  transforms as a modular form of weight  $\delta k + 2n$  on  $\Gamma$ . The fact that  $\Psi(I, f)$  is holomorphic on  $\mathfrak{H}$  is clear and also the conditions at the cusps of  $\Gamma$  are easily checked. That for fixed  $f$  the map  $I \rightarrow \Psi(I, f)$  is obtained by substitution makes it clear that it is a homomorphism. □

Note that  $\Psi(I, cf) = c^d \Psi(I, f)$  for  $I$  of degree  $d$  and  $c \in \mathbb{C}$ . Often we shall use the notation

$$\Psi_I(f) = \Psi(I, f).$$

*Remark 2.2* Of course, the theorem applies as well to modular forms with a character. We then get a map

$$\Psi_I : M_k(\Gamma, \chi) \rightarrow M_{kd+2n}(\Gamma, \chi^d),$$

where  $d$  is the degree of  $I$ .

The theorem makes the relation between transvectants and Rankin-Cohen brackets transparent. Recall that invariant theory associates to a pair  $(F, G)$  of binary forms of degree  $m$  and  $n$  a so-called transvectant in  $V_{m+n-2r}$  defined by

$$(F, G)_r = \frac{(m-r)!(n-r)!}{m!n!} \sum_{j=0}^r (-1)^j \binom{r}{j} \frac{\partial^r F}{\partial x_1^{r-j} \partial x_2^j} \frac{\partial^r G}{\partial x_1^j \partial x_2^{r-j}}.$$

We now replace  $r$  by  $2r$  and take  $m = n = 2r$  and  $F = G$ . If  $F \in V_{2r}$  denotes the universal binary form of degree  $2r$  then the invariant  $I = (F, F)_{2r}$  is of degree 2 and order  $2r$ . We apply this to the binary form  $F$  associated as in the proof of Theorem 2.1 to a modular form  $f \in M_k(\Gamma)$  and it gives

$$\Psi_I(f) = (2r)! (2\pi i)^{2r} [f, f]_{2r} \in M_{2k+4r}(\Gamma)$$

with  $[f, f]_{2r}$  the Rankin-Cohen bracket. Recall that for a pair of modular forms  $f_1 \in M_{k_1}(\Gamma)$ ,  $f_2 \in M_{k_2}(\Gamma)$  the  $r$ th Rankin-Cohen bracket  $[f, g]_r$  is defined as

$$[f, g]_r = \frac{1}{(2\pi i)^r} \sum_{n+m=r} (-1)^n \binom{k_1+r-1}{m} \binom{k_2+r-1}{n} \frac{d^n f}{d\tau^n} \frac{d^m g}{d\tau^m}$$

and is an element of  $M_{k_1+k_2+2r}(\Gamma)$  and is a cusp form for  $r > 0$ . It was introduced in [5] using results of [9]. See also [1, Sect. 5.2].

*Remark 2.3* Note that for the case  $r = 0$  we are dealing with the covariant  $FG$  and the product  $fg$ .

We give an example. The ring of invariants of  $V_3$  is generated by

$$I_3 = a_0^2 a_3^2 - 6 a_0 a_1 a_2 a_3 + 4 a_0 a_2^3 + 4 a_1^3 a_3 - 3 a_1^2 a_2^2.$$

By calculating the first few Fourier coefficients we see that it gives for the normalized Eisenstein series  $E_k = 1 - (2k/B_k) \sum_{n \geq 1} \sigma_{k-1}(n)q^n$  on  $SL(2, \mathbb{Z})$  of weights 4 and 6 the results

$$\Psi_{I_3}(E_4) = -53084160000\pi^6 E_4 \Delta^2, \quad \Psi_{I_3}(E_6) = -203928109056\pi^6 (8 E_4^3 + E_6^2) \Delta^2.$$

### 2.1 Examples from multi-invariants

As the above suggests we can also apply invariant theory for a tuple of binary forms. The invariant theory for the diagonal action of  $GL(2)$  on a direct sum  $V_{m_1} \oplus \dots \oplus V_{m_s}$  provides a plethora of invariants.

