

# LECTURES ON $\mathcal{A}_g$ , THE MODULI SPACE OF ABELIAN VARIETIES

GERARD VAN DER GEER

## 1. LECTURES

- (1) 05-09: Introduction, moduli space of complex abelian varieties
- (2) 10-09: The rank-1 compactification, Satake compactification /  $\mathbb{C}$
- (3) 12-09: The existence of  $\mathcal{A}_g$  as an DM-stack
- (4) 19-09: The Hodge bundle, Rank-1 degenerations
- (5) 24-09: Toroidal compactifications
- (6) 26-09: Satake compactification
- (7) 29-09: Tautological classes
- (8) 08-10: Kodaira dimension of  $\mathcal{A}_g$
- (9) 10-10: Stratifications on  $\mathcal{A}_g$
- (10) 15-10: Cohomology and Chow ring of  $\mathcal{A}_g$
- (11) 17-10: Canonical model
- (12) 22-10: Modular forms
- (13) 24-10: Compact subvarieties of  $\mathcal{A}_g$

## 2. SPECIFIC REFERENCES PER LECTURE

- (1) [8, 81]
- (2) [84, 95, 39, 16]
- (3) [79, 80, 78, 23, 35, 3, 4, 36, 102]
- (4) [84, 44]
- (5) [6, 85, 35, 62]
- (6) [1, 86, 35, 46]
- (7) [43, 31, 44, 27, 28]
- (8) [84, 38, 98, 24, 100]

## REFERENCES

- [1] V. Alexeev: *Complete moduli in the presence of semiabelian group action*. Ann. of Math. (2) 155 (2002), pp. 611–708.
- [2] V. Alexeev, I. Nakamura: *On Mumford's construction of degenerating abelian varieties*. Tohoku Math. J. (2) 51 (1999), pp. 399–420.
- [3] M. Artin: *Algebraization of formal moduli I*. In: Global Analysis, Papers in Honor of K. Kodaira. 21-71, Princeton Univ. Press, Princeton, 1969.
- [4] M. Artin: *Algebraization of formal moduli II. Existence of modifications*. Ann. of Math. 91 (1970), 88-135.

