Contents

Cycles on the Moduli Space of Abelian Varieties

Gerard van der Geer

Abstract

We first determine the tautological Chow ring of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties. Then we consider the Ekedahl-Oort stratification on the moduli space $\mathcal{A}_g \otimes \mathbb{F}_p$ in characteristic p and we compute the cycle classes of the strata in this stratification. The formulas for these strata can be seen as a generalization of Deuring's famous formula for the number of supersingular elliptic curves in characteristic p. The formulas for the cycle classes of the strata show that these classes lie in the tautological ring and imply effectivity of certain tautological classes.

1 Introduction

In this paper I present a number of results on cycles on the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g. The results on the tautological ring are my own work, the results on the torsion of λ_g and on the cycle classes of the Ekedahl-Oort stratification are joint work with Torsten Ekedahl and some of the results on curves are joint work with Carel Faber. Our results include:

- A description of the tautological subring of the Chow ring of $\mathcal{A}_g$, i.e. of the subring generated by the Chern classes λ_i of the Hodge bundle $\mathbb{E}$.
- A formula for the top Chern class λ_g of the Hodge bundle and a bound for the order of the torsion of this class.
- A description of the Ekedahl-Oort stratification of $\mathcal{A}_g \otimes \mathbb{F}_p$ in terms of degeneracy loci of a map between flag bundles.

- The description of the Chow classes of the strata of this stratification. This includes as special cases formulas for the classes of loci like p-rank $\leq f$ locus or a-number $\geq a$ locus. Such formulas generalize the classical formula of Deuring for the number of supersingular elliptic curves.
- The irreducibility of the locus T_a of abelian varieties of a-number $\geq a$ for $a < g$.
- A computation of this stratification for the Jacobians of hyperelliptic curves of 2-rank 0 in characteristic 2.
- A formula for the class of the supersingular locus for low genera.

Acknowledgments. It is a pleasure for me to acknowledge pleasant cooperation with and help from Torsten Ekedahl and Carel Faber and useful discussions with Hélène Esnault, Frans Oort and Piotr Pragacz. I also would like to thank Kenji Ueno for inviting me to Kyoto in 1996 where I found the time to write the first version of this paper.

2 The Tautological Subring of $\mathcal{A}_g$

Let $\mathcal{A}_g/\mathbb{Z}$ denote the moduli stack of principally polarized abelian varieties of dimension g. This is an irreducible algebraic stack of relative dimension $g(g+1)/2$. We refer to [10] for a short introduction. All the fibres $\mathcal{A}_g \otimes \mathbb{F}_p$ and $\mathcal{A}_g \otimes \mathbb{Q}$ are geometrically irreducible. This stack carries a locally free sheaf $\mathbb{E}$ of rank g ("the Hodge bundle") defined by giving for every morphism $S \to \mathcal{A}_g$ a locally free sheaf of rank g which is compatible with pullbacks. It is defined as $s^*\Omega_{A/S}$, where A/S is the principally polarized abelian variety corresponding to $S \to \mathcal{A}_g$ and s is the zero section of A/S. If $\pi : \mathcal{X}_g \to \mathcal{A}_g$ is the universal abelian variety we have $\Omega_{\mathcal{X}_g/\mathcal{A}_g} = \pi^*(\mathbb{E})$.

Let $\tilde{\mathcal{A}}_g$ be a smooth toroidal compactification of $\mathcal{A}_g$. The Hodge bundle can be extended to a locally free sheaf (again denoted by) $\mathbb{E}$ on $\tilde{\mathcal{A}}_g$.

The Chern classes λ_i of the Hodge bundle $\mathbb{E}$ are defined over $\mathbb{Z}$ and for each fibre $\mathcal{A}_g \otimes k$ they give rise to classes λ_i in $CH^*(\mathcal{A}_g \otimes k)$, and in $CH^*(\tilde{\mathcal{A}}_g \otimes k)$. They generate subrings ($\mathbb{Q}$-subalgebras) of $CH^*_{\mathbb{Q}}(\mathcal{A}_g \otimes k)$ and of $CH^*_{\mathbb{Q}}(\tilde{\mathcal{A}}_g \otimes k)$ which we shall call the *tautological subrings*.

We shall describe these tautological subrings, first for $\mathcal{A}_g$ and in the next section for $\tilde{\mathcal{A}}_g$. We derive a relation for the Chern classes λ_i on $\mathcal{A}_g$ in the *Chow ring* .

Theorem 2.1. *The Chern classes λ_i in $CH^*_{\mathbb{Q}}(\mathcal{A}_g)$ in the Chow ring with rational coefficients satisfy the relation*

$$(1 + \lambda_1 + \ldots + \lambda_g)(1 - \lambda_1 + \ldots + (-1)^g \lambda_g) = 1. \tag{1}$$

The idea of the proof is to apply the Grothendieck-Riemann-Roch theorem to the theta divisor on the universal abelian variety $\mathcal{X}_g$ over $\mathcal{A}_g$. We choose this divisor (on a level cover) so that its restriction $s^*(\Theta)$, with s the zero section, is trivial on $\mathcal{A}_g$ and apply Grothendieck-Riemann-Roch to the line bundle $L = O(\Theta)$:

$$\begin{aligned} ch(\pi_! L) &= \pi_*(ch(L)\cdot \mathrm{Td}^\vee(\Omega^1_{\mathcal{X}_g/\mathcal{A}_g})) \\ &= \pi_*(ch(L)\cdot \mathrm{Td}^\vee(\pi^*(\mathbb{E}))) \\ &= \pi_*(ch(L))\cdot \mathrm{Td}^\vee(\mathbb{E}), \end{aligned}$$

by the projection formula. Here $\mathrm{Td}^\vee$ is defined for a line bundle with first Chern class α by $\alpha/(e^\alpha - 1)$. Since $R^i\pi_*(L) = 0$ for $i > 0$ it follows that $\pi_!(L)$ is a vector bundle and it is of rank 1 since Θ is a principal polarization. We write $c_1(\pi_!(L)) = \theta$ and find

$$\sum_{k=0}^{\infty} \frac{\theta^k}{k!} = \pi_*(\sum_{k=0}^{\infty} \frac{\Theta^{g+k}}{(g+k)!})\cdot \mathrm{Td}^\vee(\mathbb{E}). \qquad (2)$$

Comparison of the terms of degree 1 gives

$$\theta = -\lambda_1/2 + \pi_*(\Theta^{g+1})/(g+1)!.$$

Replace now L by $L^{\otimes n}$. The term of degree k in $\pi_*(\sum \Theta^{g+k}/(g+k)!)$ changes by a factor n^{g+k}. But $\pi_!(L^n)$ is a numerical function of degree $\leq n^g$, cf. the arguments of Chai-Faltings on p. 26 of [6]. Or, alternatively, using the Heisenberg group, we find on a suitable cover of $\mathcal{A}_g$

$$\pi_! L^n = \pi_! L \otimes A,$$

where A is the standard irreducible representation of dimension n^g of the Heisenberg group. Therefore, up to torsion we have

$$ch(\pi_! L^n) = n^g ch(\pi_! L).$$

We thus get

$$n^g \sum_{k=0}^{\infty} \frac{\theta^k}{k!} = \pi_*(\sum_{k=0}^{\infty} \frac{n^{g+k}\Theta^{g+k}}{(g+k)!})\cdot \mathrm{Td}(\mathbb{E}^\vee). \qquad (2')$$

By comparing the coefficients of powers of n we find

$$\pi_*(\sum_{k=0}^{\infty} \frac{\Theta^{g+k}}{(g+k)!}) = 1 \in CH^*_{\mathbb{Q}}(\mathcal{A}_g). \qquad (3)$$

In particular we get

$$2\theta = -\lambda_1 \qquad \text{(the “key formula”)}.$$

Therefore, if we write λ_j as the jth symmetric function of $\alpha_1, \ldots ,\alpha_g$ and if we use $\mathrm{Td}(\mathbb{E}^\vee) = \prod(\alpha_i/(e^{\alpha_i}-1))$ the identity (2) becomes

$$\prod_{i=1}^{g} \frac{e^{\alpha_i/2} - e^{-\alpha_i/2}}{\alpha_i} = 1.$$

and implies $\mathrm{Td}(\mathbb{E} \oplus \mathbb{E}^\vee) = 1$. This is equivalent with $ch_{2k}(\mathbb{E}) = 0$ for $k \geq 1$ and with (1). $\square$

Proposition 2.2. *In the Chow ring $CH^*_{\mathbb{Q}}(\mathcal{A}_g)$ with rational coefficients we have the identity $\lambda_g = 0$.*

Proof. We apply GRR to the structure sheaf $O_{\mathcal{X}}$ of the universal abelian variety $\mathcal{X}$ over the stack $\mathcal{A}_g$. We get

$$\begin{aligned} ch(\pi_! O_{\mathcal{X}}) &= \pi_*(ch(O_{\mathcal{X}}) \cdot \mathrm{Td}^\vee(\Omega^1)) \\ &= \pi_*(1)\mathrm{Td}^\vee(\mathbb{E}), \end{aligned}$$

and this gives the relation

$$ch(1 - \mathbb{E}^\vee + \wedge^2\mathbb{E}^\vee - \ldots + (-1)^g \wedge^g \mathbb{E}^\vee) = \pi_*(1)\mathrm{Td}^\vee(\mathbb{E}) = 0.$$

Now for a vector bundle B of rank r we have in general the relation (see [1])

$$\sum_{j=0}^{r} (-1)^j ch(\wedge^j B^\vee) = c_r(B)\mathrm{Td}(B)^{-1}.$$

This gives: $\lambda_g = 0$ in $CH^*_{\mathbb{Q}}(\mathcal{A}_g)$. $\square$

Let R_g be the quotient of the graded ring $\mathbb{Q}[u_1, \ldots ,u_g]$ with generators u_i of degree i by the relation

$$(1 + u_1 + u_2 + \ldots + u_g)(1 - u_1 + u_2 - \ldots + (-1)^g u_g) - 1 = 0.$$

This relation implies (by induction)

$$u_g u_{g-1} \cdots u_{k+1} u_k^2 = 0 \qquad \text{for} \quad k = 1, \ldots ,g. \tag{4}$$

This ring R_g has additive generators

$$u_1^{\epsilon_1} u_2^{\epsilon_2} \ldots u_g^{\epsilon_g} \qquad \text{with} \quad \epsilon_j \in \{0,1\}.$$

Obviously, we have $R_g/(u_g) \cong R_{g-1}$. It follows from Theorem (2.1) and Proposition (2.2) that the tautological subring of $CH^*_{\mathbb{Q}}(\mathcal{A}_g)$ is a homomorphic image of the ring $R_g/(u_g) \cong R_{g-1}$.