We may take the invariant  $I = (F, G)_r$  for  $F, G \in V_r$  and apply it to a pair of modular forms  $(f, g)$ . By associating a binary form  $F$  to  $f$  and a binary form  $G$  to  $g$  we can apply multi-invariants and we find by applying the substitution of Theorem 2.1 to both pairs  $(E, f)$  and  $(G, g)$  that

$$\Psi_I(f, g) = (-1)^r r!(2\pi i)^r [f, g]_r.$$

We recover in a transparent way the relation between Rankin-Cohen brackets and transvectants. This relation was apparently first observed by Zagier [12, p. 74], and there is an extensive literature on this relation, see for example [8].

But there are many more invariants, bi-invariants and multi-invariants than the transvectants. To a  $s$ -tuple of modular forms  $(f_1, \dots, f_s)$  with  $f_j \in M_{k_j}(\Gamma)$  we associate a  $s$ -tuple  $(F_1, \dots, F_s)$  of binary forms  $F_j \in V_{r_j}$  as in the proof of Theorem 2.1. An invariant  $I$  of the action of  $GL(2)$  on  $\oplus V_{r_j}$  of degree  $d_j$  in the coefficients of the binary form  $F_j$  and of order  $n$  defines a map

$$\Psi_I : \oplus_{j=1}^s M_{k_j}(\Gamma) \longrightarrow M_{2n + \sum_{j=1}^s d_j k_j}(\Gamma)$$

by substituting  $i! \binom{k_j+r_j-1}{i} f_j^{(r_j-i)}$  for the coefficient  $a_i^{(j)}$  of the binary form  $F_j$ .

As a concrete example, we take  $(F, G) \in V_3 \oplus V_1$  with binary forms  $F = \sum_{i=0}^3 a_i \binom{3}{i} x_1^{3-i} x_2^i$  and  $G = b_0 x_1 + b_1 x_2$ . In this case the generators of the ring of invariants are known, see [6]. For example, there is an invariant

$$I = (F, G^3)_3 = a_0 b_1^3 - 3 a_1 b_0 b_1^2 + 3 a_2 b_0^2 b_1 - a_3 b_0^3.$$

It defines a map  $\Psi_I : M_{k_1}(\Gamma) \times M_{k_2}(\Gamma) \longrightarrow M_{k_1+3k_2+6}(\Gamma)$ , e.g.,

$$\Psi_I(E_6, E_4) = \sqrt{-1} 86016\pi^3 \Delta(E_4^3 + 2E_6^2).$$

As another example, take binary forms  $F, G, H$  of degrees 3, 2, 1 with coefficients  $a_i, b_i, c_i$  and consider the tri-invariant

$$I = a_0 b_2 c_1 - 2 a_1 b_1 c_1 - a_1 b_2 c_0 + a_2 b_0 c_1 + 2 a_2 b_1 c_0 - a_3 b_0 c_0.$$

For  $f, g, h$  modular forms of weights  $k_1, k_2, k_3$  on  $\Gamma$  we have  $\Psi_I(f, g, h) \in M_{k_1+k_2+k_3+6}(\Gamma)$ .

### 3 The second observation

Here we will be dealing with vector-valued modular forms and not with derivatives. We formulate the second observation for the case where the modular form has weight  $\rho$ , with  $\rho$  denoting the highest weight of an irreducible representation  $V$  of  $GL(g)$ . Let  $U$  be the standard representation of  $GL(g)$ . The modular form can live on a bounded symmetric domain with factor of automorphy  $\rho$ , or on a moduli space where it is a section of the vector bundle  $\mathbb{E}_\rho$  constructed from the Hodge bundle  $\mathbb{E}$  by a Schur functor defined by  $\rho$ .

