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Effective divisors on projectivized Hodge bundles and modular Forms

Gerard van der Geer[1] | **Alexis Kouvidakis**[2]

[1]Korteweg-de Vries Institute for Mathematics, University of Amsterdam, Amsterdam, The Netherlands

[2]Department of Mathematics and Applied Mathematics, University of Crete, Heraklion, Greece

Correspondence
Gerard van der Geer, Korteweg-de Vries Institute for Mathematics, University of Amsterdam, Amsterdam, The Netherlands.
Email: G.B.M.vanderGeer@uva.nl

Abstract

We construct vector-valued modular forms on moduli spaces of curves and abelian varieties using effective divisors in projectivized Hodge bundles over moduli of curves. Cycle relations tell us the weight of these modular forms. In particular, we construct basic modular forms for genus 2 and 3. We also discuss modular forms on the moduli of hyperelliptic curves. In that case, the relative canonical bundle is a pull back of a line bundle on a $\mathbb{P}^1$-bundle over the moduli of hyperelliptic curves and we extend that line bundle to a compactification so that its push down is (close to) the Hodge bundle and use this to construct modular forms. In the Appendix, we use our method to calculate divisor classes in the dual projectivized k-Hodge bundle determined by Gheorghita–Tarasca and by Korotkin–Sauvaget–Zograf.

KEYWORDS
effective divisor, Hodge bundle, modular form, moduli space

1 | INTRODUCTION

Moduli spaces of curves and of abelian varieties come with a natural vector bundle, the Hodge bundle $\mathbb{E}$. Starting from this vector bundle, one can construct other natural vector bundles by applying Schur functors, like $\mathrm{Sym}^n(\mathbb{E})$ or $\det(\mathbb{E})^{\otimes m}$. Sections of such bundles are called modular forms. For example, for the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g these are Siegel modular forms, and for the moduli space $\mathcal{M}_g$ of curves of genus g these are Teichmüller modular forms. If the Schur functor corresponds to an irreducible representation ρ, we say that a section of $\mathbb{E}_\rho$ is a modular form of weight ρ. The Hodge bundle extends to appropriate compactifications of such moduli spaces and in many cases the sections also extend automatically to the compactifications, for example, for $\mathcal{A}_g$ with $g \geq 2$ by the so-called Koecher principle.

In this paper, we try to construct modular forms in a geometric way. It is well known that an effective divisor on $\mathcal{A}_g$ or on $\overline{\mathcal{M}}_g$ with $g \geq 2$ representing the cycle class $m\lambda$ with $\lambda = c_1(\det(\mathbb{E}))$ and $m \in \mathbb{Z}_{>0}$ yields a scalar-valued modular form of weight m, that is, a section of $\det(\mathbb{E})^{\otimes m}$. We will exploit explicit effective divisors on projectivized vector bundles to construct vector-valued modular forms. In particular, we will construct in this way certain modular forms that play a pivotal role in low genera.

For example, in the case of $g = 2$ there is the modular form $\chi_{6,8}$, a section of $\mathrm{Sym}^6(\mathbb{E}) \otimes \det(\mathbb{E})^8$, that appeared in [4] as follows. Recall that the Torelli morphism $\mathcal{M}_2 \hookrightarrow \mathcal{A}_2$ has a dense image and we have an equality of standard compactifications $\overline{\mathcal{M}}_2 = \tilde{\mathcal{A}}_2$. The moduli space $\mathcal{M}_2$ has another description as a stack quotient. This derives from the fact that a smooth complete curve of genus 2 over a field k of characteristic not 2 is a double cover of $\mathbb{P}^1$ ramified at six points, so can be given as $y^2 = f$ with f a polynomial of degree 6 with nonvanishing discriminant. Writing f as a homogeneous polynomial in two variables, say $f \in \mathrm{Sym}^6(W)$ with W the k-vector space generated by x_1, x_2, and observing that we may change the basis of W, we find a presentation of $\mathcal{M}_2$ as a stack quotient

$$\mathcal{M}_2 \sim [W^0_{6,-2}/\mathrm{GL}(W)],$$

where we write $W_{a,b}$ for the $\mathrm{GL}(W)$-representation $\mathrm{Sym}^a(W) \otimes \det(W)^b$. Here, the space $W_{6,-2}$ can be seen as the vector space of binary sextics f with an action of $\mathrm{GL}(W)$ by

$$f(x_1, x_2) \mapsto (ad - bc)^{-2} f(ax_1 + bx_2, cx_1 + dx_2)$$

for a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2)$. The subspace $W^0_{6,-2}$ of $W_{6,-2}$ is the space of f with nonvanishing discriminant. The twisting by $\det(W)^{-2}$ is required to get the right stabilizer for the generic f, namely $\pm\mathrm{Id}_W$.

This interpretation of $\mathcal{M}_2$ was used in [4] to construct vector-valued Siegel modular forms of degree 2 by using invariant theory of binary sextics, thus extending and simplifying the description of scalar-valued Siegel modular forms by invariants by Igusa [17, 18]. Covariants define vector-valued modular forms and all Siegel modular forms of degree 2 on $\mathcal{A}_2$ can be constructed this way. In [4], it was shown that the most basic covariant, the universal binary sextic, defines a meromorphic Siegel modular form $\chi_{6,-2}$ of weight $(6, -2)$, that is, it defines a meromorphic section of $\mathrm{Sym}^6(\mathbb{E}) \otimes \det(\mathbb{E})^{-2}$ on $\mathcal{A}_2$. After multiplying $\chi_{6,-2}$ by Igusa's cusp form χ_{10}, one obtains the holomorphic modular form $\chi_{6,8}$, the "first" vector-valued Siegel modular cusp form of degree 2.

In the case of $g = 3$, there is an analogous form $\chi_{4,0,8}$, a section of $\mathrm{Sym}^4(\mathbb{E}) \otimes \det(\mathbb{E})^8$. Here, it derives from the description of the moduli space $\mathcal{M}_3^{nh}$ of nonhyperelliptic curves of genus 3 as a stack quotient

$$\mathcal{M}_3^{nh} \sim [W^0_{4,0,-1}/\mathrm{GL}(W)],$$

where W is now of dimension 3 and $W^0_{4,0,-1} \subset \mathrm{Sym}^4(W) \otimes \det(W)^{-1}$ represents ternary quartics defining smooth curves. In [5], this description led to the construction of a meromorphic Teichmüller modular form $\chi_{4,0,-1}$ of weight $(4, 0, -1)$ and a (holomorphic) Siegel modular form $\chi_{4,0,8}$ of degree 3 and weight (4,0,8). Also in this case all Teichmüller and Siegel modular forms of genus 3 on $\overline{\mathcal{M}}_3$ and $\mathcal{A}_3$ can be constructed from these forms by invariant theory.

This paper arises from the desire to construct these basic forms and similar forms in a geometric way. We use cycle relations for effective divisors (or almost effective divisors) on the projectivized Hodge bundle to construct our forms. It is based on the observation that an effective divisor D on the projectivized Hodge bundle $\mathbb{P}(\mathbb{E})$ with cycle class

$$[D] = [\mathcal{O}(j)] + k\,\lambda - \Delta$$

with positive integers j, k, and Δ an effective boundary class gives rise to a section of $\mathrm{Sym}^j(\mathbb{E}) \otimes \det(\mathbb{E})^k$ vanishing on boundary divisors, that is, a modular form. This method produces the basic modular forms $\chi_{6,8}$ and $\chi_{4,0,8}$ of degree 2 and 3 in an efficient way.

This connection between divisors and modular forms can also be used in the other direction, obtaining cycle classes for divisors on projectivized Hodge bundles. We give some examples of this.

Another objective of this paper is to construct modular forms on moduli spaces of hyperelliptic curves of genus g. For this we work with two descriptions of the moduli, a description as a stack quotient and a description as a Hurwitz space. The latter space $\mathcal{H}_{g,2}$ has as compactification the space $\overline{\mathcal{H}}_{g,2}$ of admissible degree 2 covers of genus g. In the stack description, modular forms pull back to covariants for the action of $\mathrm{GL}(2)$ on the space of binary forms of degree $2g + 2$.

In the Hurwitz space description, the relative canonical bundle of the universal curve over $\mathcal{H}_{g,2}$ can be viewed as the pull back of $\mathcal{O}(g - 1)$ from the trivial $\mathbb{P}^1$-bundle P over $\mathcal{H}_{g,2}$ equipped with $2g + 2$ nonintersecting sections. Using the theory of admissible covers, P is compactified to a space $\overline{P}$, a fibration of rational stable curves with $2g + 2$ marked points over $\overline{\mathcal{H}}_{g,2}$, and we show that the line bundle $\mathcal{O}(g - 1)$ on P extends to a line bundle on $\overline{P}$ with the property that its push down to $\overline{\mathcal{H}}_{g,2}$ is close to the Hodge bundle. This allows us to construct modular forms on $\overline{\mathcal{H}}_{g,2}$.

When we consider projectivized bundles, projectivization is meant in the Grothendieck sense, so that for a vector space V the projective space $\mathbb{P}(V)$ parameterizes hyperplanes in V.

In an Appendix, we apply a method used in this paper to calculate the classes of certain divisors in the dual projectivized k-Hodge bundle that were determined by Gheorghita–Tarasca and by Korotkin–Sauvaget–Zograf.

2 | THE CASE OF GENUS 2

Let k be a field of characteristic not 2. We consider the moduli space $\mathcal{M}_2$ of curves of genus 2 over k. This is a Deligne–Mumford stack and it carries a universal curve $\pi : \mathcal{C} \to \mathcal{M}_2$ of genus 2. The relative dualizing sheaf ω_π is base point free and thus defines a morphism $\varphi : \mathcal{C} \to \mathbb{P}(\mathbb{E})$. For a curve C, the map $\varphi : C \to \mathbb{P}(\mathbb{E}_C)$ associates to a point the space of differentials vanishing in that point. We have a commutative diagram with u the natural morphism

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\varphi} & \mathbb{P}(\mathbb{E}) \\ & \searrow^{\pi} & \downarrow u \\ & & \mathcal{M}_2 \end{array}$$

We let $\overline{\mathcal{M}}_2$ be the Deligne–Mumford compactification and $\overline{\pi} : \overline{\mathcal{C}} \to \overline{\mathcal{M}}_2$ the corresponding universal curve. However, the extension $\omega_{\overline{\pi}}$ of ω_π does not define an extension of the morphism φ to $\mathbb{P}(\mathbb{E})$ over the boundary component Δ_1 that parameterizes reducible curves.