Now consider the moduli space $\mathcal{A}_g \otimes \mathbb{F}_p$. It contains the loci $V_f = V_f(p)$ of abelian varieties with p-rank $\leq f$. Their closures in $\mathcal{A}_g^*$ and in $\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$ define loci again denoted V_f. These loci have been studied by Koblitz [14], Oort and Norman, cf. [22], [20]. They show that V_0 is complete in $\mathcal{A}_g \otimes \mathbb{F}_p$. We now state a slight generalization of their results. Recall that for a smooth toroidal compactification $\tilde{\mathcal{A}}_g$ one has a morphism $q : \tilde{\mathcal{A}}_g \to \mathcal{A}_g^*$ to the Satake or 'minimal' compactification as defined in [6]. The Satake compactification $\mathcal{A}_g^*$ is a disjoint union

$$\mathcal{A}_g^* = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \ldots \sqcup \mathcal{A}_0.$$

We call the inverse image under q of $\sqcup_{j \leq t} \mathcal{A}_{g-j}$ the moduli space $\tilde{\mathcal{A}}_g^{(t)}$ of *rank* $\leq t$ *degenerations.* This space parametrizes semi-stable abelian varieties whose torus-part has rank $\leq t$.

Lemma 2.3. *The subvariety V_f is complete in the moduli space $\tilde{\mathcal{A}}_g^{(f)} \subset \tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$ of rank $\leq f$ degenerations; in particular V_0 is complete in $\mathcal{A}_g$.*

Lemma 2.4. *We have $\lambda_1^{g(g-1)/2+f} \neq 0$ on $\tilde{\mathcal{A}}_g^{(f)}$.*

Proof. Observe that $\det(\mathbb{E})$ is an ample line bundle (its sections are modular forms) and so λ_1 is ample on $\mathcal{A}_g^*$, see [16]. On a complete variety of dimension d the d-th power of an ample divisor is non-zero. $\square$

Theorem 2.5. *The tautological subring of the Chow ring $CH_{\mathbb{Q}}^*(\mathcal{A}_g)$ generated by the λ_i is isomorphic to R_{g-1}.*

Proof. By the relation $(1 + \lambda_1 + \ldots + \lambda_g)(1 - \lambda_1 + \ldots + (-1)^g \lambda_g) = 1$ we get a quotient ring of R_g. Since moreover $\lambda_g = 0$ we get a quotient of $R_g/(u_g) \cong R_{g-1}$. This ring R_{g-1} is Gorenstein and its top degree elements are proportional. We now use the fact that $\mathcal{A}_g \otimes \mathbb{F}_p$ has a complete subvariety of codimension g, namely the p-rank zero locus. The existence of of a complete subvariety of dimension $g(g-1)/2$ in $\mathcal{A}_g \otimes \mathbb{F}_p$ for every prime p and the ampleness of λ_1 on $\mathcal{A}_g \otimes \mathbb{F}_p$ yield that $\lambda_1^{g(g-1)/2} \neq 0$. This implies that in the quotient of R_{g-1} the one-dimensional socle does not map to zero, hence that the quotient is isomorphic to R_{g-1}. Or more explicitly, consider the set of 2^{g-1} generators of the form

$$\lambda_1^{\epsilon_1} \lambda_2^{\epsilon_2} \cdots \lambda_{g-1}^{\epsilon_{g-1}} \quad \text{with} \quad \epsilon_i \in \{0,1\}.$$

Order these elements λ_ϵ with $\epsilon \in \{0,1\}^{g-1}$ lexicographically. Suppose we have a relation $\sum a_\epsilon \lambda_\epsilon = 0$ in $CH_{\mathbb{Q}}^*(\mathcal{A}_g)$ and suppose that ϵ' is the 'smallest' exponent. Let ϵ'' be the complementary exponent with $\epsilon' + \epsilon'' = (1, \ldots, 1)$. Then we have

$$\lambda_{\epsilon''}(\sum a_\epsilon \lambda_\epsilon) = a_{\epsilon'} \lambda_{g-1} \cdots \lambda_1 = 0,$$

and this implies that $a_{\epsilon'} = 0$. By induction the theorem follows. $\square$

Corollary 2.6. *In $CH^*_{\mathbb{Q}}(\mathcal{A}_g)$ we have: $\lambda_1^{g(g-1)/2} \neq 0$ and $\lambda_1^{1+g(g-1)/2} = 0$.*

Proof. The first statement was given in (2.4). In the ring R_{g-1} all top monomials (of degree $g(g-1)/2$) are proportional, hence $\lambda_1^{g(g-1)/2}$ is a non-zero multiple of the top monomial $\lambda_{g-1}\lambda_{g-2}\cdots\lambda_1$, so $\lambda_1^{g(g-1)/2+1}$ is a non-zero multiple of $\lambda_{g-1}\lambda_{g-2}\cdots\lambda_2\lambda_1^2$, which is zero by (4). $\square$

Corollary 2.7. *Let $\mathbb{F}$ be a field. A complete subvariety of $\mathcal{A}_g \otimes \mathbb{F}$ has codimension at least g.*

Proof. If Z is a complete subvariety of dimension m then $\lambda_1^m \neq 0$ on Z. Since $g(g-1)/2$ is the highest power of λ_1 which is not zero on $\mathcal{A}_g$ the result follows. $\square$

For a discussion of questions concerning complete subvarieties of $\mathcal{A}_g$ we refer to [23] and [5].

3 The Tautological Ring of $\tilde{\mathcal{A}}_g$

Now we come to the tautological ring of certain compactifications of $\mathcal{A}_g \otimes k$. We first choose a suitable toroidal compactification $\tilde{\mathcal{A}}_g$ as constructed in [6]. We let $\mathcal{A}_g^*$ be the minimal ('Satake') compactification as before. Furthermore, we let $\mathcal{A}'_g = \tilde{\mathcal{A}}_g^{(1)}$ be the moduli space of rank 1-degenerations, i.e. the inverse image of $\mathcal{A}_g \sqcup \mathcal{A}_{g-1} \subset \mathcal{A}_g^*$ under the natural map $q : \tilde{\mathcal{A}}_g \to \mathcal{A}_g^*$. The important point is that the space $\mathcal{A}'_g$ does *not* depend on a choice $\tilde{\mathcal{A}}_g$ of toroidal compactification of $\mathcal{A}_g$. See [19].

Let $\mathcal{G}$ be the 'universal' semi-abelian variety over $\tilde{\mathcal{A}}_g$ with zero section $s : \tilde{\mathcal{A}}_g \to \mathcal{G}$. Then we have a canonical extension defined in a geometric way by $s^*(\mathrm{Lie}\,\mathcal{G})^\vee$ of the Hodge bundle $\mathbb{E}$. By abuse of notation this canonical extension is again denoted by $\mathbb{E}$. Moreover, let $D = \tilde{\mathcal{A}}_g - \mathcal{A}_g$ be the divisor at infinity. Then we have the isomorphism (cf. [6], p. 117):

$$\mathrm{Sym}^2(\mathbb{E}) \cong \Omega^1(\log D).$$

In order to extend the relation (1) for the Chern classes of the Hodge bundle in $CH_{\mathbb{Q}}(\tilde{\mathcal{A}}_g)$ one could try to apply the Grothendieck-Riemann-Roch theorem as we did for $\mathcal{A}_g$. Unfortunately, I was not able to carry this out; for partial results I refer to the next section.

In cohomology with rational coefficients (as opposed to the Chow ring) the relation (1) is known to hold. This follows from the fact that on $\mathcal{A}_g$ we have an exact sequence

$$0 \to \mathbb{E} \longrightarrow \mathcal{H}^1_{\mathrm{dR}} \longrightarrow \mathbb{E}^\vee \to 0$$

and the Gauss-Manin connection on $\mathcal{H}^1_{\mathrm{dR}}$ and that both the exact sequence and the connection extend to $\tilde{\mathcal{A}}_g$, cf. [6], p. 225.

Now since R_g is a Gorenstein ring with socle in degree $g(g+1)/2$ and since λ_1 is ample on the Zariski-open part $\mathcal{A}_g$ it follows that $\lambda_1^{g(g+1)/2} \neq 0$. Therefore we see that the tautological subring of the cohomology ring $H^*(\tilde{\mathcal{A}}_g(\mathbb{C}),\mathbb{Q})$ is isomorphic to R_g.

Theorem 3.1. *The relation (1) holds for the Chern classes of the Hodge bundle $\mathbb{E}$ in $H^*(\tilde{\mathcal{A}}_g \otimes k,\mathbb{Q})$. The tautological subring of $H^*(\tilde{\mathcal{A}}_g \otimes k,\mathbb{Q})$ is isomorphic to R_g.*

A top monomial u^α in R_g can be written as a multiple $m_\alpha u_1 u_2 \dots u_g$ with $m_\alpha \in \mathbb{Z}$ using the relation (1). Therefore we find that the degree $\deg \lambda^\alpha$ equals $m_\alpha \deg \lambda_1 \lambda_2 \dots \lambda_g([\tilde{\mathcal{A}}_g])$.

Let Y_g be the compact dual of the Siegel upper half space $\mathbb{H}_g$. It can be identified with the Grassmann variety of Lagrangian subspaces of a $2g$-dimensional symplectic vector space. The Chow ring and the cohomology ring of Y_g are generated by the Chern classes u_i of the tautological bundle E of rank g on Y_g and are isomorphic to R_g.