Recall that an invariant relative to the action of  $GL(g)$  on the irreducible representation  $V$  of highest weight  $\rho$  is an element in the algebra on  $V$  invariant under  $SL(g)$ . In fact, an invariant can be obtained by an equivariant embedding of  $GL(g)$ -representations

$$\det(U)^{\otimes m} \hookrightarrow \text{Sym}^d(V). \tag{3.1}$$

Viewing the fibre of the dual bundle  $\mathbb{E}_\rho^\vee$  of  $\mathbb{E}_\rho$  as a  $GL(g)$ -representation and a section  $f$  of  $\mathbb{E}_\rho$  as a function on  $\mathbb{E}_\rho^\vee$ , we can evaluate  $\text{Sym}^d(f)$  via the embedding (3.1) on the subbundle  $(\det \mathbb{E}^\vee)^{\otimes m}$  and obtain a section of the bundle  $\det(\mathbb{E})^m$ .

More generally, if  $W$  is an irreducible representation of  $GL(g)$  of highest weight  $\sigma$ , an equivariant embedding

$$W \hookrightarrow \text{Sym}^d(V) \tag{3.2}$$

defines a concomitant of type  $(d, \sigma)$  for  $\rho$ . Equivalently, the equivariant embedding  $\phi : W \hookrightarrow \text{Sym}^d(V)$  may be viewed as an equivariant embedding  $\phi' : \mathbb{C} \rightarrow \text{Sym}^d(V) \otimes W^\vee$ . Then the image  $\phi'(1)$  is called a concomitant.

Again, if  $f$  is a modular form, say a section of  $\mathbb{E}_\rho$ , then we can view  $f$  as a function on  $\mathbb{E}_\rho^\vee$ , and  $\text{Sym}^d(f)$  as a function on  $\text{Sym}^d(\mathbb{E}_\rho^\vee)$ . Then by restriction to  $(\mathbb{E}^\vee)_\sigma$  via (3.2) we obtain a modular form of weight  $\sigma$  from  $f$ . Alternatively, by projection in  $\text{Sym}^d(\mathbb{E}_\rho)$  of the section  $\text{Sym}^d(f)$  we get a form of weight  $\sigma$ . By the phrase ‘applying the invariant or concomitant to  $f$ ’ we mean this evaluation. We thus obtain the following.

**Observation 3.1** *Let  $f$  be a modular form of weight  $\rho$  and let  $I$  be a concomitant of type  $(d, \sigma)$  for some  $d$  for the action of  $GL(g)$  on the representation  $\rho$ . Then by applying  $I$  to  $f$  one obtains a modular form of weight  $\sigma$ .*

Instead of spelling this out in detailed notations we will illustrate it by a couple of examples in the next section. One may formulate variants of this involving a finite set of modular forms to which invariant theory (via multi-invariants) is applied.

### 4 Illustrations

We now illustrate the second observation by a number of special cases.

#### 4.1 Siegel modular forms of degree two

Here we are dealing with the representation theory of  $GL(2)$  and the invariant theory of binary forms. There is a wealth of explicit results in invariant theory that can be applied.

Let  $\Gamma \subset Sp(4, \mathbb{Q})$  be a group commensurable with  $Sp(4, \mathbb{Z})$ . We let  $\rho$  be an irreducible representation of  $GL(2)$  of highest weight  $(j + k, k)$ . By Observation 3.1 we find for a given modular form  $f \in M_{j,k}(\Gamma)$ , that is, a section of  $\text{Sym}^j(\mathbb{E}) \otimes \det(\mathbb{E})^k$ , a homomorphism

$$\Psi_\bullet(f) : \mathcal{I}(U_j) \longrightarrow R(\Gamma), \quad J \mapsto \Psi_J(f),$$

of the ring of invariants  $\mathcal{I}(U_j)$  of binary forms of degree  $j$  to the ring  $R(\Gamma)$  of scalar-valued modular forms on  $\Gamma$ .

