

**LECTURES ON \mathcal{A}_g ,
THE MODULI SPACE OF ABELIAN VARIETIES**

GERARD VAN DER GEER

1. LECTURES

- (1) 05-09: Introduction, moduli space of complex abelian varieties
- (2) 10-09: The rank-1 compactification, Satake compactification / \mathbb{C}
- (3) 12-09: The existence of \mathcal{A}_g as an DM-stack
- (4) 19-09: The Hodge bundle, Rank-1 degenerations
- (5) 24-09: Toroidal compactifications
- (6) 26-09: Satake compactification
- (7) 29-09: Tautological classes
- (8) 08-10: Kodaira dimension of \mathcal{A}_g
- (9) 10-10: Stratifications on \mathcal{A}_g
- (10) 15-10: Cohomology and Chow ring of \mathcal{A}_g
- (11) 17-10: Canonical model
- (12) 22-10: Modular forms
- (13) 24-10: Compact subvarieties of \mathcal{A}_g

2. SPECIFIC REFERENCES PER LECTURE

- (1) [8, 82]
- (2) [85, 96, 40, 17]
- (3) [80, 81, 79, 24, 36, 3, 4, 37, 103]
- (4) [85, 45]
- (5) [6, 86, 36, 63]
- (6) [1, 87, 36, 47]
- (7) [44, 32, 45, 28, 29]
- (8) [85, 39, 99, 25, 101]
- (9) [92, 44, 30, 45]
- (10) [84, 44, 46, 16, 58, 53, 66]
- (11) [98]
- (12) [18]
- (13) [72, 57]

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KORTEWEG-DE VRIES INSTITUUT, UNIVERSITEIT VAN AMSTERDAM, SCIENCE
 PARK 904, 1098 XH AMSTERDAM, THE NETHERLANDS.
 YMSC, TSINGHUA UNIVERSITY, BEIJING, CHINA.
E-mail address: g.b.m.vandergeer@uva.nl