The fact that the tautological subring of $H^*_{\mathbb{Q}}(\tilde{\mathcal{A}}_g) \cong R_g$ gives the following famous 'principle'.

Theorem 3.2. *(Proportionality Principle of Hirzebruch-Mumford) The characteristic numbers of the Hodge bundle are proportional to those of the tautological bundle on Y_g:*

$$\lambda^\alpha([\tilde{\mathcal{A}}_g]) = (-1)^G \frac{1}{2^g} \prod_{k=1}^{g} \zeta(1-2k) \cdot u^\alpha([Y_g])$$

with $G = g(g+1)/2$.

We have $\lambda^\alpha([\tilde{\mathcal{A}}_g]) = c(g) \times u^\alpha([Y_g]$ for some constant $c(g)$. To evaluate the constant one can compare the Euler number of $\mathcal{A}_g$ (in a suitable sense; actually for $\Omega(\log D) = \mathrm{Sym}^2(\mathbb{E})$) and the Euler number of Y_g. The Euler number of Y_g equals 2^g. We know the Euler number of $\mathcal{A}_g$ by the work of Siegel and Harder:

$$\begin{aligned}\chi(Sp(2g,\mathbb{Z})) &= \frac{\# W_{Sp(\mathbb{R})}}{\# W_{U(\mathbb{R})}} \prod_{j=1}^{g} \frac{\zeta(1-2j)}{2} = \frac{2^g g!}{g!} \prod_{j=1}^{g} \frac{\zeta(1-2j)}{2} \\ &= \zeta(-1)\zeta(-3)\cdots\zeta(-2g+1),\end{aligned}$$

where $W_{Sp(\mathbb{R})}$ (resp. $W_{U(\mathbb{R})}$) denotes the Weyl group.

Define a proportionality factor by

$$p(g) = (-1)^G \prod_{j=1}^{g} \frac{\zeta(1-2j)}{2}.$$

Since $\deg u_1 u_2 \dots u_g = 1$ we find $\deg \lambda_1\lambda_2\cdots\lambda_g = p(g)$. Similarly, using the structure of R_g we have

$$\frac{\lambda_1^G}{\prod_{k=1}^g \zeta(1-2k)} = (-1)^G \frac{u_1^G}{2^g}.$$

and thus get from the relation between u_1^G and $u_1 \cdots u_g$:

$$\lambda_1^G = p(g) G! \prod_{k=1}^{g} \frac{1}{(2k-1)!!}.$$

This has to be interpreted with care (in the orbifold sense).

Some examples. We have

$$\begin{aligned} p(1) &= 1/24 & \deg \lambda_1 &= 1/24, \\ p(2) &= 1/5760 & \deg \lambda_1^3 &= 1/2880, \\ p(3) &= 1/2903040 & \deg \lambda_1^6 &= 1/181440. \end{aligned}$$

For the classical Proportionality principle of Hirzebruch we refer to his collected works, [12]. In a letter to Atiyah, Hirzebruch sketches how one obtains a copy of R_g in the cohomology of suitable compact quotients of the Siegel upper half space, cf. [13]. Mumford's extension can be found in [18].

4 Cycle Relations from Grothendieck-Riemann-Roch

In this section we derive cycle relations for the classes λ_i. We begin with λ_g. Since the class λ_g vanishes on $\mathcal{A}_g$ it can be represented in the form $\lambda_g = i_*(x)$ with x a class in $CH_{\mathbb{Q}}^{g-1}(D)$ and $i\colon D \to \tilde{\mathcal{A}}_g$ the embedding of the 'boundary'. By applying Grothendieck-Riemann-Roch to the structure sheaf on the semi-abelian variety over $\tilde{\mathcal{A}}_g$ (as in (1.2)) one can get an expression for x in $CH_{\mathbb{Q}}^*(D)$, as we shall see shortly.

On the other hand we shall show in characteristic p that a multiple of the class λ_g is represented in the Chow ring by the class of a complete subvariety V_0 of $\mathcal{A}_g$. More generally, a multiple of λ_{g-f} is represented by a complete subvariety of $\mathcal{A}_g^{(f)}$, the locus V_f of abelian varieties of p-rank $\leq f$. We thus find interesting representations of the class λ_g. We first consider the Satake compactification.

Lemma 4.1. *The class $q_*(\lambda_g)$ in $CH_{\mathbb{Q}}^g(\mathcal{A}_g^*)$ is represented by a multiple of the fundamental class of the boundary $B_g^* = \mathcal{A}_g^* - \mathcal{A}_g$.*

Indeed, λ_g is zero on $\mathcal{A}_g$; for dimension reasons $q_*\lambda_g$ is represented by a multiple of the fundamental class of B_g^*.

Proposition 4.2. *The cycle class $[B_g^*]$ of the boundary is the same in the Chow group $CH_{\mathbb{Q}}^g(\mathcal{A}_g^*)$ as a multiple of the class of the image of $\mathcal{A}_{g-1}$ in $\mathcal{A}_g$ under the map $[X] \mapsto [X \times E]$, with E a generic elliptic curve.*

Proof. Consider (for $g > 2$) the space $\mathcal{A}_{g-1,1} \sim \mathcal{A}_{g-1} \times \mathcal{A}_1$ in $\mathcal{A}_g$. Since $\mathcal{A}_1$ is the affine j-line we find a rational equivalence between the cycle class of a generic fibre $\mathcal{A}_{g-1} \times \{j\}$ and a multiple of the fundamental class of the boundary B_g^*. □

Let B_g be the cycle on $\mathcal{A}_g^{(1)}$ defined by the semi-abelian varieties which are trivial extensions

$$1 \to \mathbb{G}_m \to G \to X_{g-1} \to 0,$$

where X_{g-1} is a $(g-1)$-dimensional abelian variety. The cycle $B_g \sim \mathcal{A}_{g-1}$ is of codimension g in $\mathcal{A}_g^{(1)}$ and extends to $\tilde{\mathcal{A}}_g$.

The following proposition describes the class of B_g in *cohomology*.

Proposition 4.3. *The cohomology class of B_g and of $\mathcal{A}_{g-1} \times \{j\}$ are both equal to $(-1)^g \lambda_g / \zeta(1-2g)$.*

Proof. The intersection of the closure of $\mathcal{A}_{g-1,1}$ with the boundary divisor in $\tilde{\mathcal{A}}_g$ is the closure of B_g. Using the rational equivalence of B_g and $\mathcal{A}_{g-1} \times \{j\}$ for generic j and the fact that the class of $\mathcal{A}_{g-1} \times \{j\}$ on $\mathcal{A}_g^*$ is a multiple of $q_* \lambda_g$ the result follows by pullback. To find the multiple we look at characteristic zero, integrate the $Sp(2g,\mathbb{R})$-invariant forms representing the λ_i and use the Proportionality Theorem. □

Examples. We have in cohomology:

$$\begin{aligned} g = 1 \qquad & [B_1] = 12\lambda_1 \\ g = 2 \qquad & [B_2] = 120\lambda_2. \end{aligned}$$

Now we try to apply the Grothendieck-Riemann-Roch theorem to the Θ-divisor again on a chosen compactification $\tilde{\mathcal{X}}_g$ of the semi-abelian variety over $\tilde{\mathcal{A}}_g$ as we did for $\mathcal{A}_g$ to obtain cycle relations. We do this on a level cover of $\tilde{\mathcal{A}}_g$ for a line bundle $L = O(T)$ trivialized along the zero section. The relation (1) is equivalent to the identity:

$$\mathrm{Td}^\vee(\mathbb{E}) = e^{-\lambda_1/2}.$$

Application of GRR says

$$ch(\pi_!(L^{\otimes n})) = \pi_*(e^{nT} \cdot F) \cdot \mathrm{Td}^\vee(\mathbb{E}). \tag{5}$$

Here we have set $F := \mathrm{Td}^\vee(\mathcal{F})^{-1}$, where $\mathcal{F}$ is defined by the exact sequence

$$0 \longrightarrow \Omega_{\tilde{\mathcal{X}}_g/\tilde{\mathcal{A}}_g} \longrightarrow \pi^*(\mathbb{E}) \longrightarrow \mathcal{F} \to 0. \tag{6}$$

We know by [6] that $\pi^*(\mathbb{E})$ can be identified with the relative logarithmic differentials $\Omega^1(d\log\infty)/\pi^*(\Omega^1(d\log\infty))$ and so $\mathcal{F}$ is concentrated where π is not smooth. The problem is to have a good description of $\pi_!(L^{\otimes n})$.

By applying GRR to the case where $n = 0$, i.e. to the structure sheaf, we obtain the following expression for λ_g, cf. Proposition 2.2.

Proposition 4.4. *We have* $\pi_*(F) = \lambda_g$.

We now state the following result, to be proved in Section 9.