Let us illustrate this with the simplest non-trivial case for  $j = 2$ . The discriminant  $I = a_1^2 - 4a_0a_2$  of a quadratic form  $a_0x_1^2 + a_1x_1x_2 + a_2x_2^2$  gives an invariant of degree 2. If  $f$ , the transpose of  $(f_0, f_1, f_2)$ , is a modular form of weight  $(2, k)$  on some congruence subgroup  $\Gamma$  of  $\text{Sp}(4, \mathbb{Z})$  then we find a scalar-valued one  $\Psi_I(f) = f_1^2 - 4f_0f_2 \in M_{0,2k+2}(\Gamma)$ . The weight  $(0, 2k + 2)$  follows from the identity of  $\text{GL}(2)$ -representations

$$\text{Sym}^2(V_2) = V_4 \oplus \det(V)^2.$$

This can be extended to tuples of modular forms by using multi-invariants of binary forms. We write  $U_n = \text{Sym}^n(U)$  with  $U$  the standard representation of  $\text{GL}(2)$ . We consider the ring of invariants

$$\mathcal{I}_{n_1, \dots, n_m} = \mathcal{I}(U_{n_1} \oplus \dots \oplus U_{n_m})$$

relative to the action of  $\text{GL}(2)$ . One can consider elements of  $\mathcal{I}_{n_1, \dots, n_m}$  as (multi-)invariants of a  $m$ -tuple of binary forms  $b_1, \dots, b_m$ . Let  $J \in \mathcal{I}_{n_1, \dots, n_m}$  be a multi-invariant that is of degree  $(d_1, \dots, d_m)$  in the coefficients of the binary forms  $b_1, \dots, b_m$ . Then applying  $J$  to an  $m$ -tuple of modular forms  $f_i \in M_{j_i, k_i}(\Gamma_i)$  defines a map

$$\prod_{i=1}^m M_{j_i, k_i}(\Gamma) \longrightarrow M_{0, k}(\Gamma) \quad \text{with } k = \sum_{i=1}^m d_i(k_i + j_i/2).$$

The weights are obtained from representation theory as in the example above. For example, take  $(n_1, n_2) = (4, 2)$ . If we write

$$b_1 = \sum_{i=0}^4 \alpha_i x_1^{4-i} x_2^i, \quad b_2 = \sum_{i=0}^2 \beta_i x_1^{2-i} x_2^i,$$

where now (unlike before) for convenience we do not use binomial coefficients in the binary forms, we have the invariant

$$J_{1,2} = 6 \alpha_0 \beta_2^2 - 3 \alpha_1 \beta_1 \beta_2 + 2 \alpha_2 \beta_0 \beta_2 + \alpha_2 \beta_1^2 - 3 \alpha_3 \beta_0 \beta_1 + 6 \alpha_4 \beta_0^2.$$

The invariant  $J_{1,2}$  defines a map

$$M_{4, k_1}(\Gamma_1) \otimes M_{2, k_2}(\Gamma_2) \longrightarrow M_{0, k_1+2k_2+4}(\Gamma).$$

Next we look at covariants of binary forms. Recall that the ring of covariants  $\mathcal{C}(U_j)$  of a binary form of degree  $j$  can be identified with the ring of invariants  $\mathcal{I}(U_j \oplus U_1)$  of  $U_j \oplus U_1$ , see [10, 3.3.9]. If we write an element of  $U_1$  as  $l_1x_1 + l_2x_2$ , the isomorphism can be given explicitly by substituting  $l_1 = -x_2$  and  $l_2 = x_1$  in an invariant of  $U_j \oplus U_1$ .

The following proposition is a direct consequence of Observation 3.1.