- [5] A. Ash, D. Mumford, M. Rapoport, Y. Tai: Smooth compactification of locally symmetric varieties. Lie Groups: History, Frontiers and Applications, Vol. IV. Math. Sci. Press, Brookline, Mass., 1975. iv+335 pp.
- [6] A. Ash, D. Mumford, M. Rapoport, Y. Tai: Smooth compactification of locally symmetric varieties. Second edition. With the collaboration of Peter Scholze. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2010.
- [7] W. Baily, A. Borel: *Compactification of arithmetic quotients of bounded symmetric domains*. Ann. of Math. (2) 84 (1966), pp. 442–528.
- [8] C. Birkenhake, H. Lange: *Complex abelian varieties* Grundlehren Math. Wiss., 302 Springer-Verlag, Berlin, 2004. xii+635 pp.
- [9] M. Brandt, J. Bruce, M. Chan, M. Melo, G. Moreland, C. Wolfe: *On the top weight rational cohomology of  $\mathcal{A}_g$* . Geom. Topol. 28, No. 2, 497–538 (2024). [arXiv:2012.02892](#)
- [10] J. Bergström, C. Faber, G. van der Geer: *Siegel modular forms of degree three and the cohomology of local systems*. Selecta Math. (N.S.) 20 (2014), no. 1, 83–124. [arXiv:1108.3731](#)
- [11] A. Borel: *Stable real cohomology of arithmetic groups II*. In: Manifolds and Lie groups, Birkhäuser-Boston, 1981.
- [12] A. Borel, J.-P. Serre: *Le théorème de Riemann-Roch*. Bull. Soc. Math. France 86 (19580), pp. 97–136.
- [13] F. Brown, M. Chan, S. Galatius, S. Payne: *Hopf algebras in the cohomology of  $\mathcal{A}_g$ ,  $\mathrm{GL}_n(\mathbb{Z})$  and  $\mathrm{SL}_n(\mathbb{Z})$* . [arXiv: 2405.11528v1](#).
- [14] J. Bruce, M. Chan, G. Moreland, C. Wolfe: *On the top-weight rational cohomology of  $\mathcal{A}_g$* . Geom. Topol. 28 (2024), no. 2, 497–538.
- [15] J. Bruinier, G. van der Geer, G. Harder, D. Zagier: *The 1–2–3 of Modular Forms*. Springer Verlag, 2008.
- [16] H. Cartan: *Fonctions automorphes*. Séminaire No 10, Paris 1957/58.
- [17] F. Cléry, C. Faber, G. van der Geer: *Concomitants of ternary quartics and vector-valued Siegel and Teichmüller modular forms of genus three*. Selecta Math. (N.S.) 26 (2020), no. 4, Paper No. 55, 39 pp.
- [18] R. Charney, R. Lee: *Cohomology of the Satake compactification*. Topology 22 (1983), pp. 389–423.
- [19] J. Chen, E. Looijenga: *The stable cohomology of the Satake compactification*. Geometry and Topology 21 (2017), 2231–2241.
- [20] E. Clader, S. Grushevsky, F. Janda, D. Zakharov: *Powers of the theta divisor and relations in the tautological ring* Int. Math. Res. Not. IMRN(2018), no. 24, 7725–7754.
- [21] F. Cléry, C. Faber, G. van der Geer: *Concomitants of ternary quartics and vector-valued Siegel and Teichmüller modular forms of genus three*. Selecta Math. (N.S.) 26 (2020), no. 4, Paper No. 55, 39.
- [22] P. Deligne: *Formes modulaires et représentations  $\ell$ -adiques*. Séminaire Bourbaki 1968–69, exp. 355.
- [23] P. Deligne, D. Mumford: *The irreducibility of the moduli space of curves of given genus*. Publ. Math. I.H.E.S. 36, 75–110 (1969).
- [24] M. Dittmann, R. Salvati Manni, N. Scheithauer: *Harmonic theta series and the Kodaira dimension of  $\mathcal{A}_6$* . Algebra Number Theory 15, No. 1, 271–285 (2021).
- [25] M. Dutour Sikirić, K. Hulek, A. Schürmann: *Smoothness and singularities of the perfect form and second Voronoi compactification of  $\mathcal{A}_g$* . Algebraic Geometry 2 (2015), 642–653.
- [26] T. Ekedahl: *On supersingular curves and abelian varieties* Math. Scand. 60 (1987), 151–178.
- [27] T. Ekedahl, G. van der Geer; *The order of the top Chern class of the Hodge bundle on the moduli space of abelian varieties*. Acta Math. 192 (2004), pp. 95–109.