Proposition 4.5. *Let p be a prime. The class $(p-1)(p^2-1)\dots(p^g-1)\lambda_g \in CH^*_{\mathbb{Q}}(\mathcal{A}_g \otimes \mathbb{F}_p)$ is represented by a complete cycle in $\mathcal{A}_g \otimes \mathbb{F}_p$.*

Corollary 4.6. *In $CH^*_{\mathbb{Q}}(\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p)$ we have the relations $\lambda_g\lambda_{g-1}\dots\lambda_{k+1}\lambda_k^2 = 0$ for $k = 1,\dots,g$.*

Proof. By the two preceding propositions we have $\lambda_g^2 = 0$ in $CH^*_{\mathbb{Q}}(\tilde{\mathcal{A}}_g)$. From the relation (1) on $\mathcal{A}_g$ we see that the expressions

$$\lambda_r^2 - 2\lambda_{r-1}\lambda_{r+1} + \dots + (-1)^{g-r}2\lambda_{2r-g}\lambda_g$$

are boundary classes (living on D). Using induction the relation follows from multiplying these expressions with $\lambda_g\lambda_{g-1}\cdots\lambda_{r+1}$. □

Consider now the moduli space $\mathcal{A}'_g = \tilde{\mathcal{A}}_g^{(1)}$ of rank ≤ 1 degenerations. This is a *canonical* partial compactification of $\mathcal{A}_g$. If we want a (complete) compactification then there is not really a unique one, but we must make choices. Let D^0 be the closed subset corresponding to rank 1 degenerations. The divisor D^0 has a morphism $\phi : D^0 \to \mathcal{A}_{g-1}$ which exhibits D^0 as the universal abelian variety over $\mathcal{A}_{g-1}$. The fibre over $x \in \mathcal{A}_{g-1}$ is the dual $\hat{X}_{g-1}$ of the abelian variety X corresponding to x. The 'universal' semi-abelian variety G over $\hat{\mathcal{X}}_{g-1}$ is the $\mathbb{G}_m$-bundle obtained from the Poincaré bundle $P \to \mathcal{X}_{g-1} \times \hat{\mathcal{X}}_{g-1}$ by deleting the zero-section. We have the maps

$$G = P - \{(0)\} \to \mathcal{X}_{g-1} \times \hat{\mathcal{X}}_{g-1} \to \mathcal{A}_{g-1}.$$

By considering the cotangent bundle to G at the zero section t of $G \to \hat{X}_{g-1}$ we get an exact sequence

$$0 \to q^*\mathbb{E}_{g-1} \to \mathbb{E}_{g\,|D^0} \to U \to 0,$$

with U the pullback of a line bundle on $\mathcal{A}^*_{g-1}$. Now U is trivial since the restriction of $\mathbb{E}_g$ to B_g is a direct sum of $\mathbb{E}_{g-1}$ and a trivial line bundle.

Consider $\mathcal{X}'$, the compactified family of semi-abelian varieties over the moduli stack $\mathcal{A}'_g$ of rank ≤ 1 degenerations. A semi-abelian variety which is a $\mathbb{G}_m$-bundle over an abelian variety X_{g-1} is compactified by taking the $\mathbb{P}^1$-bundle associated to the $\mathbb{G}_m$-bundle and then identifying the 0- and the ∞-section under a shift $\xi \in X_{g-1}$. The image of the zero section of the $\mathbb{P}^1$-bundle maps to a codimension 2 cycle Δ, the locus of singular points of the fibres of $\mathcal{X}'_g \to \mathcal{A}'_g$. We have

$$\Delta \cong \mathcal{X}_{g-1} \times_{\mathcal{A}_{g-1}} \hat{\mathcal{X}}_{g-1}.$$

We analyze $\Omega^1 = \Omega_{\mathcal{X}'_g/\mathcal{A}'_g}$. In the exact sequence

$$0 \to \Omega^1 \to \pi^*(\mathbb{E}) \to \mathcal{F} \to 0, \qquad (6')$$

the sheaf $\mathcal{F}$ has support on Δ, the codimension 2 cycle. Let u be a fibre coordinate on a $\mathbb{G}_m$-bundle over $\mathcal{X}_{g-1}$. A section of $\pi^*(\mathbb{E})$ is given locally by du/u. Pull a section back to the $\mathbb{P}^1$-bundle and take the residue along the 0- and ∞-section. The residue map yields an isomorphism on $\mathcal{A}'_g$

$$\mathcal{F} \cong O_{\tilde{\Delta}},$$

where $\tilde{\Delta}$ is the etale double cover of Δ corresponding to choosing the branches (0 and ∞). Working out now the expression for F on $\mathcal{A}'_g$ one can show (cf. [3]):

Theorem 4.7. *The Chow class of B_g and that of $\mathcal{A}_{g-1} \times \{j\}$ in $CH^g_{\mathbb{Q}}(\mathcal{A}'_g)$ are both equal to $(-1)^g \lambda_g/\zeta(1-2g)$.*

The GRR theorem for the case $n = 1$ yields the identity:

$$ch(\pi_! L) = \pi_*(e^T F)\mathrm{Td}^\vee(\mathbb{E}).$$

The expression $\pi_! L$ is given by a line bundle. So to calculate it we may work on $\mathcal{A}'_g$ (instead of $\tilde{\mathcal{A}}_g$). We write

$$ch(\pi_!(L^{\otimes n})) = 1 + \theta_1^{(n)} + \theta_2^{(n)} + \dots$$

and set $\theta_1^{(1)} = \theta$. In equation (5) we compare terms of codimension 1:

$$\theta_1^{(n)} = -\frac{\lambda_1}{2} \cdot \pi_*[e^{nT} \cdot \mathrm{Td}^\vee(O_\Delta)^{-1})]_0 + \pi_*[e^{nT} \cdot \mathrm{Td}^\vee(O_\Delta)^{-1})]_1$$

Lemma 4.8. *In the Chow group $CH^1_{\mathbb{Q}}(\mathcal{A}'_g)$ we have $\theta = -\lambda_1/2 + \delta/8$, where δ is the class of the 'boundary' D.*

Proof. First we do the case $g = 1$. We have

$$1 + \theta = (\pi_*(T + T^2/2) + \pi_*(\Delta/12))(1 - \lambda_1/2).$$

Let S be the zero-section. By Kodaira's results on elliptic surfaces we have $S^2 = -\lambda_1$; so the normalized T is $T = S + \pi^*(\lambda_1)$ with $T^2 = S \cdot \pi^*(\lambda_1)$, i.e. $\pi_*(T^2/2) = \lambda_1/2$. We find $\theta = -\lambda_1/2 + \lambda_1/2 + \delta/12$. Using the relation $12\lambda_1 = \delta$ we can rewrite this as $\theta = -\lambda_1/2 + \delta/8$. For general g we have a priori $\theta = -\lambda_1/2 + a\delta$ for some $a \in \mathbb{Q}$. Restriction to the space of products of elliptic curves $\mathcal{A}_{1,\dots,1}$ gives $a = 1/8$. □

As a corollary we find for the codimension 1 terms of $ch(\pi_!(L^{\otimes n}))$ the identity

$$\theta_1^{(n)} = n^g(-\lambda_1/2) + (n^{g+1} + 2n^{g-1})\delta/24.$$

For a description of the Chow rings of $\tilde{\mathcal{A}}_g(\mathbb{C})$ for $g = 1,2,3$ see [8].

In cohomology the relation (1) holds. Substitution of this result in the GRR formula gives the following identity.

Theorem 4.9. *We have $\pi_*(e^T F) = e^{\delta/8}$ in $H^*(\tilde{\mathcal{A}}_g, \mathbb{Q})$.*

5 On the Torsion of the Class λ_g

By Proposition (2.2) we know that the top Chern class λ_g of $\mathbb{E}$ vanishes in the rational Chow group $CH^g_{\mathbb{Q}}(\mathcal{A}_g)$. So λ_g is a torsion class on $\mathcal{A}_g$. Mumford proved in [17] that the order of λ_1 in $CH^1(\mathcal{A}_1)$ is 12. In [3] we shall prove the following bound for the order of the torsion class λ_g.

Definition. Let n_g be the greatest common divisor of all $p^{2g}-1$ where p runs through all primes greater than $2g+1$.

We need a little lemma.

Lemma 5.1. *We have*

$$\prod_{i=1}^{g} n_i = \prod_{p\, prime} \big([\frac{2gp}{p-1}]!\big)_p \quad (= \text{multiple of denominator of } p(g)\,).$$

Theorem 5.2. *On $\mathcal{A}_g$ we have:* $(g-1)!\,(\prod_{i=1}^{g} n_i)\,\lambda_g = 0$.

Example i) For $g=1$ we get $24\lambda_1 = 0$ which is off by a factor 2. ii) For $g=2$ we get $24\cdot(16\cdot 3\cdot 5)\lambda_2 = 0$. iii) For $g=3$ we get $2\cdot 24\cdot(16\cdot 3\cdot 5)\cdot(8\cdot 9\cdot 7)\lambda_3 = 0$. We refer to [3] for other cycle relations in the Chow ring of $\tilde{\mathcal{A}}_g$.

6 The Ekedahl-Oort Stratification in Positive Characteristic

Ekedahl and Oort introduced a stratification of the moduli space $\mathcal{A}_g \otimes \mathbb{F}_p$, cf. Oort's paper [24] in this volume. It is defined by analyzing for abelian varieties X the action of Frobenius and Verschiebung on the group scheme $X[p]$, the kernel of multiplication by p. The strata include the well-known loci V_f of abelian varieties of p-rank $\leq f$ (for $0 \leq f \leq g$) and the loci T_a of abelian varieties with a-number $\geq a$ (for $0 \leq a \leq g$). This stratification is very useful for understanding the structure of $\mathcal{A}_g$ both in characteristic p as well as in characteristic 0.

We shall describe these strata in a somewhat different way using the Hodge bundle $\mathbb{E}$. We then can apply theorems of Porteous type to calculate the cycle classes of these loci. Such Porteous type formulas are obtained by applying results of Fulton and of Pragacz on degeneracy maps between symplectic bundles. The cycle classes all lie in the tautological ring. We get explicit formulas for the loci V_f and T_a generalizing Deuring's formula for the number of supersingular elliptic curves and the formula for the number of superspecial abelian varieties (those with $a = g$).

In the following we shall study the moduli space $\mathcal{A}_g \otimes \mathbb{F}_p$ of principally polarized abelian varieties in characteristic p. For simplicity we shall write $\mathcal{A}_g$ instead of $\mathcal{A}_g \otimes \mathbb{F}_p$.

Recall the *canonical filtration* on the de Rham cohomology of a principally polarized abelian variety X as defined by Ekedahl and Oort. They used the group scheme $X[p]$ (kernel of multiplication by p) to define it. We use de Rham-cohomology.