**Proposition 4.1** *If  $C \in \mathcal{C}(U_j)$  is a covariant of degree  $a$  in the coefficients of the binary form and of degree  $b$  in  $x_1, x_2$ , then applying  $C$  defines a map*

$$\Psi_C : M_{j, k}(\Gamma) \longrightarrow M_{b, a(k+j/2)-b/2}(\Gamma).$$

*Proof* We write  $U[n + m, m]$  for  $\text{Sym}^n(U) \otimes \det(U)^m$ . The covariant  $C$  corresponds to an equivariant embedding  $U[b + l, l] \hookrightarrow \text{Sym}^a(U[j + k, k])$  for some  $l$ . To determine  $l$  we take out a factor  $\det(U)^{\otimes k}$  and look at the irreducible representations occurring in  $\text{Sym}^a(U[j, 0])$  and these are of the form  $U[aj - r, r]$  for non-negative  $r$ . Therefore, if  $U[b + l, l]$  occurs then  $(b + l, l) = (aj - r + ak, r + ak)$ , that is,  $2l = aj + 2ak - b$ .  $\square$

If we fix a modular form  $f \in M_{j,k}(\Gamma)$  we get an induced map

$$\Psi_{\bullet}(f) : \mathcal{C}(U_j) \rightarrow M(\Gamma), \quad C \mapsto \Psi_C(f),$$

where  $M(\Gamma)$  is the ring of vector-valued Siegel modular forms on  $\Gamma$  of degree 2.

In the paper [2] it was shown that for  $\Gamma = \text{Sp}(4, \mathbb{Z})$ ,  $j = 6$  and  $f = \chi_{6,8}$ , a generator of the space of cusp forms  $S_{6,8}(\text{Sp}(4, \mathbb{Z}))$ , every vector-valued modular form on  $\text{Sp}(4, \mathbb{Z})$  can be obtained from a form  $\Psi_C(\chi_{6,8})$  for a  $C \in \mathcal{C}(U_6)$  after dividing by an appropriate power of the cusp form  $\chi_{10}$  of weight 10.

Alternatively, using the meromorphic modular form  $\chi_{6,-2} = \chi_{6,8}/\chi_{10}$ , we found maps

$$M(\text{Sp}(4, \mathbb{Z})) \longrightarrow \mathcal{C}(U_6) \xrightarrow{\Psi_{\bullet}(\chi_{6,-2})} M(\text{Sp}(4, \mathbb{Z}))[1/\chi_{10}],$$

the composition of which is the identity.

A variation of this deals with multi-covariants. We can also allow modular forms with a character.

**Proposition 4.2** *If  $C$  is a covariant in  $\mathcal{C}(U_{j_1} \oplus \dots \oplus U_{j_m})$  of degree  $a_i$  in the coefficients of the binary form in  $U_{j_i}$  and degree  $b$  in  $x_1, x_2$ , then  $C$  defines a map*

$$\otimes_{i=1}^m M_{j_i, k_i}(\Gamma, \chi_i) \longrightarrow M_{b,k}(\Gamma, \chi_1^{a_1} \dots \chi_m^{a_m})$$

with  $k = \sum_{i=1}^m a_i(k_i + j_i/2) - b/2$ .

#### 4.2 Siegel and Teichmüller modular forms of degree three

If  $U_{\rho}$  is an irreducible representation of  $\text{GL}(3)$  of highest weight  $\rho$  and  $f$  is a Siegel modular form of weight  $\rho$  on some group  $\Gamma \subset \text{Sp}(6, \mathbb{Q})$  commensurable with  $\text{Sp}(6, \mathbb{Z})$ , then we get a homomorphism

$$\Psi_{\bullet}(f) : \mathcal{I}(U_{\rho}) \longrightarrow R(\Gamma)$$

of the ring  $\mathcal{I}(U_{\rho})$  of invariants to  $R(\Gamma)$ , the ring of scalar-valued modular forms on  $\Gamma$ .

We can extend this map. For a given irreducible representation  $\sigma$  of  $\text{GL}(3)$  we let  $\mathcal{C}_{\sigma}(U_{\rho})$  be the  $\mathcal{I}(U_{\rho})$ -module of covariants obtained from equivariant embeddings of  $U_{\sigma}$  into the symmetric algebra on  $U_{\rho}$ . We get for a given form  $f \in M_{\rho}(\Gamma)$  a map

$$\Psi_{\bullet}(f) : \mathcal{C}_{\sigma}(U_{\rho}) \longrightarrow M_{\sigma}(\Gamma), \quad C \mapsto \Psi_C(f),$$

where  $M_{\sigma}(\Gamma)$  is the  $R(\Gamma)$ -module  $\oplus_k M_{\sigma \otimes_{\det} k}(\Gamma)$ .