We consider the branch divisor $D \subset \mathbb{P}(\mathbb{E})$ of the morphism φ. The divisor D is of relative degree 6 in the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathbb{E})$ over the base $\mathcal{M}_2$. We define $\overline{D}$ to be the closure of D in $\mathbb{P}(\mathbb{E})$ over $\overline{\mathcal{M}}_2$. In the rational Picard group of $\mathbb{P}(\mathbb{E})$, we can write

$$[\overline{D}] = [\mathcal{O}(6)] + u^*(A)$$

with A a class in the rational Picard group of $\overline{\mathcal{M}}_2$ and $u : \mathbb{P}(\mathbb{E}) \to \overline{\mathcal{M}}_2$ the natural projection.

We want to determine A in terms of the generators λ, δ_0 of the Picard group of $\overline{\mathcal{M}}_2$. We write λ for the first Chern class of $\mathbb{E}$ and δ_1 (resp. δ_0) for the class of Δ_1 (resp. Δ_0) in the Picard group of the stack $\overline{\mathcal{M}}_2$; here Δ_0 is the boundary component that parameterizes irreducible curves with a double point.

In order to do this, we extend the morphism φ. It extends over a Zariski open part of Δ_0 since ω_π has no base points there. However, over Δ_i with $i > 0$ this system has base points. We then use a base change as described in Appendix A. After a base change, we have in an open neighborhood U_i of the generic point of Δ_i a semistable family. If we take the base to be 1-dimensional, we get a semistable family $\tilde{f} : \tilde{C} \to \tilde{B}$ with as central fiber a chain $C' + R + C''$ with R a (-2)-curve and C' and C'' of genus 1. The extension φ' of φ is given by $\omega_{\tilde{f}}(-R)$ with $\tilde{f}_*(\omega_{\tilde{f}}(-R)) = \mathbb{E}_{\tilde{B}}$ and the morphism $\varphi' : \tilde{C} \to \mathbb{P}(\mathbb{E})$ contracts C' and C'' and is of degree 2 on R. We refer to Appendix A for the details. The morphism φ' has the property

$$\varphi'^*(\mathcal{O}_{\mathbb{P}(\mathbb{E}_{\tilde{B}})}(1)) = \omega_{\tilde{f}}(-R)\,.$$

Proposition 2.1. *We have* $[\overline{D}] = 6\,[\mathcal{O}(1)] + u^*(8\,\lambda - \delta_0 - \delta_1)$.

Proof. We write $[\overline{D}] = 6\,[\mathcal{O}(1)] + u^*(A)$. We work with the above two types of 1-dimensional families $f : C \to B$. The morphism φ is ramified over D, and thus φ' is ramified over $\overline{D}$ and contracts C' and C''. We denote the ramification divisor by S. We thus get (writing abusively line bundles and divisors for the corresponding divisor classes)

$$\omega_{\tilde{f}} = \varphi'^*\omega_u + S + 2\,(C' + C''), \quad \varphi'^*\overline{D}/2 = S + 3\,(C' + C'')\,,$$

where the first equation comes from adjunction $\omega_{\tilde{f}} + C'_{|C'} = \mathcal{O}_{C'}$ for C' and similarly for C'', and the second one from $C' \cdot \varphi'^*\overline{D} = 0 = C'' \cdot \varphi'^*\overline{D}$. This gives

$$\begin{aligned} \omega_{\tilde{f}} &= \varphi'^*(\omega_u + \overline{D}/2) - (C' + C'') \\ &= \varphi'^*(\mathcal{O}(-2) + u^*(\lambda) + \mathcal{O}(3) + u^*(A/2)) - (C' + C'') \\ &= \varphi'^*(\mathcal{O}(1) + u^*(\lambda + A/2)) - (C' + C'') \\ &= \omega_{\tilde{f}} - R + \tilde{f}^*(\lambda + A/2) - (C' + C'') \\ &= \omega_{\tilde{f}} + \tilde{f}^*(\lambda + A/2 - b_1) \end{aligned}$$

with b_1 the special point of $\tilde{B}$. This shows that $A = -2\lambda + 2b_1$. Because of the base change that we executed, we have $2b_1 = \delta_1$ and we obtain $A = -2\lambda + \delta_1$. Now we use the well-known relation $10\lambda = \delta_0 + 2\delta_1$ (see [24]) and thus get $A = -2\lambda + \delta_1 = 8\lambda - \delta_0 - \delta_1$. □

Remark 2.2. We indicate an alternative proof of this result in Remark 13.5.

An important remark is now that $u_*(\mathcal{O}(1)) = \mathbb{E}$ and $u_*(\mathcal{O}(m)) = \mathrm{Sym}^m(\mathbb{E})$ for $m \geq 1$. The divisor $\overline{D}$ with

$$[\overline{D}] = [\mathcal{O}(6)] + u^*(8\lambda - \delta_0 - \delta_1)$$

is an effective divisor on $\mathbb{P}(\mathbb{E})$. We apply u_* to the corresponding section 1 of $\mathcal{O}(\overline{D})$. By Proposition 2.1, we see that we get a regular section $\chi_{6,8}$ of the vector bundle $\mathrm{Sym}^6(\mathbb{E}) \otimes \det(\mathbb{E})^8$ over $\overline{\mathcal{M}}_2$. Moreover, this section vanishes on the divisors Δ_0 and Δ_1. Note that the Torelli map extends to an isomorphism $\overline{\mathcal{M}}_2 \cong \tilde{\mathcal{A}}_2$ with $\tilde{\mathcal{A}}_2$ the standard smooth compactification of $\mathcal{A}_2$. Therefore, our section defines a Siegel modular form $\chi_{6,8}$ of weight (6,8) that is a cusp form.

Corollary 2.3. *Let $\overline{D}$ be the closure in $\mathbb{P}(\mathbb{E})$ of the branch divisor of the canonical map for the universal curve over $\mathcal{M}_2$. The push forward $u_*(s)$, with s the natural section 1 of $\mathcal{O}(\overline{D})$ on $\mathbb{P}(\mathbb{E})$, defines a Siegel modular cusp form $\chi_{6,8}$ of degree 2 and weight (6,8).*

The relation $10\lambda = \delta_0 + 2\delta_1$ quoted above implies that there exists a Siegel modular cusp form χ_{10} of degree 2 and of weight 10 with divisor $\delta_0 + 2\delta_1$. The quotient $\chi_{6,-2} := \chi_{6,8}/\chi_{10}$ defines a meromorphic section of $\mathrm{Sym}^6(\mathbb{E}) \otimes \det \mathbb{E}^{-2}$ that is regular outside Δ_1.

We now analyze the orders of vanishing along δ_1 of $\chi_{6,8}$ and $\chi_{6,-2}$. When identifying $\overline{\mathcal{M}}_2$ with $\tilde{A}_2$, we also write $\mathcal{A}_{1,1}$ for δ_1; it is the locus of products of elliptic curves.

We analyze the orders by working locally on a family over a local base B with central fiber a general point b_1 of the boundary divisor Δ_1. As we mentioned before, the map $\varphi : C \to \mathbb{P}(\mathbb{E})$ defined over $\mathcal{M}_2$ does not extend to the whole $\overline{C}$ over $\overline{\mathcal{M}}_2$ due to the fact that the canonical system has base points at the nodes of the curves over the boundary divisor Δ_1. On the other hand, by the theory of admissible covers, the ramification divisor of the above map φ extends to a divisor S on $\overline{C}$ in a way that avoids the above nodal locus. Namely, over $b_1 \in \Delta_1$ the fiber is a nodal curve C, which is the union of two elliptic curves C_1 and C_2 meeting at a point p. The restriction of the ramification divisor on each component is the union of the three—additional to p—ramification points of the system $|\mathcal{O}(2p)|$. Therefore, the extension of the map φ is defined on the ramification divisor S. The map φ maps $C_1\backslash\{p\}$ and $C_2\backslash\{p\}$ to two distinct points p_1 and p_2, respectively, which are defined as follows. The fiber of $\mathbb{P}(\mathbb{E})$ over b_1 can be identified with $\mathbb{P}(H^0(C, \omega_C))$. The points of $H^0(C, \omega_C)$ have the form (s_1, s_2), with s_i a section of $H^0(C_i, \mathcal{O}(p))$. Then, the point p_1 corresponds to the hyperplane $\{(0, s_2), s_2 \in H^0(C_2, \mathcal{O}(p))\}$ and the point p_2 corresponds to the hyperplane $\{(s_1, 0), s_1 \in H^0(C_1, \mathcal{O}(p))\}$.

The divisor $\overline{D}$, the image of the ramification divisor under the extended map φ, splits then into six irreducible components denoted by $D_1, \ldots, D_6$. Over our local base B, we thus have the six local sections D_i $(i = 1, \ldots, 6)$ of the family $\mathbb{P}(\mathbb{E}) \to B$. By the above description of the extension of the map φ, we may conclude that D_1, D_2, D_3 pass through p_1 and D_4, D_5, D_6 through p_2.

Lifting the sections D_i locally to sections σ_i of $\mathbb{E}$ and choosing a basis e_1, e_2 of $\mathbb{E}$ over B such that e_1 and e_2 determine p_1 and p_2 in the fiber of $\mathbb{P}(\mathbb{E})$ over $z = 0$, we can write $\sigma_i = a_i e_1 + b_i e_2$ for $i = 1, \ldots, 6$. Then at $z = 0$, the functions b_1, b_2, b_3 and a_4, a_5, a_6 vanish, while $a_1, a_2, a_3, b_4, b_5, b_6$ do not vanish. Since by blowing up once we can separate, we may assume that these sections vanish with order 1 at $z = 0$. By construction the section χ of $\mathrm{Sym}^6(\mathbb{E}) \otimes \det(\mathbb{E})^{-2}$ is locally given by

$$\frac{\sigma_1 \cdots \sigma_6}{z}.$$

We may write $\sigma_1 \cdots \sigma_6$ as

$$a_1 \cdots a_6\, e_1^6 + (a_1a_2a_3a_4a_5b_6 + \cdots + b_1a_2a_3a_4a_5a_6)\, e_1^5 e_2 +$$
$$(a_1a_2a_3a_4b_5b_6 + \cdots + b_1b_2a_3a_4a_5a_6)\, e_1^4 e_2^2 +$$
$$(a_1a_2a_3b_4b_5b_6 + \cdots + b_1b_2b_3a_4a_5a_6)\, e_1^3 e_2^3 + \cdots + b_1 \cdots b_6\, e_2^6.$$

The order at $z = 0$ of the coefficient of $e_1^i e_2^{6-i}$ equals

$$\min_{\#\Lambda=i}(\#\Lambda^c \cap \{1,2,3\} + \#\Lambda \cap \{4,5,6\})$$

with Λ running though the subsets of $\{1,\dots,6\}$ of cardinality i and Λ^c denoting the complement. We find for these orders $(3,2,1,0,1,2,3)$ for $i = 0,\dots,6$, hence for the section χ given by $\sigma_1 \cdots \sigma_6/z$, we find the orders $(2,1,0,-1,0,1,2)$.