We write $G = H^1_{dR}(X)$ on which we have a σ-homomorphism F and a σ^{-1}-homomorphism V. For a moment we shall ignore the $\sigma^{\pm}$-linearity. We have $FV = VF = 0$. The $2g$-dimensional space is provided with a symplectic form $\langle\, ,\rangle$ and F and V are adjoints: $\langle Vg,g'\rangle = \langle g,Fg'\rangle$. For any subspace H of G we have $(VH)^{\perp} = F^{-1}(H^{\perp})$. The spaces $V(G) = F^{-1}(0)$ and $F(G) = V^{-1}(0)$ are maximally isotropic subspaces of dimension g.

To construct the canonical filtration one starts with $0 \subset G$ and constructs finer filtrations by adding the images of V and the orthogonal complements of the spaces present. This process stops. The filtration obtained is stable under V and under $^{\perp}$, hence under F^{-1} as well and is called the *canonical filtration*. The canonical filtration

$$0 \subset C_1 \subset ... \subset C_r \subset C_{r+1} \subset ... \subset C_{2r}$$

satisfies $C_r = V(C_{2r})$ and $C_{r-i}^{\perp} = C_{r+i}$. This filtration can be refined to a so-called *final filtration* by choosing a V- and $^{\perp}$-stable filtration of length $2g$ which refines the canonical one. In general, there is no unique choice for a final filtration. We thus get a filtration

$$0 \subset G_1 \subset ... \subset G_g \subset G_{g+1} \subset ... \subset G_{2g}$$

which satisfies $V(G_{2g}) = G_g$, $G_{g-i}^{\perp} = G_{g+i}$ and now also $\dim(G_i) = i$. The associated *final type* is the increasing and surjective map

$$\nu : \{0,1,2,\dots ,2g\} \to \{0,1,2,\dots ,g\}$$

satisfying

$$\nu(2g-i) = \nu(i) - i + g \qquad \text{for} \quad 0 \leq i \leq g \tag{9}$$

obtained by $\nu(i) = \dim(V(G_i))$ and $\nu(0) = 0$. The *canonical type* is the restriction of ν to the integers which arise as dimensions of the C_i. Although the final filtration is not unique, the final type is. (The dimensions between two steps of the canonical filtration either remain constant or grow each step by 1.)

Example. Let X be an abelian variety with p-rank f and $a(X) = 1$ (equivalently, on G_g the operator V has rank $g-1$ and semi-simple rank $g-f$). Then the canonical type is given by the numbers $\{\mathrm{rank}(C_i)\} =$

$$\{0 < f < f+1 < \dots < g-1 < g < g+1 < \dots < 2g-f-1 < 2g-f < 2g\}$$

and

$$\nu(f) = f,\, \nu(f+1) = f,$$
$$\nu(f+2) = f+1,\, \dots ,\nu(g) = g-1,\, \dots ,\nu(2g-f-1) = g-1,$$
$$\nu(2g-f) = g,\, \nu(2g) = g.$$

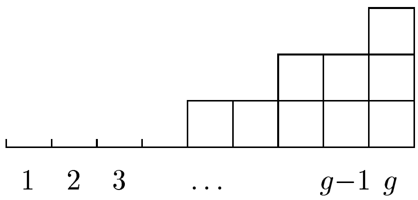
It is not difficult to see that there is a bijection between the set of final types and the set of canonical types and we have 2^g of them. Note that in view of (9) the function ν is determined by its restriction to $\{1,2,\dots ,g\}$. This restriction is again denoted by ν.

The Ekedahl-Oort stratification of $\mathcal{A}_g$ is obtained by looking for each geometric point of $\mathcal{A}_g$ what the canonical type (or final type) is. The set Z_ν of all abelian varieties which have given final type ν is locally closed and these Z_ν define a stratification, cf. [24].

I prefer to describe the combinatorial datum of a final type ν by a *partition* $\mu = \{g \geq \mu_1 > \mu_2 > \dots > \mu_r\}$ as follows:

$$\mu_j = \#\{i : 1 \leq i \leq g, \nu(i) \leq i-j\};$$

equivalently, we can visualize it by the associated Young-type diagram with μ_j squares in the j-th layer (i.e. by putting a stack of $i - \nu(i)$ squares in position i):



This example corresponds to $\{\nu(i) : i = 1,...,g\} = \{1,2,\dots ,g-5,g-5,g-4,g-4,g-3,g-3\}$ and to $\mu = \{5,3,1\}$.

Definition. We call a partition μ *admissible* if $g \geq \mu_1 > \mu_2 > \dots > \mu_r > 0$. We call the function $\nu\colon \{1,\dots ,g\} \to \{1,\dots ,g\}$ *admissible* if

$$0 \leq \nu(i) \leq \nu(i+1) \leq \nu(i)+1 \leq g+1$$

for $1 \leq i \leq g$. The number $|\mu| := \sum_i \mu_i$ is called the *area of the diagram.*

The notions 'admissible' diagram (or partition) and 'admissible function' ν and final type are all equivalent. The set of admissible diagrams carries a partial ordering in the obvious way: $\mu \geq \mu'$ if $\mu_i \geq \mu'_i$ for all i.

7 The Ekedahl-Oort Strata as Degeneracy Loci

We shall now give another approach to these strata by defining them globally on a flag space over $\mathcal{A}_g$. Our starting point is the observation that if we define for a final filtration E_i $(1 \leq i \leq g)$ another filtration by $F_g = \ker(V) = V^{-1}(0)$, by $F_{g+i} = V^{-1}(E_i)$ for $i = 1,...,g$ and by $F_{g-i} = F_{g+i}^{\perp}$ then we have

$$V(E_i) \subseteq E_{\nu(i)} \Longleftrightarrow E_i \subseteq V^{-1}(E_{\nu(i)}) = F_{g+\nu(i)} \Longleftrightarrow \dim(E_i \cap F_{g+\nu(i)}) \geq i. \quad (10)$$

Working now globally, we let S be a scheme in characteristic p and let $\mathcal{X} \to S$ be an abelian variety over S with principal polarization. Then we consider the de Rham cohomology sheaf $\mathcal{H}^1_{dR}(\mathcal{X}/S)$. It is defined as the hyper-direct image $\mathcal{R}^1\pi_*(O_{\mathcal{X}} \to \Omega^1_{\mathcal{X}/S})$. It is a locally free sheaf of rank $2g$ on S equipped with a non-degenerate alternating form (cf. [21]).

$$\langle\, ,\rangle \colon \mathcal{H}^1_{dR}(\mathcal{X}/S) \times \mathcal{H}^1_{dR}(\mathcal{X}/S) \to O_S.$$

Indeed, the polarization (locally in the étale topology given by a relatively ample line bundle on $\mathcal{X}/S$) provides us with a symmetric homomorphism $\rho\colon \mathcal{X} \to \hat{\mathcal{X}}$ and the Poincaré bundle defines a perfect pairing between $\mathcal{H}^1_{dR}(\mathcal{X}/S)$ and $\mathcal{H}^1_{dR}(\hat{\mathcal{X}}/S)$. Moreover, we have an exact sequence of locally free sheaves

$$0 \to \pi_*(\Omega^1) \to \mathcal{H}^1_{dR}(\mathcal{X}/S) \to R^1\pi_* O_{\mathcal{X}} \to 0.$$

We shall write $\mathbb{H}$ for the sheaf $\mathcal{H}^1_{dR}(\mathcal{X}_g/\mathcal{A}_g)$ and $\mathcal{X}$ for $\mathcal{X}_g$. We thus have an exact sequence

$$0 \to \mathbb{E} \to \mathbb{H} \to \mathbb{E}^\vee \to 0.$$

The relative Frobenius $F : \mathcal{X} \to \mathcal{X}^{(p)}$ and the Verschiebung $V : \mathcal{X}^{(p)} \to \mathcal{X}$ satisfy $F \cdot V = p \cdot \mathrm{id}_{\mathcal{X}^{(p)}}$ and $V \cdot F = p \cdot \mathrm{id}_{\mathcal{X}}$ and they induce maps in cohomology, again denoted by F and V:

$$F : \mathbb{H}^{(p)} \to \mathbb{H} \qquad \text{and} \qquad V : \mathbb{H} \to \mathbb{H}^{(p)}.$$

Of course, we have $FV = 0$ and $VF = 0$ and F and V are adjoints. This implies that $\mathrm{Im}(F) = \ker(V)$ and $\mathrm{Im}(V) = \ker(F)$ are maximally isotropic subbundles of $\mathbb{H}$ and $\mathbb{H}^{(p)}$. Moreover, since $dF = 0$ on $\mathrm{Lie}(\mathcal{X})$ it follows that $F = 0$ on $\mathbb{E}$ and thus $\mathrm{Im}(V) = \mathbb{E}^{(p)}$. Verschiebung thus provides us with a bundle map (again denoted by V): $V : \mathbb{H} \to \mathbb{E}^{(p)}$. Consider the space of symplectic flags $\mathcal{F} = \mathrm{Flag}(\mathbb{H})$ on the bundle $\mathbb{H}$. This space is fibred by the spaces $\mathcal{F}^{(i)}$ of partial flags

$$\mathbb{E}_i \subsetneq \mathbb{E}_{i+1} \subsetneq \ldots \subsetneq \mathbb{E}_g.$$

So $\mathcal{F} = \mathcal{F}^{(1)} = \mathrm{Flag}(\mathbb{H})$ and $\mathcal{F}^{(g)} = \mathcal{A}_g$ and there are natural maps

$$\pi_{i,i+1} \colon \mathcal{F}^{(i)} \to \mathcal{F}^{(i+1)}.$$

The fibres are Grassmann varieties of dimension i. The space $\mathcal{F}^i$ is equipped with a universal flag. On $\mathcal{F}$ the Chern classes of the bundle $\mathbb{E}$ decompose into their roots:

$$\lambda_i = \sigma_i(\ell_1, \ldots ,\ell_g) \qquad \text{with} \qquad \ell_i = c_1(\mathbb{E}_i/\mathbb{E}_{i-1}).$$

Given an arbitrary flag of subbundles

$$0 \subsetneq \mathbb{E}_1 \subsetneq \ldots \subsetneq \mathbb{E}_g = \mathbb{E} \tag{11}$$

with $\mathrm{rank}(\mathbb{E}_i) = i$ we can extend this to a symplectic filtration on $\mathbb{H}$ by putting

$$\mathbb{E}_{g+i} = (\mathbb{E}_{g-i})^{\perp}.$$

By base change we can transport this filtration to $\mathbb{H}^{(p)}$.