The moduli space  $\overline{\mathcal{M}}_3$  of stable curves of genus 3 carries a Hodge bundle, also denoted by  $\mathbb{E}$ . Its restriction to  $\mathcal{M}_3$  is the pullback of the Hodge bundle on  $\mathcal{A}_3$  under the Torelli map. For given irreducible representation  $\rho$  of  $\text{GL}(3)$  we have a vector bundle  $\mathbb{E}_{\rho}$  obtained by a Schur functor from  $\mathbb{E}$  and  $\rho$ . Sections of  $\mathbb{E}_{\rho}$  on  $\overline{\mathcal{M}}_3$  are called Teichmüller forms of degree or genus 3 and weight  $\rho$ . We have a graded ring of scalar-valued Teichmüller forms  $T_3 = \oplus_k H^0(\overline{\mathcal{M}}_3, \det(\mathbb{E})^k)$  of genus 3. The ring  $T_3$  is a quadratic extension of the ring of scalar-valued Siegel modular forms  $R(\text{Sp}(6, \mathbb{Z}))$  by  $\chi_9$ , with  $\chi_9$  the Teichmüller modular cusp form of weight 9 that vanishes simply on the closure of the hyperelliptic locus. It was introduced by Ichikawa, see [7]. Its square is a Siegel cusp form of weight 18, the product of the 36 even theta constants.

Given a Teichmüller modular form  $f$  of weight  $\rho$  we have a similar map

$$\Psi_{\bullet}(f) : \mathcal{C}_{\sigma}(U_{\rho}) \longrightarrow T_{\sigma}(\Gamma),$$



where  $T_\sigma(\Gamma)$  is the  $T_3$ -module

$$\bigoplus_k H^0(\overline{\mathcal{M}}_3, \mathbb{E}_\sigma \otimes \det \mathbb{E}^{\otimes k}).$$

With  $\chi_{4,0,8}$  a generator of the space of cusp forms  $S_{4,0,8}(\mathrm{Sp}(6, \mathbb{Z}))$ , the quotient  $\chi_{4,0,-1} = \chi_{4,0,8}/\chi_9$  is a meromorphic section of  $\mathrm{Sym}^4(\mathbb{E}) \otimes \det(\mathbb{E})^{-1}$  on  $\overline{\mathcal{M}}_3$ . In [3] we used this form to construct maps

$$H^0(\overline{\mathcal{M}}_3, \mathbb{E}_\sigma) \longrightarrow C_\sigma(\mathrm{Sym}^4(U)) \xrightarrow{\Psi \cdot (\chi_{4,0,-1})} H^0(\overline{\mathcal{M}}_3, \mathbb{E}_\sigma)[1/\chi_9]$$

the composition of which is the identity. Here  $U$  is the standard representation of  $\mathrm{GL}(3)$ . This enables one to construct all Teichmüller and all Siegel modular forms of genus 3 on  $\mathrm{Sp}(6, \mathbb{Z})$  by concomitants for the action of  $\mathrm{GL}(3)$  on ternary quartics using  $\chi_{4,0,-1}$ .

### 4.3 Teichmüller forms of genus 3 and 4

In [11] a Teichmüller modular form  $f$  of weight  $(2, 0, 0, 8)$ , a section of  $\mathrm{Sym}^2(\mathbb{E}) \otimes \det(\mathbb{E})^8$  on  $\overline{\mathcal{M}}_4$ , is constructed. Here  $\mathbb{E}$  is the Hodge bundle on  $\overline{\mathcal{M}}_4$ . This Teichmüller modular form can not be obtained by pulling back a Siegel modular form under the Torelli morphism. It is associated to the quadric containing the canonical image of the generic curve of genus 4.