- [28] T. Ekedahl, G. van der Geer: *Cycles representing the top Chern class of the Hodge bundle on the moduli space of abelian varieties*. Duke Math. J. 129 (2005), pp. 187–199.
- [29] T. Ekedahl, G. van der Geer: *Cycle classes of the E-O stratification on the moduli of abelian varieties*. In: Algebra, Arithmetic, and Geometry: in honor of Yu. I. Manin. Vol. I, pp. 567–636, Progr. Math., 269, Birkhäuser Boston, Inc., Boston, MA, 2009
- [30] C. Erdenberger, S. Grushevsky, K. Hulek: *Some intersection numbers of divisors on toroidal compactifications of  $\mathcal{A}_g$* . J. Algebraic Geom. 19 (2010), pp. 99–132
- [31] H. Esnault, E. Viehweg: *Chern classes of Gauss-Manin bundles of weight 1 vanish*. K-theory 26 (2002), pp. 287–305.
- [32] C. Faber, G. van der Geer: *Sur la cohomologie des systèmes locaux sur les espaces des modules des courbes de genre 2 et des surfaces abéliennes*, I, II. C.R. Acad. Sci. Paris, Sér. I, **338** (2004), 381–384, 467–470.
- [33] G. Faltings: *On the cohomology of locally symmetric Hermitian spaces*. Paul Dubreil and Marie-Paule Malliavin algebra seminar, 35th year (Paris, 1982), p. 55–98, Lecture Notes in Math., 1029, Springer, Berlin, 1983.
- [34] G. Faltings: *Arithmetische Kompaktifizierung des Modulraums der abelschen Varietäten* Workshop Bonn 1984 (Bonn, 1984), pp. 321–383, Lecture Notes in Math., 1111, Springer, Berlin, 1985.
- [35] G. Falting, C-L. Chai: *Degeneration of abelian varieties*. Ergebnisse der Mathematik 22. Springer Verlag.
- [36] B. Fantechi: *Stacks for everybody*
- [37] E. Freitag: *Der Körper der Siegelschen Modulfunktionen*. Abh. Math. Sem. Univ. Hamburg 47 (1975), 25–41.
- [38] E. Freitag: *Die Kodairadimension von Körpern automorpher Funktionen*. J. reine und angewandte Math. 296 (1977), pp. 162–170.
- [39] E. Freitag: *Siegelsche Modulfunktionen*. Grundlehren der Mathematischen Wissenschaften 254. Springer-Verlag, Berlin, 1983.
- [40] E. Freitag, K. Pommerening: *Reguläre Differentialformen des Körpers der Siegelschen Modulfunktionen*. J. Reine Angew. Math. 331 (1982), pp. 207–220.
- [41] W. Fulton: *Intersection Theory*. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Springer-Verlag, Berlin, 1998.
- [42] W. Fulton, P. Pragacz: *Schubert varieties and degeneracy loci*. Lecture Notes in Math. 1689. Springer Verlag 1998.
- [43] G. van der Geer: *Cycles on the moduli space of abelian varieties* In: Moduli of curves and abelian varieties. Editors: C. Faber and E. Looijenga. Aspects Math. E33, pp. 65–89.
- [44] G. van der Geer: *The cohomology of the moduli space of abelian varieties*, Adv. Lect. Math. (ALM), 24 International Press, Somerville, MA, 2013, 415457.
- [45] G. van der Geer: *The Chow ring of the moduli space of abelian threefolds* J. Algebraic Geom. 7 (1998), pp. 753–770. Corrigendum: J. Algebraic Geom. 18 (2009), pp. 795–796,
- [46] G. van der Geer: *Siegel modular forms and their applications*. The 1-2-3 of Modular Forms, 181–245, Universitext, Springer, Berlin, 2008.
- [47] G. van der Geer: *Rank one Eisenstein cohomology of local systems on the moduli space of abelian varieties*. Sci. China Math., 54 (2011), doi 10.1007/s11425-010-4159-4.
- [48] E. Getzler: *Euler characteristics of local systems on  $\mathcal{M}_2$* . Compositio Math. 132 (2002), pp. 121–135.
- [49] A. Grothendieck: *Classes de Chern et représentations linéaires des groupes discrets*. In: Dix Exposés sur la Cohomologie des Schémas. Advanced Studies in Pure Math. 3, pp. 215–305.