We introduce a second filtration by starting with the isotropic subbundle

$$\mathbb{F}_g = \ker(V) = V^{-1}(0) \subset \mathbb{H}$$

and continuing with

$$\mathbb{F}_{g+i} = V^{-1}(\mathbb{E}_i^{(p)}) \qquad \text{for} \quad 1 \leq i \leq g.$$

We extend it to a symplectic filtration by setting $\mathbb{F}_{g-i} = (\mathbb{F}_{g+i})^{\perp}$. We thus have two filtrations $\mathbb{E}_{\bullet}$ and $\mathbb{F}_{\bullet}$ on $\mathbb{H}$.

Example. i) Let X be an ordinary abelian variety. Then $\mathrm{Lie}(X) = \mathrm{Lie}(\mu)$ with μ the multiplicative subgroup scheme of $X[p]$ of rank p^g. It follows that V is invertible on $\mathrm{Lie}(X)^{\vee} = \omega(X)$, i.e. $\mathbb{F}_g \cap \mathbb{E}_g = (0)$.

ii) If X is a superspecial abelian variety (i.e. equivalently, $a = g$ or X without its polarization is a product of supersingular elliptic curves) then $V = 0$ on $\omega(X)$ so that $\mathrm{rk}(\mathbb{E}_i \cap \mathbb{F}_g) = i$.

These two (extreme) examples show that the respective position of the two filtrations $\mathbb{E}_{\bullet}$ and $\mathbb{F}_{\bullet}$ for an abelian variety X gives information on the structure of the kernel of multiplication by p on X. These respective positions are encoded by a combinatorial datum, e.g. ν or to be more precise, by an element of a Weyl group. We shall associate strata to such data.

To either ν or μ corresponds an element of the Weyl group of the symplectic group. The Weyl group W_g of type C_g in Cartan's terminology is isomorphic to the semi-direct product $S_g \ltimes (\mathbb{Z}/2\mathbb{Z})^g$, where S_g acts on $(\mathbb{Z}/2\mathbb{Z})^g$ by permuting the g factors. Another description of this group is as the subgroup of S_{2g} of elements which map any symmetric 2-element subset of the form $\{i, 2g+1-i\}$ of $\{1, \ldots, 2g\}$ to a subset of the same type:

$$W_g = \{\sigma \in S_{2g} \colon \sigma(i) + \sigma(2g+1-i) = 2g+1 \text{ for } i = 1, \ldots, g\}.$$

An element in this Weyl group has a *length* and a *codimension*:

$$\ell(w) = \#\{i < j : w(i) > w(j)\} + \#\{i < j : w(i) + w(j) > 2g+1\}$$

and

$$\mathrm{codim}(w) = \#\{i < j : w(i) < w(j)\} + \#\{i < j : w(i) + w(j) < 2g+1\}.$$

We have the equality

$$\ell(w) + \mathrm{codim}(w) = g^2.$$

To a function ν we associate the following element of the Weyl group, a permutation of $\{1,2,\ldots,2g\}$: let

$$S = \{i_1,i_2,\ldots\} = \{1 \leq i \leq g : \nu(i) = \nu(i-1)\}$$

with $i_1 < i_2 < \ldots$ given in increasing order. Let

$$S^c = \{j_1,j_2,\ldots\}$$

be the elements of $\{1,2,\ldots,g\}$ not in S, in increasing order. Then one gets the permutation of S_{2g} defined by ν by writing $g+k$ at position j_k for $k = 1,2,\ldots$ and k at position i_k for $k = 1,2,\ldots$. We finish off by putting $2g+1-k$ at position $2g+1-i$ if k is written at position i. We obtain a sequence s which is a permutation of $\{1,2,\ldots,2g\}$.

Alternatively, using diagrams, we can describe the element $w = w_\mu$ as follows. Let t_i be the operator on the set of diagrams which is 'remove the top box of the i-th column'. We let the t_i act from the right. Given an admissible diagram μ we consider the complementary diagram

$$\mu^c = \{g,g-1,\ldots,1\} - \mu$$

which is also an admissible diagram (partition). We can successively apply operators t_i to it such that after every step we obtain an admissible diagram and such that after $|\mu^c|$ steps we obtain the empty diagram $(\mu^c) \cdot t_{i_1} \cdots t_{i_\ell} = \emptyset$. We remove first the top layer, then the next one and so on. For $1 \leq i < g$ let $s_i \in S_{2g}$ be the permutation $(i,i+1)(2g-i,2g+1-i)$ and let $s_g = (g,g+1) \in S_{2g}$. Now associate to the diagram μ the element of the Weyl group:

$$\mu \mapsto w_\mu = s_{i_1} \cdots s_{i_\ell}.$$

Each diagram thus yields an element in the Weyl group. The admissible partitions yield 2^g elements of the $2^g g!$ elements of W_g.

Example. $g = 3$

μ	ν	$[1,2,3,4,5,6] \mapsto$	ℓ	w_ℓ
$\emptyset$	$\{1,2,3\}$	$[4,5,6,1,2,3]$	6	$s_3s_2s_3s_1s_2s_3$
$\{1\}$	$\{1,2,2\}$	$[4,5,1,6,2,3]$	5	$s_2s_3s_1s_2s_3$
$\{2\}$	$\{1,1,2\}$	$[4,1,5,2,6,3]$	4	$s_3s_1s_2s_3$
$\{3\}$	$\{0,1,2\}$	$[1,4,5,2,3,6]$	3	$s_3s_2s_3$
$\{2,1\}$	$\{1,1,1\}$	$[4,1,2,5,6,3]$	3	$s_1s_2s_3$
$\{3,1\}$	$\{0,1,1\}$	$[1,4,2,5,3,6]$	2	s_2s_3
$\{3,2\}$	$\{0,0,1\}$	$[1,2,4,3,5,6]$	1	s_3
$\{3,2,1\}$	$\{0,0,0\}$	$[1,2,3,4,5,6]$	0	1

We can associate to the map $V : \mathbb{H} \to \mathbb{E}^{(p)}$ and an element w a degeneracy locus $\mathcal{U}_w$ and in particular to an admissible diagram μ a degeneracy locus $\mathcal{U}_\mu = \mathcal{U}_{w_\mu}$ in $\mathcal{F} = \mathrm{Flag}(\mathbb{H})$. Intuitively, $\mathcal{U}_w$ is defined as the locus of points x such that at x we have

$$\dim(\mathbb{E}_i \cap \mathbb{F}_j) \geq \#\{a \leq i : w(a) \leq j\} \quad \text{for all} \quad 1 \leq i,j \leq g$$

or equivalently, that $\ker(V) \cap \mathbb{E}_i \geq i - \nu(i)$ (= the number of squares in the diagram μ in position i). For the precise definition of degeneracy loci of bundle maps we refer to Fulton [7]. Note that for the admissible diagrams we do not use the full filtration of $\mathbb{F}$, but only $\mathbb{F}_g$. Note that our strata (the degeneracy loci) are closed.

8 The Degeneracy Locus $\mathcal{U}_\emptyset$ and $\Gamma_0(p)$

We first look at the case of the empty diagram $\mu = \emptyset$, or equivalently, that $\nu = \{1,2,3,\ldots,g\}$. The degeneracy conditions say that (cf. (10))

$$V(\mathbb{E}_i) \subseteq \mathbb{E}_i^{(p)} \qquad i = 1,\ldots,g.$$

i.e. we are looking at the space $\mathcal{U}_{\{\emptyset\}}$ of symplectic filtrations on $\mathbb{E}$ which are compatible with the action of V. The codimension of this space in $\mathcal{F}$ is $g(g-1)/2$, hence $\dim(U_{\{\emptyset\}}) = g(g+1)/2 = \dim(\mathcal{A}_g)$. We have a finite map $\pi : U_{\{\emptyset\}} \to \mathcal{A}_g$ of degree

$$\deg(\pi) = (1+p)(1+p+p^2)\ldots(1+p+p^2+\ldots+p^{g-1}).$$

This space $\mathcal{U} = U_{\{\emptyset\}}$ can be seen as a component of the moduli space $\Gamma_0(p)$. Indeed, a filtration of the subgroup scheme $X[p]$ corresponds to a filtration on H^1_{dR}. Fix g and let $\Gamma_0(p)'$ be the functor that associates to a scheme S the set of isomorphism classes of principally polarized abelian schemes X over S plus a V-stable symplectic filtration on $X[p]$.

Proposition 8.1. *The degeneracy cycle $\mathcal{U}_\emptyset$ is the algebraic stack representing the functor $\Gamma_0(p)'$. The stack $\mathcal{U}_\emptyset$ is fibred by finite morphisms*

$$\mathcal{U}_\emptyset = \mathcal{U}^{(1)} \xrightarrow{\pi_{1,2}} \mathcal{U}^{(2)} \xrightarrow{\pi_{2,3}} \ldots \xrightarrow{\pi_{g-1,g}} \mathcal{U}^{(g)} = \mathcal{A}_g.$$

with $\deg(\pi_{i,i+1}) = 1 + p + \ldots + p^i$. *It contains the degeneracy loci $\mathcal{U}_\mu$ for all 2^g partitions μ.*

9 The Cycle Classes of the Strata

In this section we try to compute the cycle classes of the stratification on $\mathcal{A}_g \otimes \mathbb{F}_p$ and show that these classes lie in the tautological ring.