As an invariant we now take the discriminant  $I$  of a quadratic form in four variables and apply it to  $f$ . It yields a scalar-valued Teichmüller modular form of weight 34 that vanishes on the closure of the locus of non-hyperelliptic curves of genus 4 for which the unique quadric that contains the canonical image is singular. Its square is the pull back of a Siegel modular form of degree 4 and weight 68 that is the product of all the even theta characteristics.

An analogous case is given by the section  $\chi_{2,0,4}$  of  $\mathrm{Sym}^2(\mathbb{E}) \otimes \det(\mathbb{E})^4$  on the Hurwitz space  $\overline{\mathcal{H}}_{3,2}$  of admissible covers of genus 3 and degree 2, constructed in [11] and  $\mathbb{E}$  the corresponding Hodge bundle. Its discriminant is a modular form of weight 14, related to the discriminant of binary octics, whose square is the pullback of a Siegel modular cusp form of weight 28.

### 4.4 Picard modular forms

Here we fix an imaginary quadratic field  $F$  with ring of integers  $O_F$  and consider a non-degenerate Hermitian form  $h = z_1 z'_2 + z'_1 z_2 + z_3 z'_3$  on the vector space  $F^3$  with the prime denoting complex conjugation. The group of similitudes of  $h$  is an algebraic group  $G$  over  $\mathbb{Q}$  of type  $\mathrm{GU}(2, 1)$ . The connected component  $G^+(\mathbb{R})$  of  $G(\mathbb{R})$  acts on the set  $\mathfrak{B}$  of negative complex lines

$$\mathfrak{B} = \{L : L \subset F^3 \otimes_{\mathbb{Q}} \mathbb{R}, \dim_{\mathbb{C}} L = 1, h|_L < 0\}$$

which can be identified with the complex 2-ball  $\{(u, v) \in \mathbb{C}^2 : v + \bar{v} + u\bar{u} < 0\}$ . If  $\Gamma$  is an arithmetic subgroup of  $G$ , the quotient  $\Gamma \backslash \mathfrak{B}$  is a moduli space of 3-dimensional abelian varieties with multiplication by  $F$ . It carries a Hodge bundle  $\mathbb{E}$  that splits as  $W \oplus L$  of rank 2 and 1 and we obtain two factors of automorphy. We have  $\det(W)^6 = L^6$ . We then have modular forms that can be seen as sections of  $\mathrm{Sym}^j(W) \otimes L^k$ . If  $\det(W) \neq L$  we have to deal with modular forms with a character.

In the paper [4] we considered the case where  $F = \mathbb{Q}(\sqrt{-3})$  and  $\Gamma = \Gamma[\sqrt{-3}]$  is a certain congruence subgroup. We constructed modular forms  $\chi_{1,1} \in M_{1,1}(\Gamma[\sqrt{-3}], \det)$  and  $\chi_{4,4} \in M_{4,4}(\Gamma[\sqrt{-3}], \det^2)$ . We also have a scalar-valued cusp form  $\zeta \in S_{0,6}(\Gamma[\sqrt{-3}], \det)$ .



We refer to [4] for the notation. The quotient  $\chi_{4,-2} = \chi_{4,4}/\zeta$  is a meromorphic modular form of weight  $(4, -2)$ . We can apply the second observation to this situation starting from the two forms  $\chi_{1,1}$  and  $\chi_{4,-2}$ . We constructed in [4] maps

$$M(\Gamma) \rightarrow \mathcal{C}(V_1 \oplus V_4) \xrightarrow{\Psi_{\bullet}(\chi_{1,1}, \chi_{4,-2})} M(\Gamma)[1/\zeta]$$

from the ring  $M(\Gamma)$  of vector-valued modular forms to the ring  $\mathcal{C}(V_1 \oplus V_4)$  of bi-covariants for the action of  $GL(2)$  on  $V_1 \oplus V_4$ . The map  $\Psi_{\bullet}(\chi_{1,1}, \chi_{4,-2})$  illustrates Observation 3.1. The composition of the maps is the identity and this shows that we can obtain all modular forms this way.