Corollary 2.4. *The section 1 of the line bundle $\mathcal{O}(\overline{D})$ on $\mathbb{P}(\mathbb{E})$ over $\overline{\mathcal{M}}_2$ pushes down via $\mathbb{P}(\mathbb{E}) \to \overline{\mathcal{M}}_2$ to the meromorphic modular form $\chi_{6,-2}$ on $\overline{\mathcal{M}}_2 = \tilde{\mathcal{A}}_2$. The orders of the seven coordinates of $\chi_{6,-2}$ along $\mathcal{A}_{1,1}$ in $\mathcal{A}_2$ are* $(2,1,0,-1,0,1,2)$.

These orders are in agreement with the result of [4] where $\chi_{6,-2}$ was constructed by invariant theory and properties of modular forms were used to determine these orders.

A different way to construct the form $\chi_{6,8}$ uses the so-called Weierstrass divisor W in the dual bundle:

$$W := \{(C,\eta) \in \mathbb{P}(\mathbb{E}^\vee) : \text{div}(\eta) \text{ contains a Weierstrass point}\}$$

over $\mathcal{M}_2$. Here, C denotes a curve of genus 2 and η a differential form on C. We let $\overline{W}$ be the closure of W over $\overline{\mathcal{M}}_2$. We then have an identity due to Gheorghita [12, Theorem 1]

$$[\overline{W}] = 6\,[\mathcal{O}_{\mathbb{P}(\mathbb{E}^\vee)}(1)] + 34\,\lambda - 3\,\delta_0 - 5\,\delta_1\,,$$

where we write λ and δ_i for the pull back of λ and δ_i to $\mathbb{P}(\mathbb{E}^\vee)$. Now $\overline{W}$ is an effective divisor and the push forward of the section 1 of $\mathcal{O}(\overline{W})$ is a section of $\text{Sym}^6(\mathbb{E}^\vee) \otimes \det(\mathbb{E})^{34} \otimes \mathcal{O}(-3\delta_0 - 5\delta_1)$. For $g = 2$, we have $\mathbb{E}^\vee \cong \mathbb{E} \otimes \det(\mathbb{E})^{-1}$, hence $\text{Sym}^6(\mathbb{E}^\vee) \cong \text{Sym}^6(\mathbb{E}) \otimes \det(\mathbb{E})^{-6}$. This implies that under the isomorphism of $\mathbb{P}^1$-bundles $\mathbb{P}(\mathbb{E}) \cong \mathbb{P}(\mathbb{E}^\vee)$ the isomorphism identifies $[\overline{W}]$ with $[\overline{D}]$, and we get in the dual bundle

$$[\overline{W}] = 6\,[\mathcal{O}(1)] + 28\,\lambda - 3\,\delta_0 - 5\,\delta_1 = 6\,[\mathcal{O}(1)] + 8\,\lambda - \delta_0 - \delta_1\,.$$

Using push forward, we find again a form of weight (6,8) vanishing on δ_1 and δ_0. Up to a multiplicative nonzero constant, this is $\chi_{6,8}$.

Remark 2.5. The identity $[\overline{W}] = 6\,[\mathcal{O}(1)] + 28\,\lambda - 3\,\delta_0 - 5\,\delta_1$ implies that there exists a modular form of weight (6,28) vanishing with multiplicity 3 on Δ_0 and multiplicity 5 on Δ_1, but this is (up to a multiplicative constant) the form $\chi_{10}^2\,\chi_{6,8}$ with χ_{10} the form of weight 10 with divisor $\delta_0 + 2\delta_1$ that displays the relation $10\,\lambda = \delta_0 + 2\delta_1$.

3 | THE CLASS OF THE k-CANONICALLY EMBEDDED CURVE

For the calculation of the classes of effective divisors in $\mathbb{P}(\mathbb{E})$ related to the canonical image of the universal curve, it is helpful to have (part of) the class of the closure of the canonical image in $\mathbb{P}(\mathbb{E})$ over $\overline{\mathcal{M}}_g$. Without extra effort we can and will extend the calculation to the case of the k-canonically embedded curve for $k \geq 1$.

We consider the universal family $\pi : \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$. This comes with a natural vector bundle $\mathbb{E}_k = \pi_*(\omega_\pi^{\otimes k})$ for $k \in \mathbb{Z}_{\geq 1}$ and for $k = 1$ this is the Hodge bundle $\mathbb{E}_1 = \mathbb{E}$. We write $u : \mathbb{P}(\mathbb{E}_k) \to \overline{\mathcal{M}}_g$ for the natural map. For $k \geq 2$, the sheaf $\omega_\pi^{\otimes k}$ is base point free for stable curves and the surjection $\pi^*\mathbb{E}_k \to \omega_\pi^{\otimes k}$ defines a morphism $\varphi_k : \overline{\mathcal{C}}_g \to \mathbb{P}(\mathbb{E}_k)$. For $k = 1$, the sheaf ω_π is base point free on $\mathcal{M}_g \cup \Delta_0^0$, with $\Delta_0^0 \subset \Delta_0$ the open locus with only disconnecting nodes, but it has base points on the nodes lying over the generic points of the boundary components Δ_i for $i > 0$. Appendix A describes the closure of the image over an open neighborhood of the generic point of Δ_i.

We denote by Γ_k the image of $\varphi_k(\mathcal{C}_g)$ over $\mathcal{M}_g \cup \Delta_0^0$ and by $\overline{\Gamma}_k$ the closure of Γ_k in $\mathbb{P}(\mathbb{E}_k)$ over $\overline{\mathcal{M}}_g$. We can write the class of Γ_k as a cycle on $\mathbb{P}(\mathbb{E}_k)$ as

$$[\overline{\Gamma}_k] = \sum_{i=0}^{r-2} h_k^i u^*(\beta_{r-2-i}), \tag{1}$$

with $r = \text{rank}(\mathbb{E}_k)$, $h_k = c_1(\mathcal{O}_{\mathbb{P}(\mathbb{E}_k)}(1))$ and β_j a codimension j cycle on $\overline{\mathcal{M}}_g$.

Proposition 3.1. *We have* $\beta_0 = 2\,k(g-1)$ *and* $\beta_1 = k^2\kappa_1 - 2k(g-1)c_1(\mathbb{E}_k) - \epsilon$ *with* $\epsilon = \sum_{i=1}^{[g/2]} \delta_i$ *for* $k = 1$ *and* $\epsilon = 0$ *else.*

Using $c_1(\mathbb{E}_k) = \frac{k(k-1)}{2}\kappa_1 + \lambda$ and $\kappa_1 = 12\lambda - \sum_{i=0}^{[g/2]} \delta_i$ (by [23]), we can write β_1 as

$$\beta_1 = \left((1-g)(12k^3 - 12k^2 + 2k) + 12\,k^2\right)\lambda + ((g-1)k - g)\,k^2 \sum_{i=0}^{[g/2]} \delta_i - \epsilon\,.$$

Proof. We start with the case $k \geq 2$. In this case, $\overline{\Gamma}_k$ is the image of $\overline{C}_g$ under φ_k and $\varphi_k^*(h_k) = k\,\omega_\pi$. Since the image of the generic fiber has degree $2k(g-1)$, we find $\beta_0 = 2k(g-1)$. The hyperplane class h_k satisfies $\sum_{i=0}^{r}(-1)^i h_k^i c_{r-i}(\mathbb{E}_k) = 0$. For dimension reasons we have $u_*(h_k^i) = 0$ for $i \leq r-2$ and $u_*(h_k^{r-1}) = 1$. Thus, we get

$$u_*(\varphi_{k*}[1]h_k^2) = u_*\varphi_{k*}(k^2\omega_\pi^2) = k^2\pi_*(\omega_\pi^2) = k^2\,\kappa_1,$$

with $\pi_*(\omega_\pi^2) = \kappa_1$, while on the other hand by (1)

$$u_*((\varphi_{k*}[1])h_k^2) = (u_* h_k^r)\beta_0 + (u_* h_k^{r-1})\beta_1 = 2k(g-1)\,c_1(\mathbb{E}_k) + \beta_1,$$

giving $\beta_1 = k^2\kappa_1 - 2k(g-1)c_1(\mathbb{E}_k)$ and this settles the case $k \geq 2$. The same argument works for $k = 1$ as long as we work on $\mathcal{M}_g \cup \Delta_0^0$. To get the coefficients of the δ_i for $i > 0$ we work over a 1-dimensional base B in an open neighborhood U_i of the generic point of Δ_i with special fiber $C' + R + C''$ as in Appendix A where the extension φ' of φ is defined by $\omega_{\pi'}(-R)$. The contribution of δ_i to $2\beta_1$ is now $\pi'_*(\omega(-R)^2)$, where the coefficient 2 of β_1 comes from the fact that φ' is of degree 2 on R. We get $\pi'_*(\omega_{\pi'}(-R)^2) = \pi'_*(\omega_{\pi'}^2) + \pi'_*(R^2) = -2\,\delta_i - 2\,\delta_i = -4\,\delta_i$, as the fiber has two singular points and $R^2 = -2$. Putting everything together results in the given formula. □

4 | THE CASE OF GENUS 3

Here, there is no restriction on the characteristic. We consider the moduli stack $\mathcal{M}_3$ of curves of genus 3 over our field k and the universal curve $\pi : C \to \mathcal{M}_3$. The canonical map defines a morphism $\varphi : C \to \mathbb{P}(\mathbb{E})$ and we thus obtain the image divisor D in $\mathbb{P}(\mathbb{E})$ over $\mathcal{M}_3$. We have a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & \mathbb{P}(\mathbb{E}) \\ & {\scriptstyle\pi}\searrow & \downarrow{\scriptstyle u} \\ & & \mathcal{M}_3 \end{array}$$

We consider the closure $\overline{D}$ of D in $\mathbb{P}(\mathbb{E})$ over $\overline{\mathcal{M}}_3$. The canonical image of the generic curve is a quartic curve. Thus, we have a relation $[\overline{D}] = [\mathcal{O}(4)] + u^*(A)$ in the rational Picard group of $\mathbb{P}(\mathbb{E})$ with A a divisor class on $\overline{\mathcal{M}}_3$ given in Proposition 3.1.