- [50] S. Grushevsky: *Geometry of  $\mathcal{A}_g$  and its compactifications*. In: Algebraic geometry, Seattle 2005. Part 1, pp. 193–234, Proc. Sympos. Pure Math., 80, Part 1, Amer. Math. Soc., Providence, RI, 2009.
- [51] S. Grushevsky, R. Salvati Manni: *The loci of abelian varieties with points of high multiplicity on the theta divisor*. Geom. Dedicata, 139:233–247, 2009.
- [52] S. Grushevsky, K. Hulek: *The intersection cohomology of the Satake compactification of  $\mathcal{A}_g$  for  $g \leq 4$* . Math. Ann. (2017) 369:1353–1381. DOI 10.1007/s00208–016–1491–1.
- [53] S. Grushevsky, K. Hulek, O. Tommasi: *Stable Betti numbers of (partial) toroidal compactifications of the moduli space of abelian varieties*. Geometry and physics. Vol. II, Oxford Univ. Press, Oxford, 2018. With an appendix by Mathieu Dutour Sikirić, pp. 581609.
- [54] S. Grushevsky, K. Hulek, O. Tommasi: *Stable cohomology of the perfect cone toroidal compactification of  $\mathcal{A}_g$* . J. reine angew. Math. 741 (2018), 211–254.
- [55] S. Grushevsky, T. Ibukiyama, G. Mondello, R. Salvati Manni: *Differentiating Siegel modular forms and the moving slope of  $\mathcal{A}_g$* . Int. Math. Res. Not. IMRN(2024), no. 4, 3442–3486.
- [56] S. Grushevsky, G. Mondello, R. Salvati Manni, J. Tsimerman: *Compact Subvarieties of the Moduli Space of Complex Abelian Varieties* [arXiv](#)
- [57] R. Hain: *The rational cohomology ring of the moduli space of abelian 3-folds*. Math. Res. Lett. 9 (2002), pp. 473–491.
- [58] G. Harder: *A Gauss-Bonnet formula for discrete arithmetically defined groups*. Ann. Sc. E.N.S. Paris, 4 (1971), 409–455.
- [59] G. Harder: *A congruence between a Siegel and an elliptic modular form*. The 1-2-3 of Modular Forms, pp. 247–262, Universitext, Springer, Berlin, 2008.
- [60] G. Harder: *The Eisenstein motive for the cohomology of  $\mathrm{GSp}_2(\mathbb{Z})$* , pp. 143–164, in: Geometry and arithmetic (Island of Schiermonnikoog, 2010), edited by C. Faber et al., European Mathematical Society, Zürich, 2012.
- [61] F. Hirzebruch: *Automorphe Formen und der Satz von Riemann-Roch*. In: 1958 Symposium internacional de topología algebraica, pp. 129–144. Universidad Nacional Autónoma de México, Mexico City.
- [62] K. Hulek, G. Sankaran: *The geometry of Siegel modular varieties* Advanced Studies in Pure Mathematics 35, 2002. Higher Dimensional Birational Geometry pp. 89–156.
- [63] K. Hulek, O. Tommasi: *Cohomology of the toroidal compactification of  $\mathcal{A}_3$* . In: Vector bundles and complex geometry, pp. 89–103, Contemp. Math., 522, Amer. Math. Soc., Providence, RI, 2010.
- [64] K. Hulek, O. Tommasi: *Cohomology of the second Voronoi compactification of  $\mathcal{A}_4$* . [arXiv:1103.6169](#).
- [65] K. Hulek, O. Tommasi: *The topology of  $\mathcal{A}_g$  and its compactifications*. Abel Symp., 14 Springer, Cham, 2018, 135–193.
- [66] T. Ichikawa: *Teichmüller modular forms of degree 3*. American Journal of Mathematics 117 (1995), 1057–1061.
- [67] J.-I. Igusa: *On Siegel modular forms of genus two*. American Journal of Mathematics 84 (1962), 175–200.
- [68] J.-I. Igusa: *Modular forms and projective invariants*. American Journal of Mathematics 89 (1967), 817–855.
- [69] J. Igusa: *A desingularization problem in the theory of Siegel modular functions*. Math. Ann. 168 (1967), pp. 228–260.
- [70] W. van der Kallen, E. Looijenga: *Spherical complexes attached to symplectic lattices* [arXiv:1001.0883](#). Geometriae Dedicata.
- [71] S. Keel, L. Sadun: *Oort’s conjecture for  $\mathcal{A}_g \otimes \mathbb{C}$* . JAMS 16 (2003), pp. 887–900.