### 5 An example involving both observations

The example we now treat deals with Picard modular forms living on the 2-ball and is inspired by both observations and constructs new vector-valued Picard modular forms from given Picard modular forms and their derivatives. For simplicity’s sake we deal with modular forms of weight  $(1, k)$  on the Picard modular group  $\Gamma[\sqrt{-3}]$ , see [4] for definitions and notation. Let  $f \in M_{1,k}(\Gamma[\sqrt{-3}])$  be a modular form. We write  $f$  as the transpose of  $(f_0, f_1)$ . We can view  $f$  as defining a linear form  $l = f_0x_1 + f_1x_2$ . We put  $\partial f$  as the transpose of

$$(f_{0u}, (f_{1u} + f_{0v})/2, f_{1v}),$$

where  $f_{iu} = \partial f_i / \partial u$  and  $f_{iv} = \partial f_i / \partial v$  for  $i = 0, 1$ . We view  $\partial f$  as the vector of coefficients of a binary form  $q$  of degree 2.

Writing  $V = \langle x_1, x_2 \rangle$  and  $l = a_0x_1 + a_1x_2$  and  $q = b_0x_1^2 + 2b_1x_1x_2 + b_2x_2^2$  we consider now multi-invariants for the action of  $GL(2)$  on  $V_1 \oplus V_2$ .

**Proposition 5.1** *Let  $I$  be a multi-invariant of the binary forms  $l$  and  $q$ . Assume that  $I$  has degree  $d_1$  in the  $a_i$  and degree  $d_2$  in the  $b_i$  and has order  $n$ . Then for  $f \in M_{1,k}(\Gamma)$  the expression  $\Psi_I(f, \partial f)$  obtained by evaluating the multi-invariant  $I$  defines a Picard modular form of weight  $(0, d_1 k + d_2(k + 1) + n)$  on  $\Gamma$ .*

The proof follows the pattern of the proof of Theorem 2.1. It seems complicated to extend it to the case where higher derivatives are involved.

We finish by giving an example. If we let  $I$  be the bi-invariant

$$I = a_0^2 b_2 - 2a_0 a_1 b_1 + a_1^2 b_0$$

and  $f = E_{1,1}$ , a vector-valued Eisenstein series defined in [4], then it turns out that  $\Psi_I(f)$  is a multiple of the scalar-valued modular form  $\zeta$  of weight 6 that occurred above. More precisely, with  $f = E_{1,1} \in M_{1,1}(\Gamma[\sqrt{-3}], \det)$  with Fourier expansion

$$E_{1,1}(u, v) = \sum_{\alpha \in O_F} \left[ \frac{X'(\alpha u)}{\sqrt{3} \tilde{\alpha} X(\alpha u)} \right] q_v^{N(\alpha)} = \begin{bmatrix} X'(0) \\ 0 \end{bmatrix} + 6 \begin{bmatrix} X'(u) \\ \frac{2\pi}{\sqrt{3}} X(u) \end{bmatrix} q_v + 6 \begin{bmatrix} (XYZ)'(u) \\ \frac{2\pi}{\sqrt{3}} 3XYZ(u) \end{bmatrix} q_v^3 + \dots$$

with  $X, Y, Z$  elliptic modular functions, see [4] for the notation, we get

$$\Psi_I(E_{1,1})(u, v) = 8\pi^2 a^2 \left( Xq_v + \frac{9X}{a^2} (4XX'' - (X')^2 + a^2YZ) \right) q_v^3 + \dots$$

with  $a = X'(0)$ , but we know that  $4XX'' - (X')^2 = -a^2YZ$ , so we get

$$\Psi_I(E_{1,1})(u, v) = 8\pi^2 a^2 (Xq_v - 27XYZq_v^3 + \dots) = 8\pi^2 a^2 \zeta(u, v).$$

Here  $\zeta \in S_{0,6}(\Gamma[\sqrt{-3}], \det)$  is the form that appeared in the preceding section.

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