Corollary 4.1. *We have* $[\overline{D}] = [\mathcal{O}(4)] + u^*(8\,\lambda - \delta_0 - 2\,\delta_1)$.

The divisor $\overline{D}$ is effective over $\overline{\mathcal{M}}_3 - \Delta_1$. Because of the relation $[\overline{D}] = [\mathcal{O}(4)] + u^*(8\,\lambda - \delta_0 - 2\,\delta_1)$, the corresponding section 1 of $\mathcal{O}(\overline{D})$ maps under u_* to a section ψ of $\mathrm{Sym}^4(\mathbb{E}) \otimes \det(\mathbb{E})^8$ that is regular outside Δ_1 and vanishes on Δ_0. In view of the even powers Sym^4 and 8, this section ψ is invariant under the action of -1 on the fibers of $\mathbb{E}$. As the action of -1 defines the involution of the double covering of stacks $\mathcal{M}_3 \to \mathcal{A}_3$, the section ψ descends to a section $\chi_{4,0,8}$ of $\mathrm{Sym}^4(\mathbb{E}) \otimes \det(\mathbb{E})^8$ on the image of $\overline{\mathcal{M}}_3 - \Delta_1$ under the Torelli morphism $\overline{\mathcal{M}}_3 \to \tilde{\mathcal{A}}_3$, with $\tilde{\mathcal{A}}_3$ the standard second Voronoi compactification of $\mathcal{A}_3$. Since the image of Δ_1 in $\tilde{\mathcal{A}}_3$ is of codimension 2, the section $\chi_{4,0,8}$ extends to a regular section of $\mathrm{Sym}^4(\mathbb{E}) \otimes \det(\mathbb{E})^8$ on all of $\mathcal{A}_3$, and then by the Koecher Principle it extends to $\tilde{\mathcal{A}}_3$. Thus, it defines a regular Siegel modular cusp form $\chi_{4,0,8}$ of degree 3 and weight (4,0,8).

Corollary 4.2. *Let $\overline{D}$ be the closure of the canonical curve over $\mathcal{M}_3$ in $\mathbb{P}(\mathbb{E})$ and s the natural section 1 of $\mathcal{O}(\overline{D})$. Then, $\chi_{4,0,8} = u_*(s)$ is a Teichmüller modular form and it descends to a Siegel modular cusp form of degree 3 and weight (4,0,8).*

The class $\mathfrak{H}$ of the hyperelliptic locus $\overline{\mathcal{H}}_3$ in $\overline{\mathcal{M}}_3$ satisfies [14, p. 140]

$$\mathfrak{h} = 9\,\lambda - \delta_0 - 3\,\delta_1\,. \tag{2}$$

Relation (2) shows that there exists a scalar-valued Teichmüller modular form χ_9 of weight 9 on $\overline{\mathcal{M}}_3$. Its square is invariant under the action of -1 on the fibers of $\mathbb{E}$, hence descends to a Siegel modular form of weight 18. Up to a multiplicative scalar, this is Igusa's modular form χ_{18}.

If we divide $\chi_{4,0,8}$ by χ_9 we obtain a meromorphic section of $\mathrm{Sym}^4(\mathbb{E}) \otimes \det(\mathbb{E})^{-1}$ on $\overline{\mathcal{M}}_3$ that is regular on $\mathcal{M}_3$ outside the hyperelliptic locus. This form was used in [5] to construct Teichmüller modular forms and Siegel modular forms by invariant theory.

5 | MODULI OF HYPERELLIPTIC CURVES AS A STACK QUOTIENT

In this section, we discuss the stack quotient description of the moduli of hyperelliptic curves. We consider hyperelliptic curves in characteristic not 2. A hyperelliptic curve of genus g is a morphism $\alpha : C \to \mathbb{P}^1$ of degree 2 where C is a smooth curve of genus g. A morphism $a : \alpha \to \alpha'$ between two hyperelliptic curves is a commutative diagram

$$\begin{array}{ccc} C & \longrightarrow & C' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

A hyperelliptic curve C of genus g can be written as $y^2 = f(x)$ with $f \in \mathbb{C}[x]$ of degree $2g + 2$. In fact, choosing a basis (x_1, x_2) of the g^1_2 defines the morphism α. Let $W = \langle x_1, x_2\rangle$, a vector space (over our algebraically closed base field) of dimension 2, and $L = \alpha^*(O_{\mathbb{P}^1}(1))$. By Riemann–Roch, we have $\dim H^0(C, L^{g+1}) = g + 3$, while $\dim \mathrm{Sym}^{g+1}(W) = g + 2$, so we have a nonzero element $y \in H^0(C, L^{g+1})$, which is anti-invariant under the involution corresponding to α. The anti-invariant subspace of $H^0(C, L^{g+1})$ has dimension 1. Then y^2 is invariant and lies in $\mathrm{Sym}^{2g+2}(W)$. Thus, we find the equation $y^2 = f(x_1, x_2)$ with f homogeneous of degree $2g + 2$ and with nonzero discriminant.

We have made two choices here: a generator y of $H^0(C, L^{g+1})^{(-1)}$, a space of dimension 1, and a basis of W. We can change the choice of y (by a nonzero scalar) and the choice of a basis of W by $\gamma = (a, b; c, d) \in \mathrm{GL}(W)$. The action of $\mathrm{GL}(W)$ is on the right via

$$f(x_1, x_2) \mapsto f(ax_1 + bx_2, cx_1 + dx_2)\,.$$

If we let $\mathrm{GL}(W)$ act on y by a power of the determinant, then this action preserves the type of equation. In inhomogeneous form, the action by $\mathrm{GL}(W)$ is by

$$f \mapsto f\left(\frac{ax + b}{cx + d}\right), \quad y \mapsto y/(cx + d)^{g+1}\,,$$

with the following effect on the equation:

$$y^2 = f(x) \mapsto y^2 = (cx + d)^{2g+2} f\left(\frac{ax + b}{cx + d}\right).$$

The last expression on the right-hand side can be written as binary form of degree $2g + 2$.

The stabilizer of a generic $f \in \mathrm{Sym}^{2g+2}(W)$ is μ_{2g+2}, the roots of unity of order dividing $2g+2$. Since we want a stabilizer of order 2 for the generic element, we consider a twisted action: Define the $\mathrm{GL}(W)$-representation

$$W_{a,b} = \mathrm{Sym}^a(W) \otimes \det(W)^{\otimes b}\,.$$

This can be identified with $\mathrm{Sym}^a(W)$ as a vector space, but the action by $\mathrm{GL}(W)$ is different. Inside this space $W_{a,b}$ we have the open subspace $W^0_{a,b}$ of homogeneous polynomials of degree a with nonzero discriminant. We now distinguish two cases.

Case 1. g even. Here, we consider the stack quotient

$$[W^0_{2g+2,-g}/\mathrm{GL}(W)]\,.$$

This stack quotient can be identified with the moduli stack $\mathcal{H}_g$ of hyperelliptic curves of genus g for g even. Indeed, the action of $t\cdot \mathrm{Id}_W$ is (on inhomogeneous equations) by

$$f \mapsto t^{-2g} f, \quad y \mapsto y/t^{g+1}\,,$$

hence $y^2 = f$ maps to $y^2 = t^2 f$, so that the stabilizer is μ_2, as required. Note also that the action of $-1 \in \mathrm{GL}(W)$ is by $y \mapsto -y$, so y is an odd element. A basis of $H^0(C, \Omega^1_C)$ is given by

$$x^i dx/y, \quad (i = 0, \dots, g-1)\,.$$

The action on dx is by $(ad-bc)dx/(cx+d)^2$ resulting in the action on the space of differentials by

$$x^i dx/y \mapsto (ad-bc)(cx+d)^{g-1-i}(ax+b)^i\, dx/y\,.$$

If we forget the twisted action on y, we can identify $H^0(C, \Omega^1_C)$ with $W_{g-1,1}$. But y^2 must be viewed as an element of $W_{2g+2,-g}$, so the action of $t\,1_W$ on y should be twisted by $t^{-g} = \det^{-g/2}$. We get

$$H^0(C, \Omega^1_C) \cong W_{g-1,(2-g)/2} \quad \text{for g even.}$$

We see that under the identification $h : [W^0_{g+2,-g}/\mathrm{GL}(W)] \xrightarrow{\sim} \mathcal{H}_g$ the pullback $h^*(\mathbb{E})$ of the Hodge bundle $\mathbb{E}$ is the equivariant bundle $W_{g-1,(2-g)/2}$. The action of -1_W is by -1 on $W_{g-1,(2-g)/2}$. We also observe $h^*(\det(\mathbb{E})) = \det(W)^{g/2}$.

Case 2. g odd. Here, we take $W_{2g+2,-g+1}$.

Remark 5.1. If we consider $W_{2g+2,r}$, then r has to be even, since as above we later view y^2 as an element of $W_{2g+2,r}$ and we need an action by $\det^{r/2}$ on y.

Here the stabilizer of a generic element is μ_4. Now on inhomogeneous equations the action is by

$$f \mapsto t^{-2g+2} f, \quad y \mapsto y/t^{g+1}\,,$$

hence $y^2 = f$ maps to $y^2 = t^4 f$. Note that here -1_W acts by $f \mapsto f$ and $y \mapsto y$. But $\sqrt{-1}_W$ acts by $f \mapsto f$ and $y \mapsto -y$. To get the right stack quotient with stabilizer of the generic element of order 2, we take

$$[W^0_{2g+2,1-g}/(\mathrm{GL}(W)/(\pm 1_W))]\,.$$

The action on the differentials $x^i dx/y$ with $i = 0, \dots, g-1$ is by

$$x^i dx/y \mapsto (ad-bc)^{(1-g)/2}(cx+d)^{g-1-i}(ax+b)^i\, dx/y\,,$$

hence without twisting we get $H^0(C,\Omega^1_C)=W_{g,1}$. Since we view y^2 as element of $W_{g+3,1-g}$ we find under $h: [W^0_{2g+2,1-g}/(\mathrm{GL}(W)/(\pm 1_W))] \xrightarrow{\sim} \mathcal{H}_g$ that $h^*(\mathbb{E})=W_{g-1,(3-g)/2}$. The element $\sqrt{-1}\,1_W$ acts on the differentials as $(-1)^{(3-g)/2}(\sqrt{-1})^{g-1}=-1$.