The stacks $\mathcal{U}^{(i)}$ come with universal (partial) flags $\mathbb{E}_i \subsetneq \ldots \subsetneq \mathbb{E}_g$. We denote by $\lambda_j(i) = c_j(\mathbb{E}_i)$ the Chern class in $CH^j_{\mathbb{Q}}(\mathcal{U}^{(i)})$. We also have tautological quotient bundles $L_i = \mathbb{E}_i/\mathbb{E}_{i-1}$. We denote by l_i the Chern class of L_i. We have $(\lambda_j)(i) = \sigma_j(l_1, \ldots, l_i)$, the j-th elementary symmetric function of the $l_1, \ldots, l_i$.

Next we look at the cases where μ is a partition of the form $\mu = \{\mu_1 = g-f\}$. The corresponding degeneracy loci classify the loci of p-rank $\leq f$. It is well-known by Koblitz (cf. [14], [20]) that the codimension of V_f in $\mathcal{A}_g$ equals $g-f$. These loci admit the following explicit description. The pullback of V_{g-1} to $\mathcal{U}_\emptyset$ consists of g components, say $Z_1, \ldots, Z_g$. An abelian variety of p-rank $g-1$ and $a=1$ has a unique subgroup scheme α_p. The index i of Z_i indicates where for the generic point of Z_i the α_p can be found: $\alpha_p \subset G_i$, $\alpha_p \not\subset G_{i-1}$.

Then the pullback of V_f consists of

$$\pi^{-1}(V_f) = \sum_{\#S=g-f} Z^S,$$

where for a subset $S \subset \{1, \ldots, g\}$ the cycle Z^S is defined as

$$Z^S = \cap_{i \in S} Z_i.$$

Then one sees easily that the cycle V_{g-f} on $\mathcal{A}_g$ is obtained as the pushforward of $Z_g \cap \ldots \cap Z_{g-f+1}$.

Lemma 9.1. *The cycle class of Z_i is equal to $(p-1)\ell_i$.*

Proof. It is described as the locus where $\det(V) : L_i \to L_i^{(p)}$ vanishes. By viewing $\det(V)$ as a section of $L_i^{(p)} \otimes L_i^{-1}$ the result follows. □

We find after a calculation:

$$\pi^*([V_f]) = (p-1)^{g-f}\lambda_{g-f}$$

on $\mathcal{U}_\emptyset$. Using the pushforward of the $\pi_{i,i+1}$ on the classes ℓ_i and extending the result to a compactification $\tilde{\mathcal{A}}_g$ we get:

Theorem 9.2. *The cycle class $[V_f]$ of V_f, the p-rank $\leq f$ locus, in the Chow ring $CH^*_{\mathbb{Q}}(\tilde{\mathcal{A}}_g)$, is given by $[V_f] = (p-1)(p^2-1)\ldots(p^{g-f}-1)\lambda_{g-f}$.*

By applying this formula in the simplest case where $g=1$ and $f=0$ we get a classical result.

Corollary 9.3. *(Deuring Mass Formula) Let $g=1$. We have*

$$\sum_E \frac{1}{\#\mathrm{Aut}(E)} = \frac{p-1}{24},$$

where the sum is over the isomorphism classes over $\overline{\mathbb{F}}_p$ of supersingular elliptic curves.

Proof. The formula gives $(p-1)\lambda_1$ and λ_1 equals $\delta/12$. The class of δ is equivalent to $1/2$ times the class of a 'physical' point of the j-line because the degenerate elliptic curve corresponding to δ has 2 automorphisms. □

Another case where we can find an explicit formula is that of the locus T_a. This corresponds to $\mu = \{a,a-1,\ldots,2,1\}$. But here we can work directly on $\mathcal{A}_g$. The locus T_a on $\mathcal{A}_g$ may be defined as the locus

$$\{x \in \mathcal{A}_g : \mathrm{rank}(V)|_{\mathbb{E}_g} \leq g-a\}.$$

We have $T_{a+1} \subset T_a$ and $\dim(T_g)=0$.

Pragacz and Ratajski, cf. [26], have developed formulas for the degeneracy locus for the rank of a self-adjoint bundle map of symplectic bundles globalizing the results in isotropic Schubert calculus from [25]. Before we apply their result to our case we have to introduce some notation.

Define for a vector bundle A with Chern classes a_i the expression

$$Q_{ij}(A) := a_i a_j + 2\sum_{k=1}^{j}(-1)^k a_{i+k}a_{j-k} \quad \text{for} \quad i>j.$$

For a admissible partition $\beta = (\beta_1,\ldots,\beta_r)$ (with r even, β_r may be zero) we set

$$Q_\beta = \mathrm{Pfaffian}(x_{ij}),$$

where the (x_{ij}) is an anti-symmetric matrix with $x_{ij} = Q_{\beta_i,\beta_j}$. Applying the Pragacz-Ratajski formula to our situation gives the following result:

Theorem 9.4. *The cycle class $[T_a]$ of the reduced locus T_a of abelian varieties with a-number $\geq a$ is given by*

$$\sum Q_\beta(\mathbb{E}^{(p)}) \cdot Q_{\rho(a)-\beta}(\mathbb{E}^*),$$

where the sum is over the admissible partitions β contained in the partition $\rho(a) = (a,a-1,a-2,\ldots,1)$.

Example:

$$\begin{aligned}[T_1] &= p\lambda_1 - \lambda_1\\ [T_2] &= (p-1)(p^2+1)(\lambda_1\lambda_2) - (p^3-1)2\lambda_3\\ &\dots\\ [T_g] &= (p-1)(p^2+1)\dots(p^g+(-1)^g)\lambda_1\lambda_2\cdots\lambda_g.\end{aligned}$$

As a corollary we find a classical result of Ekedahl (cf. [2]) on the number of principally polarized abelian varieties with $a = g$. Such abelian varieties are sometimes called *superspecial*.

Corollary 9.5. *We have*

$$\sum_X \frac{1}{\#\mathrm{Aut}(X)} = (-1)^G 2^{-g}\big[\prod_{j=1}^{g}(p^j+(-1)^j)\big]\cdot\zeta(-1)\zeta(-3)\dots\zeta(1-2g),$$

where the sum is over the isomorphism classes (over $\overline{\mathbb{F}}_p$) of principally polarized abelian varieties of dimension g with $a = g$.

Proof. Combine the formula for T_g with the Proportionality Theorem. $\square$

For each element w we now find a degeneracy locus $\mathcal{U}_w$. In particular we find such a locus $\mathcal{U}_\mu$ for each partition μ in $\mathcal{F}$ and the $\mathcal{U}_\mu$ actually lie in $\mathcal{U}_\emptyset$. The fact that T_g is zero-dimensional implies that the codimension of each $\mathcal{U}_\mu$ equals $\mathrm{codim}(w(\mu)) = |\mu|$. We can apply a theorem of Fulton to determine the class of the degeneracy locus U_μ in $CH^*_{\mathbb{Q}}(\mathcal{F})$.

For an admissible diagram $\mu = \{\mu_1 > \dots > \mu_r > 0\}$ Fulton defines a determinant ('Schur function')

$$\Delta_\mu(x_i) = \det(x_{\mu_i+j-i})_{1\le i,j\le r};$$

this is a polynomial with integer coefficients in the commuting variables $x_1,x_2,\dots$. Define a 'double Schubert function' by putting

$$\Delta(x,y) = \Delta_{g,g-1,\dots,1}\big(\sigma_i(x_1,\dots,x_g)+\sigma_i(y_1,\dots,y_g)\big).$$

The operators ∂_i ('divided difference operators') on the ring $\mathbb{Z}[x_1,x_2,\dots,]$ are defined by setting for $F(x)\in\mathbb{Z}[x_1,\dots,x_g]$

$$\partial_i(F(x)) = \begin{cases}\frac{F(x)-F(s_i(x))}{x_i-x_{i+1}} & \text{if } i<g\\ \frac{F(x)-F(s_g(x))}{2x_g} & \text{if } i=g\end{cases}$$

Put

$$\Delta = \Delta(x,y) = \Delta_{g,g-1,\dots,1}\big(\sigma_i(x_1,\dots,x_g)+\sigma_i(y_1,\dots,y_g)\big).$$

We shall apply the result of Fulton [7] and state an abstract formula for our cycle classes. In principle one then can calculate the pushforward algorithmically. But it seems difficult to get closed formulas for the cycle classes of these pushforwards.

We find:

Theorem 9.6. *Let μ be an admissible diagram whose corresponding element w_μ in the Weyl group is written as $s_{i_\ell}\cdots s_{i_1}$. The cycle class of the degeneracy locus U_μ in $CH^*_{\mathbb{Q}}(\mathcal{F})$ is given by*

$$u_\mu = \partial_{i_\ell}\cdots\partial_{i_1}\big(\big(\prod_{i+j\leq g}(x_i-y_i)\big)\cdot\Delta(x,y)\big)\big|_{\{x_i=pl_i,y_i=-l_i\}}$$

Corollary 9.7. *The pushforward of the class of $[\mathcal{U}_\mu]$ under π is given by $\pi_*(u_\mu)$ and is a multiple of the class of the reduced cycle Z_μ, the multiple being equal to the number of final filtrations refining the canonical filtration associated to μ. The class belongs to the tautological ring.*

We list here the formulas for $g=3$. The multiplicity is given by the factor in square brackets.

Formulas for $g=3$.

$$\pi_*(\{\emptyset\}) = [(1+p)(1+p+p^2)]$$

$$\begin{aligned}
\pi_*(\{1\}) &= [(1+p)]\times\,(1-p)\,\lambda_1\\
\pi_*(\{2\}) &= (p-1)(p^2-1)\,\lambda_2\\
\pi_*(\{1,2\}) &= [(1+p)]\times\{\,(1-p)\,(1+p^2)\,\lambda_1\,\lambda_2-2\,(-1+p^3)\,\lambda_3\}\\
\pi_*(\{3\}) &= (p-1)(p^2-1)(p^3-1)\,\lambda_3\\
\pi_*(\{1,3\}) &= [(1+p)]\times(-1+p)^2\,\,(1-p+p^2)\,\lambda_1\,\lambda_3\\
\pi_*(\{2,3\}) &= (-1+p)^3\,(1+p)\,(-1+p-p^2)\,(1+p+p^2)\,\lambda_2\,\lambda_3\\
\pi_*(\{1,2,3\}) &= [(1+p^3)]\times(p-1)\,(p^2+1)(p^3-1)\,\lambda_1\,\lambda_2\,\lambda_3.
\end{aligned}$$

The canonical filtration on $H^1_{dR}(X)$ is the most economical one with respect to the operation V. The types of this filtration correspond to the elements of $(\mathbb{Z}/2\mathbb{Z})^g$ in the Weyl group $(\mathbb{Z}/2\mathbb{Z})^g\rtimes S_g$. The other types of filtrations are obtained by applying an element of S_g to the right. Therefore, in the closure of a stratum $\mathcal{U}_\mu$ we find also $\mathcal{U}_w$ with $w\notin(\mathbb{Z}/2\mathbb{Z})^g$. This explains the phenomenon observed by Oort in [24], 4.6.