- [72] N. Koblitz: *p-adic variation of the zeta function over families of varieties defined over finite fields*. *Comp. Math.* 31 (1975), pp. 119–218.
- [73] A. Kresch: *Cycle groups for Artin stacks*. *Invent. Math.* 138 (1999), pp. 495–536.
- [74] E. Looijenga: Goresky-Pardon lifts of Chern classes and associated Tate extensions. *Compositio Math.* 153 (2017) 1349–1371.
- [75] Y. Matsushima, S. Murakami: *On vector bundle valued harmonic forms and automorphic forms on symmetric riemannian manifolds*. *Ann. of Math. (2)* 78 (1963) 365–416.
- [76] S. Morel: *On the cohomology of certain noncompact Shimura varieties*. With an appendix by Robert Kottwitz. *Annals of Mathematics Studies*, 173. Princeton University Press, Princeton, NJ, 2010.
- [77] L. Moret-Bailly: *Pinceaux de variétés abéliennes*. In *Astérisque* 129 (1995), pp. 266.
- [78] D. Mumford: *Picard groups of moduli problems*. In: *Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963)*, pp. 33–81, 1965, Harper & Row, New York
- [79] D. Mumford: *Geometric invariant theory*. *Ergebnisse der Math.* **34** Springer Verlag, 1965.
- [80] D. Mumford, J. Fogarty, F. Kirwan: *Geometric Invariant theory*. Third enlarged edition, Springer Verlag, 1994.
- [81] D. Mumford: *Abelian varieties*. Oxford University Press, Tata Institute of Fundamental Research Studies in Mathematics 5, 1970.
- [82] D. Mumford: *Hirzebruch’s Proportionality Theorem in the non-compact case*. *Inv. Math.*
- [83] D. Mumford: *Towards an enumerative geometry of the moduli space of curves*. *Arithmetic and geometry*, Vol. II, pp. 271–328, *Progr. Math.*, 36, Birkhäuser Boston, Boston, MA, 1983.
- [84] D. Mumford: *On the Kodaira dimension of the Siegel modular variety*. In: *Algebraic geometry: open problems (Ravello, 1982)*, pp. 348–375, *Lecture Notes in Math.*, 997, Springer, Berlin, 1983.
- [85] Y. Namikawa: *Toroidal compactification of Siegel spaces*. *Lecture Notes in Math.* 812. Springer Verlag 1980.
- [86] M. Olsson: *Compactifying moduli spaces for abelian varieties*. *Lecture Notes in Mathematics*, 1958. Springer-Verlag, Berlin, 2008.
- [87] F. Oort: *Subvarieties of moduli spaces*. *Inv. Math.* 24 (1974), pp. 95–119.
- [88] F. Oort: *Which abelian surfaces are products of elliptic curves?* *Mathematische Annalen* **214** (1975), 35–47.
- [89] F. Oort: *Complete subvarieties of moduli spaces*. In: *Abelian Varieties (Egloffstein, 1993)*, de Gruyter, Berlin 1995, pp. 225–235.
- [90] F. Oort: *Newton polygons and formal groups*. *Annals of Mathematics* **152** (2000), 183–206.
- [91] F. Oort: *A stratification of a moduli space of abelian varieties*. In: *Moduli of Abelian varieties (Texel Island)*. Editors: C. Faber, G. van der Geer, F. Oort. *Progress in Math.* 195, Birkhäuser 195, Basel, 2001, pp. 345–416.
- [92] F. Oort: *Foliations in moduli spaces of abelian varieties*. *J. Amer. Math. Soc.* 17 (2004), pp. 267296.
- [93] D. Petersen: *Cohomology of local systems on the moduli of principally polarized abelian surfaces*. *Pacific Journal of Mathematics* **275** (2015), no. 1, 39–61.
- [94] P. Pragacz: *Enumerative geometry of degeneray loci*. *Ann. Scient. Ec. Norm. Sup.* 21 (1988), pp. 413–454.
- [95] I. Satake: *On the compactification of the Siegel space*. *J. Indian Math. Soc.* 20 (1956), 259–281.
- [96] A.J. Scholl: *Motives for modular forms*. *Invent. Math.* 100 (1990), pp. 419–430.

- [97] N. Shepherd-Barron: *Perfect forms and the moduli space of abelian varieties*. Invent. Math. 163 (2006), pp. 25–45.
- [98] Y-S. Tai: *On the Kodaira dimension of the moduli space of abelian varieties*. Invent. Math. 68 (1982), 425–439.
- [99] R. Tsushima: *An explicit dimension formula for the spaces of generalized automorphic forms with respect to  $\mathrm{Sp}(2, \mathbb{Z})$* . Proc. Japan Acad. Ser. A, Math. Sci. 59 (1983), pp. 139–142.
- [100] A. Verra: *A short proof of the unirationality of  $\mathcal{A}_5$* . Nederl. Akad. Wetensch. Indag. Math. 46 (1984), 339–355.
- [101] R. Weissauer: *Vektorwertige Siegelsche Modulformen kleinen Gewichtes*. J. Reine Angew. Math. 343 (1983), pp. 184–202.
- [102] A. Vistoli: *Intersection theory on algebraic stacks* Invent. Math. 97, 613–670 (1989).
- [103] R. Weissauer: *The trace of Hecke operators on the space of classical holomorphic Siegel modular forms of genus two*. [arXiv:0909.1744](https://arxiv.org/abs/0909.1744)
- [104] D. Zagier: *Correction to: The Eichler-Selberg trace formula on  $\mathrm{SL}_2(\mathbb{Z})$*  (Introduction to modular forms, Appendix, pp. 44–54, Springer, Berlin, 1976) by S. Lang. In: Modular functions of one variable, VI (Bonn, Bonn, 1976), pp. 171–173. Lecture Notes in Math., Vol. 627, Springer, Berlin, 1977.

KORTEWEG-DE VRIES INSTITUUT, UNIVERSITEIT VAN AMSTERDAM, SCIENCE  
PARK 904, 1098 XH AMSTERDAM, THE NETHERLANDS.  
YMSC, TSINGHUA UNIVERSITY, BEIJING, CHINA.  
*E-mail address:* [g.b.m.vandergeer@uva.nl](mailto:g.b.m.vandergeer@uva.nl)