We summarize.

Proposition 5.2. *Writing $W_{a,b}=\mathrm{Sym}^a(W)\otimes\det(W)^b$ we have the identification of stacks*

$$h^{-1}:\mathcal{H}_g \xrightarrow{\sim} \begin{cases} [W^0_{2g+2,-g}/\mathrm{GL}(W)] & g \text{ even} \\ [W^0_{2g+2,1-g}/(\mathrm{GL}(W)/(\pm 1_W))] & g \text{ odd}, \end{cases}$$

and

$$h^*(\mathbb{E})\cong\begin{cases} W_{g-1,(2-g)/2} & g \text{ even} \\ W_{g-1,(3-g)/2} & g \text{ odd}, \end{cases} \qquad h^*(\det(\mathbb{E}))=\begin{cases} \det(W)^{g/2} & g \text{ even} \\ \det(W)^g & g \text{ odd}. \end{cases}$$

For a somewhat different description see [2, Corollary 4.7, p. 654].

Recall that the moduli stack $\mathcal{H}_g$ has as compactification the closure $\overline{\mathcal{H}}_g$ of $\mathcal{H}_g$ inside the moduli stack $\overline{\mathcal{M}}_g$. The Picard group of $\mathcal{H}_g$ is known by [2] to be finite cyclic for $g\geq 2$ of order $4g+2$ if g is even and $8g+4$ else. The rational Picard group of $\overline{\mathcal{H}}_g$ is known by Cornalba (see [7]) to be free abelian of rank g generated by classes δ_i and ζ_j for $i=0,\dots,\lfloor g/2\rfloor$ and $j=1,\dots\lfloor (g-1)/2\rfloor$. Cornalba gives also the first Chern class λ of the Hodge bundle $\mathbb{E}$ on $\overline{\mathcal{H}}_g$

$$(8g+4)\,\lambda = g\,\delta_0 + 4\sum_{i=1}^{\lfloor g/2\rfloor} i(g-i)\,\delta_i + 2\sum_{i=1}^{\lfloor (g-1)/2\rfloor}(i+1)(g-i)\,\zeta_i\,,$$

where the generic point of the divisor ζ_i has an admissible model $C'\cup C''$ with two nodes $C'\cap C''=\{p,q\}$ mapping to a union of two $\mathbb{P}^1$, with $2i+2$ marked points on C', see Figures 1 and 2 in Section 7.

6 | MODULAR FORMS ON THE HYPERELLIPTIC LOCUS OF GENUS 3

Let $\mathbb{E}$ be the Hodge bundle on $\overline{\mathcal{H}}_3$. By a modular form of weight k on $\overline{\mathcal{H}}_3$ we mean a section of $\det(\mathbb{E})^{\otimes k}$. The construction in the preceding section shows that a modular form of weight k on $\overline{\mathcal{H}}_3$ when pulled back to the stack $[W^0_{2,0}/(\mathrm{GL}(W)/\pm\mathrm{id}_W)]$ gives rise to an invariant of degree $3k/2$. Indeed, it defines a section of the equivariant bundle $\det(W)^{3k}$ invariant under $\mathrm{SL}(W)$, but in view of the fact that we divide by the action of $\mathrm{GL}(W)/(\pm\mathrm{id}_W)$, this yields an invariant of degree $3k/2$.

Let $M_k(\Gamma_3)=H^0(\mathcal{A}_3,\det(\mathbb{E})^k)$ be the space of Siegel modular forms of degree 3 on $\Gamma_3=\mathrm{Sp}(6,\mathbb{Z})$. In [19], Igusa considered an exact sequence

$$0\to M_{k-18}(\Gamma_3)\xrightarrow{\cdot\chi_{18}} M_k(\Gamma_3)\to I_{3k/2}(2,8)$$

with $I_d(2,8)$ the vector space of invariants of degree d of binary octics. We can interpret Igusa's sequence in the following way. A Siegel modular form of weight k defines by restriction to the hyperelliptic locus a modular form of weight k on $\overline{\mathcal{H}}_3$ and it thus defines an invariant of degree $3k/2$.

For each irreducible representation ρ of $\mathrm{GL}(3)$, we have a vector bundle $\mathbb{E}_\rho$ made from $\mathbb{E}$ by a Schur functor. By a modular form of weight ρ on $\overline{\mathcal{H}}_3$, we mean a section of a vector bundle $\mathbb{E}_\rho$. We can pull back to the stack $[W^0_{8,-2}/(\mathrm{GL}(W)/\pm\mathrm{id}_W)]$, but the situation is more involved as $\mathrm{Sym}^n(\mathrm{Sym}^2(W))$ decomposes as a representation of $\mathrm{GL}(W)$. For example, we have with $W_{a,b}=\mathrm{Sym}^a(W)\otimes\det(W)^b$

$$h^*(\mathrm{Sym}^4(\mathbb{E}))=\mathrm{Sym}^4(\mathrm{Sym}^2(W))=W_{8,0}\oplus W_{4,2}\oplus W_{0,4}\,.$$

Here and in the rest of this section, we assume that the characteristic is 0, or not 2, and high enough for the representation theory (plethysm) to work.[1] In this case, we can consider the restriction of the Siegel modular form $\chi_{4,0,8}$ to the hyperelliptic locus and we know that it does not vanish identically by [5, Lemma 7.7]. On the other hand, we have the basic covariant $f_{8,-2}$, the diagonal section of $W_{8,-2}$ over the stack $[W^0_{8,-2}/(\mathrm{GL}(W)/\pm \mathrm{id}_W)]$.

The discriminant form $\mathfrak{d}$ of binary octics, an invariant of degree 14, does not define a modular form, but its third power $\mathfrak{d}^3$ does. It defines a modular form of weight 28, see [27, p. 811] and also Remark 13.1.

Via the projection $p_{8,0} : \mathrm{Sym}^4(\mathrm{Sym}^2(W)) \to W_{8,0}$, a section of $\mathrm{Sym}^4(\mathbb{E}) \otimes \det(\mathbb{E})^8$ defines a covariant of bidegree $(8, 24/2) = (8, 12)$ for the action of $\mathrm{GL}(W)$.

Proposition 6.1. *The restriction to the hyperelliptic locus of the section $\chi_{4,0,8}$ corresponds via the projection $p_{8,0}$ to a multiple of the covariant $f_{8,-2} \cdot \mathfrak{d}$ with $\mathfrak{d}$ the discriminant of binary octics.*

Proof. By restricting and projecting we obtain a covariant of bidegree (8,12). This covariant is divisible by the discriminant and does not vanish on the locus of smooth hyperelliptic curves. Therefore, division by $\mathfrak{d}$ gives a nonvanishing covariant of bidegree $(8, -2)$. Taking into account the "twisting" by $\det(W)^{-2}$, this must be a multiple of the universal binary octic. □

We will discuss the other two projections later in Lemma 14.3. Note that the divisor D, the canonical image of the universal curve in $\mathbb{P}(\mathbb{E})$ that defines $\chi_{4,0,8}$, has a restriction to the locus of smooth hyperelliptic curves, which is divisible by 2. Indeed, the canonical image of a hyperelliptic curve is a double conic. This suggests that we can take the "square root" of the restriction of $\chi_{4,0,8}$ to the hyperelliptic locus. However, the boundary divisors prevent this. If we take a level cover of the moduli space we can construct a modular form of weight (2,0,4). We will carry this out later (in Corollary 13.4), working on a Hurwitz space that we shall introduce in the next section.

7 | THE HURWITZ SPACE OF ADMISSIBLE COVERS OF DEGREE 2

This and the following sections will use the other description of the moduli of hyperelliptic curves, namely the moduli space $\overline{\mathcal{H}}_{g,2}$ of admissible covers of degree 2 and genus g in the sense of [16], see [15]. Thus, we are looking at covers $f : C \to P$ of degree 2 with C nodal of genus g and P a stable b-pointed curve of genus 0. Here, the $b = 2g + 2$ branch points are ordered and $\mathcal{H}_{g,2} \to \mathcal{H}_g$ is a Galois cover with Galois group, the symmetric group $\mathfrak{S}_{2g+2}$.

The boundary $\overline{\mathcal{H}}_{g,2} - \mathcal{H}_{g,2}$ consists of finitely many divisors that we shall denote by $\Delta_b^\Lambda = \Delta^\Lambda$, where we omit the index b if g is clear. Here the index Λ defines a partition $\{1, 2, \dots, b\} = \Lambda \sqcup \Lambda^c$, and the generic point of Δ^Λ corresponds to an admissible cover that maps to a stable curve of genus 0 that is the union of two copies of $\mathbb{P}^1$, one containing the points with mark in Λ, the other one those with mark in Λ^c. Here, we will assume that $\#\Lambda = j$ with $2 \le j \le g + 1$.

The parity of $\#\Lambda$ plays an important role here. If $\#\Lambda = 2i + 2$ is even, then the generic admissible cover corresponding to a point of Δ^Λ is a union $C_i \cup C_{g-i-1}$ that is a double cover of a union of two rational curves $\mathbb{P}_1$ and $\mathbb{P}_2$ with C_i lying over $\mathbb{P}_1$ and C_{g-i-1} over $\mathbb{P}_2$. Here C_i (resp. C_{g-i-1}) has genus i (resp. $g - i - 1$) with $0 \le i \le (g-1)/2$ and is ramified over the points of Λ (resp. Λ^c).

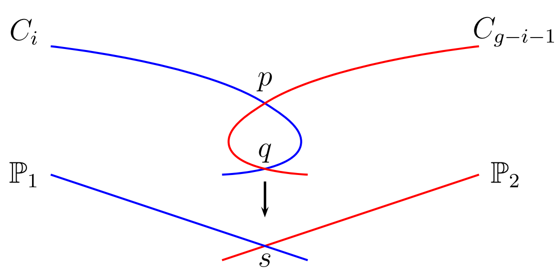

FIGURE 1 Λ even.

If $\#\Lambda = 2i + 1$ is odd with $1 \le i \le g/2$, then we have a union $C_i \cup C_{g-i}$ lying over $\mathbb{P}_1 \cup \mathbb{P}_2$, where C_i (resp. C_{g-i}) of genus i (resp. $g - i$) is ramified over Λ and in p, the intersection of C_i and C_{g-i} (resp. over Λ^c and in p). Note that p is a simple node.