We can use Pieri-type formulas to get results on the boundary of strata. In particular one obtains the result of Ekedahl-Oort that the class λ_1 is torsion on the open strata. We refer to [3].

Some strata can be described in detail. Consider the locus associated to the partition

$$\mu=\{g,g-1,\ldots,3,2\}.$$

This is the penultimate stratum and it is of dimension 1. It is proved by Ekedahl and Oort that this locus is connected.

Define $h(g)$ by

$$h(1) = 1, \qquad h(g) = h(g-1) \times \frac{p^g + (-1)^g}{p + (-1)^g} \quad \text{for} \quad g > 1.$$

Then we have

$$\pi_*(U_{[g,g-1,\ldots,1]}) = h(g)[T_g]$$

with $[T_g]$ given as above.

Theorem 9.8. *The cycle class of* $\pi_*(\mathcal{U}_\mu)$ *for* $\mu = \{g,g-1,\ldots,3,2\}$ *is given by*

$$\frac{(p-1)}{(p^2+1)} \times h(g) \times \left(\prod_{i=1}^{g} (p^i + (-1)^i) \right) \times \lambda_2 \lambda_3 \ldots \lambda_g .$$

In $\mathcal{U}_\emptyset$ *this locus consists of a configuration of* $\mathbb{P}^1$*'s.*

Since the degree of λ_1 on each copy of $\mathbb{P}^1$ is $p-1$ one can compute the number of components. This locus is highly reducible. But some loci are irreducible:

Theorem 9.9. *For all* $a < g$ *the locus* T_a *is irreducible. In particular, the locus* $T_1 = V_{g-1}$ *is irreducible.*

Proof. By Ekedahl-Oort (cf. [24]) we know that the locus corresponding to the diagram $\{g,g-1,\ldots,3,2\}$ is connected. This implies that T_a is connected for $a < g$. We know that $\mathrm{Sing}(T_a) \subseteq T_{a+1}$, hence of codimension > 1. Actually, one can describe the normal bundle to $T_a - T_{a+1}$. By a theorem of Hartshorne (cf. [11]) this implies that $T_a - \mathrm{Sing}(T_a)$ is connected. □

The special case of the irreducibility of V_{g-1} was also shown by Ekedahl and Oort, cf. [24], (6.5).

10 Effectivity of Tautological Classes

As a corollary of the results in the preceding section we obtain the effectivity of the expressions in the tautological ring that are the cycle classes of the Ekedahl-Oort strata. In particular we obtain the following result.

Theorem 10.1. *The Chern classes* $\lambda_i \in CH_{\mathbb{Q}}(\mathcal{A}_g \otimes \mathbb{F}_p)$ *of the Hodge bundle are represented by effective classes (with* $\mathbb{Q}$*-coefficients).*

Proof. The class of the locus of abelian varieties with p-rank $\leq f$ is a multiple of λ_{g-f}. □

Remark 10.2. The Hodge bundle has an ample determinant bundle with first Chern class λ_1, but the bundle $\mathbb{E}$ is not positive in positive characteristic for $g \geq 2$. For example, for $g = 2$ the restriction of the Hodge bundle to a line from the p-rank 0 locus is isomorphic to $O(p) \oplus O(-1)$ which is evidently not positive. In general, $\mathbb{E}$ is not positive on the stratum defined by $\{g,g-1,\ldots,3,2\}$.

11 Some Additional Results

We describe as an example the canonical type for hyperelliptic curves of 2-rank 0 in characteristic $p = 2$.

Lemma 11.1. *A hyperelliptic curve C of genus g and of 2-rank 0 over $k = \overline{k}$ of characteristic $p = 2$ can be written as*

$$y^2 + y = xP(x^2) = x(a_1x^2 + a_2x^4 + ... + a_{g-1}x^{2g-2} + x^{2g}). \tag{13}$$

Theorem 11.2. *For a hyperelliptic curve as in (13) the canonical type is described by the corresponding partition $\mu = [g, g-2, g-4, ...]$. In particular, the canonical filtration is independent of the coefficients a_i.*

We refer for the proof to [3].

For $g = 1$ and $g = 2$ the supersingular locus coincides with V_0 and thus occurs as a stratum in the Ekedahl-Oort stratification. For $g \geq 3$ this is no longer true. The definition of supersingular locus involves also the higher filtrations, i.e. it uses not only $X[p]$, but the higher group schemes $X[p^i]$ as well. But in a number of cases we can determine the class of this locus. We state here the result for the case $g = 3$.

Theorem 11.3. *The class of the supersingular locus S_3 in $\mathcal{A}_3^* \otimes \mathbb{F}_p$ is*

$$[S_3] = (p-1)(p^2-1)(p^3-1)(p-1)(p^2+1)\lambda_1\lambda_3.$$

The proof uses the explicit description by Li and Oort of the moduli of principally polarized supersingular abelian varieties, cf. [15]. We refer the reader to [3] for the proof and formulas for other genera.

One can define analogues of the stratification defined here on $\mathcal{A}_g \otimes \mathbb{F}_p$ for other moduli spaces, e.g. for the moduli space of K3 surfaces of given genus in characteristic p. We refer to a joint paper with T. Katsura, see [9].

Bibliography

[1] A. Borel, J-P. Serre: Le théorème de Riemann-Roch (d'après Grothendieck). *Bull. Soc. Math. France* **86** (1958), 97–136.

[2] T. Ekedahl: On supersingular curves and abelian varieties. *Math. Scand.* **60** (1987), 151–178.

[3] T. Ekedahl, G. van der Geer: Manuscripts in preparation.

[4] T. Ekedahl, F. Oort : Connected subsets of a moduli space of abelian varieties. Preliminary version of a preprint.

[5] C. Faber, G. van der Geer: Complete subvarieties of $\mathcal{M}_g$ and the Prym map. In preparation.

[6] G. Faltings, C-L. Chai: Degeneration of abelian varieties. Ergebnisse der Math. 22. Springer Verlag 1990.

[7] W. Fulton: Determinantal formulas for orthogonal and symplectic degeneracy loci. *J. Diff. Geom.* **43** (1996), 276–290.

[8] G. van der Geer: The Chow ring of the moduli space of abelian threefolds. *J. of Alg. Geometry* **7** (1998), 753–770.

[9] G. van der Geer, T. Katsura: A stratification of the moduli space of K3 surfaces in positive characteristic. In preparation.

[10] G. van der Geer, F. Oort: Moduli of abelian varieties : a short introduction and survey. This volume.

[11] R. Hartshorne: Complete intersections and connectedness. *Amer. J. Math.* **84** (1962), 497–508.

[12] F. Hirzebruch: Automorphe Formen und der Satz von Riemann-Roch. In *Symposium Internacional de Topologia Algebrica* (México 1956), 129–144. México: La Universidad Nacional Autónoma de México 1958. (= Collected Papers I , Springer Verlag, p. 345)

[13] F. Hirzebruch: Kommentar zu 'Elliptische Differentialoperatoren auf Mannigfaltigkeiten'. In: Collected Papers II, Springer Verlag, p. 773.

[14] N. Koblitz: p-adic variation of the zeta function over families of varieties defined over finite fields. *Compositio Math.* **31**, (1975), 119–218.

[15] K-Z. Li, F. Oort: Moduli of supersingular abelian varieties. Lecture Notes in Mathematics 1680, Springer Verlag, Berlin-New York, 1998.

[16] L. Moret-Bailly: Pinceaux de variétés abéliennes. Astérisque 129 (1985).

[17] D. Mumford: Picard groups of moduli problems. In *Arithmetical Algebraic Geometry.* Ed. O.F.G. Schilling. Harper and Row, 1965, p. 33–81.

[18] D. Mumford: Hirzebruch's Proportionality Theorem in the non-compact case. *Inv. Math.* **42** (1977) , 239–272.

[19] D. Mumford: On the Kodaira dimension of the Siegel modular variety. In: Algebraic Geometry—Open Problems. SLNM 997, 348–375.

[20] P. Norman, F. Oort: Moduli of abelian varieties. *Ann. Math.* **112** (1980), 413–439.

[21] T. Oda : The first de Rham cohomology group and Dieudonné modules. Ann. Sci. ENS (1969), 63–135.

[22] F. Oort: Subvarieties of moduli spaces. *Inv. Math.* **24** (1974), 95–119.

[23] F. Oort: Complete subvarieties of moduli spaces. In: *Abelian Varieties* (W. Barth, K. Hulek, H. Lange, eds.), de Gruyter Verlag, Berlin, 1995, p. 225–235.

[24] F. Oort: A stratification of a moduli space of polarized abelian varieties in positive characteristic. Preprint May 1996. This volume.

[25] P. Pragacz: Algebro-geometric applications of Schur S- and Q- polynomials. In: Topics in Invariant Theory. Séminaire d'Algèbre Dubreil-Malliavin 1989–1990. SLNM 1478, 130–191 (1991).

[26] P. Pragacz, J. Ratajski: Formulas for Lagrangian and orthogonal degeneracy loci; the $\tilde{Q}$-polynomial approach. *Comp. Math.* **107** (1997), 11–87.