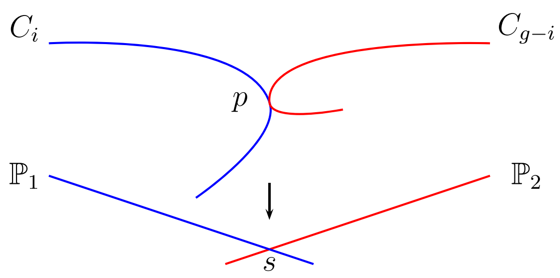

FIGURE 2 Λ odd.

Assuming that g and $b = 2g + 2$ are fixed, we will write

$$\Delta_j = \sum_{\#\Lambda=j} \Delta^\Lambda \qquad \text{with } 2 \le j < b/2$$

and provide the symmetric case with a factor 1/2, that is, $\Delta_{b/2} = \frac{1}{2} \sum_{\#\Lambda=b/2} \Delta^\Lambda$.

8 | DIVISORS ON THE MODULI OF STABLE CURVES OF GENUS 0

For later use, we recall some notation and facts concerning divisors on the moduli spaces $\overline{\mathcal{M}}_{0,n}$. We refer to [20]. The boundary divisors on $\overline{\mathcal{M}}_{0,n}$ are denoted by S_n^Λ and are indexed by partitions $\{1, \dots, n\} = \Lambda \sqcup \Lambda^c$ into two disjoint sets with $2 \le \#\Lambda \le n-2$ and we have $S_n^\Lambda = S_n^{\Lambda^c}$. Via the natural map $\pi_{n+1} : \overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n}$, we may view $\overline{\mathcal{M}}_{0,n+1}$ as the universal curve and π_{n+1} has n sections. The generic point of S_n^Λ corresponds to a stable curve with two rational components, one of which contains the points marked by Λ. For pull back by π_{n+1}, we have the relation

$$\pi_{n+1}^*(S_n^\Lambda) = S_{n+1}^{\{\Lambda,n+1\}} \cup S_{n+1}^{\{\Lambda^c,n+1\}}.$$

The n sections of π_{n+1} have images $S_{n+1}^{\{i,n+1\}}$ with $i = 1, \dots, n$.

We can collect these boundary divisors on $\overline{\mathcal{M}}_{0,n+1}$ via

$$T_{n+1,j} = \sum_{\#\Lambda=j} S_{n+1}^{\{\Lambda,n+1\}}, \quad T_{n+1,j}^c = \sum_{\#\Lambda=j} S_{n+1}^{\{\Lambda^c,n+1\}},$$

with the convention that in view of the symmetry we add a factor 1/2 for even n and $j = n/2$

$$T_{n+1,n/2} = \frac{1}{2} \sum_{\#\Lambda=n/2} S_{n+1}^{\{\Lambda,n+1\}}, \quad T_{n+1,n/2}^c = \frac{1}{2} \sum_{\#\Lambda=n/2} S_{n+1}^{\{\Lambda^c,n+1\}}.$$

Later, when a fixed index k is given, we will split these divisors as $T = T(k^+) + T(k^-)$ where (k^+) (resp. (k^-)) indicates that the sum is taken over Λ containing k (resp. not containing k). So $T_{n+1,j}(k^+) = \sum_{\#\Lambda=j, k\in\Lambda} S_{n+1}^{\{\Lambda,n+1\}}$ (and with a factor 1/2 if $j = n/2$).

9 | A GOOD MODEL

We now will work with a "good model" of the universal admissible cover over $\overline{\mathcal{H}}_{g,2}$. Such a model was constructed in [11, section 4]. We start with the observation that the space $\overline{\mathcal{H}}_{g,2}$ is not normal, and we therefore normalize it. The result $\widetilde{\mathcal{H}}_{g,2}$ is now a smooth stack over which we have a universal curve $\tilde{C} \to \widetilde{\mathcal{H}}_{g,2}$.

We have a natural map $h : \widetilde{\mathcal{H}}_{g,2} \to \overline{\mathcal{M}}_{0,b}$ with $b = 2g + 2$ and the universal curve now fits into a commutative diagram

$$\begin{array}{ccccc} & & \widetilde{C} & \xrightarrow{c} & \overline{\mathcal{M}}_{0,b+1} \\ & & \downarrow{\scriptstyle \varpi} & & \downarrow{\scriptstyle \pi_{b+1}} \\ \overline{\mathcal{H}}_{g,2} & \xleftarrow{\nu} & \widetilde{\mathcal{H}}_{g,2} & \xrightarrow{h} & \overline{\mathcal{M}}_{0,b} \end{array}$$

We can construct a proper flat map that extends the relative canonical morphism $C \to \mathbb{P}^1_{\mathcal{H}_{g,2}}$ by taking the fiber product $\mathbb{P}$ of $\overline{\mathcal{M}}_{0,b+1}$ and $\widetilde{\mathcal{H}}_{g,2}$ over $\overline{\mathcal{M}}_{0,b}$ and thus obtain a commutative diagram

$$\begin{array}{ccccc} \widetilde{C} & \xrightarrow{\alpha} & \mathbb{P} & \xrightarrow{c'} & \overline{\mathcal{M}}_{0,b+1} \\ & {\scriptstyle \varpi}\searrow & \downarrow{\scriptstyle \varpi'} & & \downarrow{\scriptstyle \pi_{b+1}} \\ & & \widetilde{\mathcal{H}}_{g,2} & \xrightarrow{h} & \overline{\mathcal{M}}_{0,b} \end{array}$$

The resulting space $\mathbb{P}$ is not smooth, but has rational singularities. Resolving these in a minimal way gives a model $\widetilde{\mathbb{P}}$; taking the resolution $\widetilde{Y}$ of the normalization Y of the fiber product of $\widetilde{\mathbb{P}}$ and $\widetilde{C}$ over $\mathbb{P}$ gives us finally a commutative diagram

$$\begin{array}{cccc} \widetilde{Y} & \xrightarrow{f} & \widetilde{\mathbb{P}} & \\ & {\scriptstyle t}\searrow & \downarrow{\scriptstyle \pi} & \\ & & B & \xrightarrow{\beta} \widetilde{\mathcal{H}}_{g,2} \end{array} \qquad (0.1) \qquad (3)$$

where B is our base $\widetilde{\mathcal{H}}_{g,2}$ or any other base mapping to it. We write π for the resulting morphism $\widetilde{\mathbb{P}} \to B$, h for the natural map $B \to \overline{\mathcal{M}}_{0,b}$, and ν for $B \to \overline{\mathcal{H}}_{g,2}$. We refer to [11, section 4] for additional details.

In the following, we will assume that we have a physical family over a base B. We will abuse the notation Δ^Λ for the pull back of the divisor Δ^Λ under $\nu : B \to \overline{\mathcal{H}}_{g,2}$.

In the case that $\#\Lambda$ is even, say $\#\Lambda = 2i + 2$ with $0 \le i \le (g-1)/2$, the pull back of Δ^Λ decomposes as

$$\pi^*(\Delta^\Lambda) = \Pi^\Lambda + \Pi^{\Lambda^c},$$

corresponding to the two components of a general fiber of π, with Π^Λ mapping to $S_{b+1}^{\{\Lambda,b+1\}}$ under $\widetilde{\mathbb{P}} \to \overline{\mathcal{M}}_{0,b+1}$, and similarly Π^{Λ^c} mapping to $S_{b+1}^{\{\Lambda^c,b+1\}}$. Note that we restrict $\#\Lambda$ by $\le g+1$, hence the notation Π^Λ should not lead to confusion.

In the case $\#\Lambda$ is odd, we find a similar decomposition

$$\pi^*(\Delta^\Lambda) = \Pi^\Lambda + R^\Lambda + \Pi^{\Lambda^c},$$

corresponding now to the fact that the general fiber of π has three components, one coming from the blowing up.

We notice

$$h^*(S_b^\Lambda) = \begin{cases} \Delta^\Lambda & \#\Lambda \equiv 0 \,(\mathrm{mod}\ 2) \\ 2\,\Delta^\Lambda & \#\Lambda \equiv 1 \,(\mathrm{mod}\ 2). \end{cases}$$

If we use the notation $\Delta_j = \sum_{\#\Lambda=j} \Delta^\Lambda$, we find for the tautological classes $\lambda = c_1(\mathbb{E})$ and $h^*(\psi_k)$, simply denoted by ψ_k and defined as the first Chern class of the line bundle given by the cotangent space at the kth point of our pointed curve

$(k = 1, \ldots, b)$, and the following formulas on our base B (see [10])

$$\lambda = \sum_{i=0}^{(g-1)/2} \frac{(i+1)(g-i)}{2(2g+1)} \Delta_{2i+2} + \sum_{i=1}^{g/2} \frac{i(g-i)}{2g+1} \Delta_{2i+1} \tag{4}$$

and

$$\begin{aligned} \psi_k = & \sum_{i=0}^{(g-1)/2} \left(\frac{(g-i)(2g-2i-1)}{g(2g+1)} \Delta_{2i+2}(k^+) + \frac{(i+1)(2i+1)}{g(2g+1)} \Delta_{2i+2}(k^-) \right) \\ & + 2 \sum_{i=1}^{g/2} \left(\frac{(g-i)(2g-2i+1)}{g(2g+1)} \Delta_{2i+1}(k^+) + \frac{i(2i+1)}{g(2g+1)} \Delta_{2i+1}(k^-) \right), \end{aligned} \tag{5}$$

where we use the notation (k^+) (resp. (k^-)) to denote the condition $k \in \Lambda$ (resp. $k \notin \Lambda$) as above. The relation (4) implies the following.

Corollary 9.1. *There exists a scalar-valued modular form of weight $2(2g+1)$ on the moduli space $\widetilde{\mathcal{H}}_{g,2}$ whose divisor is a union of boundary divisors. It descends to the hyperelliptic locus $\overline{\mathcal{H}}_g$ and corresponds to a power of the discriminant of the binary form of degree $2g+2$.*

10 | EXTENDING THE LINEAR SYSTEM

The canonical system on a hyperelliptic curve is defined by the pull back of the sections of the line bundle of $\mathcal{O}(g-1)$ of degree $g-1$ on the projective line. We now try to extend this line bundle over our compactification.

A first attempt would be to consider the divisor $(g-1)\tilde{S}_k$ with $\tilde{S}_k$ the pull back to $\widetilde{\mathbb{P}}$ of the section S_k of $\pi_{b+1} : \overline{\mathcal{M}}_{0,b+1} \to \overline{\mathcal{M}}_{0,b}$. Recall the morphism $t = \pi f : \widetilde{Y} \to B$. We can add a boundary divisor Ξ_k to it such that $f^*\mathcal{O}_{\widetilde{\mathbb{P}}}(D_k)$ with $D_k = (g-1)\tilde{S}_k + \Xi_k$ coincides with ω_t on the fibers of t, namely in view of the intersection numbers take Ξ_k equal to

$$\sum_{i=0}^{(g-1)/2} \left((g-1-i)\, \Pi_{2i+2}(k^+) + i\, \Pi^c_{2i+2}(k^-)\right) +$$

$$\sum_{i=1}^{g/2} \left((g-i-1)\, \Pi_{2i+1}(k^+) - (g-i)\, \Pi^c_{2i+1}(k^+)\right) + \sum_{i=1}^{g/2} \left((i-1)\, \Pi^c_{2i+1}(k^-) - i\, \Pi_{2i+1}(k^-)\right).$$

Here, $\Pi_j = \sum_{\#\Lambda=j} \Pi^\Lambda$ and $\Pi^c_j = \sum_{\#\Lambda=j} \Pi^{\Lambda^c}$ and (k^+) (resp. (k^-)) indicates the condition that $k \in \Lambda$ (resp. $k \notin \Lambda$); moreover, we add a factor $1/2$ in case $j = b/2$.

Now $f^*\mathcal{O}(D_k)$ and ω_t agree on the fibers of t, so they differ by a pull back under $t = \pi \circ f$, see diagram (2).

To see the above, when, for example, $\#\Lambda = 2i+2$ is even, in that case the fiber of $\tilde{Y}$ over t is as in Figure 1 and $\omega_C = (\omega_{C_i} + p + q, \omega_{C_{g-i-1}} + p + q) = (i(p+q), (g-i-1)(p+q)) = f^*(i, g-i-1)$, where we indicate by i the line bundle of degree i on $\mathbb{P}^1$. One then checks that with the above choice of Ξ_k the restriction of D_k on the corresponding fiber of π is of type $(i, g-1-i)$. Indeed, in case $k \in \Lambda$, then $(g-1)\tilde{S}_k + (g-1-i)\Pi_{2i+2}(k^+)$ restricts to $((g-1)-(g-i-1), g-i-1) = (i, g-i-1)$ and in case $k \notin \Lambda$, then $(g-1)\tilde{S}_k + i\Pi^c_{2i+2}(k^-)$ restricts to $(i, (g-1)-i) = (i, g-i-1)$. The case where $\#\Lambda = 2i+1$ is odd, although a little more complicated, is treated similarly.

We therefore will change D_k by a pull back under π. Define a divisor class on B by

$$\begin{aligned} E_k = \frac{2g-1}{2}\psi_k & - \sum_{i=0}^{(g-1)/2} \left((g-i-1)\, \Delta_{2i+2}(k^+) + i\, \Delta_{2i+2}(k^-)\right) \\ & - \sum_{i=1}^{g/2} \left((g-i-1)\, \Delta_{2i+1}(k^+) + (i-1)\, \Delta_{2i+1}(k^-)\right) \end{aligned}$$

and define a line bundle on $\tilde{\mathbb{P}}$ by

$$M = \mathcal{O}(D_k + \pi^* E_k)\,. \tag{6}$$

Lemma 10.1. *The line bundle M does not depend on k, satisfies $f^*(M) = \omega_t$, and restricts to the general fiber $\mathbb{P}^1$ of π as $\mathcal{O}(g-1)$. For $\#\Lambda = 2i+2$, its restriction to the general fiber $\mathbb{P}_1 \cup \mathbb{P}_2$ over Δ^Λ is of degree $(i, g-i-1)$, while for $\#\Lambda = 2i+1$, its restriction to the general fiber $\mathbb{P}_1 \cup R \cup \mathbb{P}_2$ is of degrees $(i, -1, g-i)$.*

Proof. We use the section τ_k of $t : \widetilde{Y} \to B$ with $f\tau_k = \tilde{s}_k$, with $\tilde{s}_k$ the natural sections of the map π with image $\tilde{S}_k$. Then, we have $\tau_k^*\omega_t = \psi_k/2$ and $\tau_k^* f^* D_k = \tilde{s}_k^* D_k$ for which we have

$$\tilde{s}_k^* D_k = -(g-1)\psi_k + \sum_{i=0}^{(g-1)/2} \big((g-i-1)\,\Delta_{2i+2}(k^+) + i\,\Delta_{2i+2}(k^-)\big)$$
$$+ \sum_{i=1}^{g/2} \big((g-1-i)\,\Delta_{2i+1}(k^+) + (i-1)\,\Delta_{2i+1}(k^-)\big)\,.$$

From this we obtain $\tau_k^*(\omega_t) - \tau_k^* f^* D_k = \pi^*(E_k)$, so that $\omega_t = f^*(\mathcal{O}(D_k + E_k))$. We also see that the restriction of M on the fibers of π does not depend on k. Moreover, we have

$$\tilde{s}_j^*(D_k + \pi^* E_k) = \tau_j^* f^*(D_k + E_k) = \tau_j^*(\omega_t) = \psi_j/2 = \tilde{s}_j^*(D_j + \pi^* E_j)\,,$$

showing that the restrictions of $\mathcal{O}(D_k + \pi^* E_k)$ and $\mathcal{O}(D_j + \pi^* E_j)$ agree on $\tilde{S}_j$. The restrictions of the fibers of π over the general points of Δ_b^Λ are easily checked. □

We now want to compare $\pi_*(M)$ with the Hodge bundle $\mathbb{E} = t_*(\omega_t)$ on B. The next proposition shows that these agree up to codimension 2.

Proposition 10.2. *We have an exact sequence $0 \to \pi_*(M) \to \mathbb{E} \to \mathcal{T} \to 0$, where $\mathcal{T}$ is a coherent sheaf that is a torsion sheaf supported on the boundary. Moreover, we have $c_1(\pi_*(M)) = \lambda$.*

Proof. By Lemma 10.1, we have $\omega_t = f^*(M)$. But $R^1\pi_*(M) = (0)$, so we have

$$\pi_*(M \otimes f_*\mathcal{O}_{\tilde{Y}}) = \pi_* f_*(f^*(M)) = \pi_* f_*(\omega_t) = t_*(\omega_t)\,.$$

We have an exact sequence $0 \to \mathcal{O}_{\tilde{\mathbb{P}}} \to f_*\mathcal{O}_{\tilde{Y}} \to \mathcal{F} \to 0$ with $\mathcal{F}$ a coherent sheaf of rank 1 that restricted to the smooth fibers of π has degree $-(g+1)$, as one sees by applying Riemann–Roch to f and $\mathcal{O}_{\tilde{Y}}$. Tensoring the sequence with M and applying π_* gives the exact sequence

$$0 \to \pi_*(M) \to \pi_*(M \otimes f_*\mathcal{O}_{\tilde{Y}}) \to \pi_*(M \otimes \mathcal{F}) \to 0\,.$$

On the smooth fibers of π the sheaf $M \otimes \mathcal{F}$ restricts to a line bundle of degree $(g-1)-(g+1) = -2$, hence $\pi_*(M \otimes \mathcal{F})$ is a torsion sheaf.

We now calculate $c_1(\pi_*(M))$. We apply Grothendieck–Riemann–Roch to π and $\mathcal{O}(D_k)$. It says

$$\mathrm{ch}(\pi_!(\mathcal{O}(D_k))) = \pi_*(\mathrm{ch}(\mathcal{O}(D_k)))\mathrm{Td}^\vee(\omega_\pi)\,,$$

which by $\pi_*(\mathrm{Td}_2^\vee(\omega_\pi)) = 0$ gives

$$c_1(\pi_*(\mathcal{O}(D_k))) = \frac{1}{2}\pi_*(-D_k\,\omega_\pi + D_k^2)\,.$$

We calculate

$$\pi_*(D_k\omega_\pi) = (g-1)\psi_k - \sum_{i=0}^{(g-1)/2} \big((g-i-1)\,\Delta_{2i+2}(k^+) + i\,\Delta_{2i+2}(k^-)\big) + \sum_{i=1}^{g/2} \Delta_{2i+1}\,, \tag{7}$$

and

$$\begin{aligned}\pi_*(D_k^2) = &-(g-1)^2\psi_k + \sum_{i=0}^{(g-1)/2} (g-i-1)(g+i-1)\,\Delta_{2i+2}(k^+)\\ &+ \sum_{i=0}^{(g-1)/2} (2g-2-i)i\,\Delta_{2i+2}(k^-) + \sum_{i=1}^{g/2}((2g-i-1)(i-1)-i^2)\,\Delta_{2i+1}\,.\end{aligned} \tag{8}$$

Adding $\pi_*(\pi^*E_k)$ gives

$$\begin{aligned}c_1(\pi_*(M)) = \frac{g^2}{2}\,\psi_k - \frac{1}{2}\sum_{i=0}^{(g-1)/2} \big((g-i-1)(g-i)\,\Delta_{2i+2}(k^+) + i(i+1)\,\Delta_{2i+2}(k^-)\big)\\ - \sum_{i=1}^{g/2} \big((g-i)^2\,\Delta_{2i+1}(k^+) + i^2\,\Delta_{2i+1}(k^-)\big)\,.\end{aligned}$$

Substituting the formula for ψ_k, we find

$$c_1(\pi_*(M)) = \sum_{i=0}^{(g-1)/2} \frac{(g-i)(i+1)}{2(2g+1)}\Delta_{2i+2} + \sum_{i=1}^{g/2} \frac{i(g-i)}{2g+1}\Delta_{2i+1} = \lambda\,.$$

□

The line bundle M on $\tilde{\mathbb{P}}$ is not base point free as Proposition 10.2 shows; the restriction to the R-part has negative degree. We can make it base point free by defining

$$N = M(-R) = \mathcal{O}(D_k + \pi^*E_k - R)\,. \tag{9}$$

Now the restriction of N to a general fiber over Δ_{2i+1}, which is a chain of three rational curves $\mathbb{P}_1, R, \mathbb{P}_2$, has degrees $(i-1, 1, g-i-1)$ and one checks that N is base point free.

Lemma 10.3. *Up to codimension 2, we have on B that $\pi_*(N) = \mathbb{E}$.*

Proof. We have $R^1\pi_*(N) = 0$. Therefore, the exact sequence $0 \to N \to M \to M_{|R} \to 0$ yields the exact sequence

$$0 \to \pi_*(N) \to \pi_*(M) \to \pi_*(M_{|R}) \to 0\,.$$

We now show that $c_1(\pi_*(N)) = c_1(\pi_*(M))$. Since $R^1\pi_*(M) = 0 = R^1\pi_*(N)$, we find by Grothendieck–Riemann–Roch that

$$c_1(\pi_*(M)) = \frac{1}{2}\pi_*(c_1(M)^2 - c_1(M)\omega_\pi), \quad c_1(\pi_*(N)) = \frac{1}{2}\pi_*(c_1(N)^2 - c_1(N)\omega_\pi)\,.$$

By the definition of N and the fact that R is a (-2)-curve if we take a base B of dimension 1, and thus has intersection number 0 with a fiber, we have

$$\pi_*(c_1(N))^2 = \pi_*(c_1(M))^2$$

and $c_1(N)\,\omega_\pi = c_1(M)\,\omega_\pi$ since the restriction of ω_π to R is trivial. □

11 | THE RATIONAL NORMAL CURVE

The image of a hyperelliptic curve by the canonical map is a rational normal curve, that is, $\mathbb{P}^1$ embedded in $\mathbb{P}^{g-1}$ via the linear system of degree $g-1$.

In our setting, we can see the rational normal curve and its degenerations using the extension N of the line bundle of degree $g-1$, as defined in (9), to the compactification as constructed in the preceding section.

We let $u : \mathbb{P}(\mathbb{E}) \to B$ be the natural projection. Now N is base point free and up to codimension 2 we have $\pi_*(N) = \mathbb{E}$, so the global-to-local map $\pi^*\pi_*(N) \to N$ induces a surjective map $\nu : \pi^*(\mathbb{E}) \to N$ over $\tilde{\mathbb{P}}$. This induces a morphism $\phi : \tilde{\mathbb{P}} \to \mathbb{P}(\mathbb{E})$ by associating to a point of $\tilde{\mathbb{P}}$ the kernel of ν. It fits into a diagram

$$\begin{array}{ccc} \tilde{\mathbb{P}} & \xrightarrow{\phi} & \mathbb{P}(\mathbb{E}) \\ & {\scriptstyle \pi}\searrow & \downarrow{\scriptstyle u} \\ & & B \end{array}$$

Proposition 11.1. *For a point of B with smooth fiber under π, the image of ϕ is a rational normal curve of degree $g-1$. For a general point $\beta \in \Delta_{2i+2}$ with fiber $\mathbb{P}_1 \cup \mathbb{P}_2$, the image is a union of two rational normal curves of degree i and $g-i-1$. For a general point $\beta \in \Delta_{2i+1}$ with fiber $\mathbb{P}_1, R, \mathbb{P}_2$ the image is a union of three rational normal curves of degree $i-1$, 1, and $g-i-1$. Here, we interpret the case of degree 0 as a contracted curve.*

Proof. The proposition follows almost immediately from Lemma 10.1. □

Remark 11.2. If $i=1$, then $\mathbb{P}_1$ is contracted. If also $g=2$, then both $\mathbb{P}_1$ and $\mathbb{P}_2$ are contracted and the image of R coincides with the fiber of $\mathbb{P}(\mathbb{E})$.

Remark 11.3. The sections $\tilde{s}_i : B \to \tilde{\mathbb{P}}$ for $i = 1, \dots, b$ induce sections $\sigma_i = \phi \circ s_i : B \to \mathbb{P}(\mathbb{E})$ by sending β to the kernel of $\mathbb{E} = \tilde{s}_i^* \pi^*(\mathbb{E}) \to s_i^*(N)$.

Remark 11.4. In the case $g=2$, the map ϕ is a birational map $\tilde{\mathbb{P}} \to \mathbb{P}(\mathbb{E})$ that blows down boundary components. More precisely, over Δ_2 it blows down Π_2 and over Δ_3 the components supported at $\Pi_3 = \Pi_3^c$.

12 | SYMMETRIZATION

We have been working with the moduli space $\mathcal{H}_{g,2}$ and $\mathcal{M}_{0,b}$ and their compactifications. Here the symmetric group $\mathfrak{S}_b$ acts. We therefore make our construction symmetric.

We put $D = \sum_{k=1}^{b} D_k$ and $E = \sum_{k=1}^{b} E_k$ and set

$$\tilde{M} = \mathcal{O}(D+E), \quad \psi = \sum_{k=1}^{b} \psi_k, \quad \text{and} \quad \tilde{S} = \sum_{k=1}^{b} \tilde{S}_k .$$

We find

$$\psi = 4 \sum_{i=0}^{(g-1)/2} \frac{(g-i)(i+1)}{2g+1} \Delta_{2i+2} + 2 \sum_{i=1}^{g/2} \frac{(2g-2i+1)(2i+1)}{2g+1} \Delta_{2i+1}$$

and

$$D = (g-1)\tilde{S} + 2 \sum_{i=0}^{(g-1)/2} \big((g-i-1)(i+1)\Pi_{2i+2} + i\,(g-i)\,\Pi_{2i+2}^c\big) +$$

$$\sum_{i=1}^{g/2} \big((g-4i-1)\Pi_{2i+1} - (3g-4i+1)\,\Pi_{2i+1}^c\big)$$

and

$$E = \frac{2g-1}{2}\psi - 2\sum_{i=0}^{(g-1)/2}((g-i)(2i+1)-(i+1))\Delta_{2i+2} - \sum_{i=1}^{g/2}(4i(g-i)-(g+2))\Delta_{2i+1}$$

13 | THE CASE OF HYPERELLIPTIC GENUS 3

The Hurwitz space $\mathcal{H}_{3,2}$ admits a compactification $\overline{\mathcal{H}}_{3,2}$ with boundary components Δ^{Λ} with $\#\Lambda \in \{2,3,4\}$. Taking the components with Λ of fixed cardinality together gives boundary components Δ_2, Δ_3, and Δ_4. Under the morphism $\overline{\mathcal{H}}_{3,2} \to \overline{\mathcal{M}}_3$, the components Δ_2 and Δ_4 are mapped to δ_0, while Δ_3 goes to δ_1. The formulas (4) and (5) specialize to

$$\lambda = \frac{3}{14}\Delta_2 + \frac{2}{7}\Delta_3 + \frac{2}{7}\Delta_4 \tag{10}$$

and

$$\psi_k = \frac{5}{7}\Delta_2(k^+) + \frac{1}{21}\Delta_2(k^-) + \frac{20}{21}\Delta_3(k^+) + \frac{2}{7}\Delta_3(k^-) + \frac{2}{7}\Delta_4 .$$

Remark 13.1. Equation (10) shows that on $\overline{\mathcal{H}}_{3,2}$ there exists a scalar-valued modular form of weight 14 whose square equals χ_{28}, a form mentioned in Section 6. Since on $\overline{\mathcal{H}}_3$ we have $28\,\lambda = 3\,\delta_0 + 8\,\delta_1 + 8\,\zeta_1$, an integral class not divisible by 2, there is not a modular form of weight 14 on $\overline{\mathcal{H}}_3$ with square χ_{28}. Compare with Cornalba's formula at the end of Section 5.

We have the line bundle M on $\tilde{\mathbb{P}}$ defined in (6) corresponding to the divisor class $D_k + E_k$ given by

$$D_k = 2\,\tilde{S}_k + 2\,\Pi_2(k^+) + 2\,\Pi_4(k^+) + \Pi_3(k^+) - 2\,\Pi_3^c(k^+) - \Pi_3(k^-)$$

and

$$E_k = \frac{5}{2}\psi_k - (2\,\Delta_2(k^+) + \Delta_3(k^+) + \Delta_4)\,,$$

where ψ_k is given in (6). Define the rational divisor class

$$U := \frac{1}{14}\Delta_2 + \frac{3}{7}(\Delta_3 + \Delta_4) = \frac{3}{2}\psi_k - (\Delta_2(k^+) + \Delta_3(k^+))\,.$$

The divisor class of $D_k + E_k$ is independent of k as observed in Lemma 10.1, but this can be seen also directly from the next lemma.

Lemma 13.2. *We have the linear equivalence* $D_k + E_k \sim -\omega_\pi + \Pi_2 + \Pi_3 + \pi^*(U)$.

Proof. One checks that $-\omega_\pi + \Pi_2 + \Pi_3$ and $D_k + E_k$ have the same restriction to fibers of π. We have $s_k^*(-\omega_\pi + \Pi_2 + \Pi_3) = -\psi_k + \Delta_2(k^+) + \Delta_3(k^+)$ and $s_k^*(D_k + E_k) = \psi_k/2$. □

Let Q be the image of $\phi : \tilde{\mathbb{P}} \to \mathbb{P}(\mathbb{E})$, see Proposition 11.1. The map ϕ is the composition of a map $\phi' : \tilde{\mathbb{P}} \to Q$ with the inclusion map $\iota : Q \hookrightarrow \mathbb{P}(\mathbb{E})$. The generic fiber of $Q \to B$ is a conic, hence $\mathcal{O}(Q) = \mathcal{O}(2) \otimes \mathcal{O}(u^*A)$ for some divisor A on B. We determine A.

Lemma 13.3. *On* $\mathbb{P}(\mathbb{E})$ *we have the linear equivalence*

$$[Q] \sim [\mathcal{O}(2)] + u^*(4\,\lambda - (\Delta_2 + \Delta_3 + \Delta_4))\,.$$

Proof. We have $\omega_{\mathbb{P}(\mathbb{E})} \otimes u^*(\omega_B^{-1}) = \mathcal{O}(-3) \otimes u^*(\det \mathbb{E})$ and by adjunction $\omega_Q = \iota^*(\mathcal{O}(Q) \otimes \omega_{\mathbb{P}(\mathbb{E})})$. Since ϕ' is a blow down, we have $\omega_{\tilde{\mathbb{P}}} = (\phi')^* \omega_Q \otimes \mathcal{O}(\Pi_2 + \Pi_3)$. We get

$$\begin{aligned}\phi^*(\mathcal{O}(Q)) =& {\phi'}^* \omega_Q \otimes \phi^* \omega_{\mathbb{P}(\mathbb{E})}^{-1} \\ =& \omega_{\tilde{\mathbb{P}}} \otimes \mathcal{O}(-\Pi_2 - \Pi_3) \otimes \phi^* \mathcal{O}(3) \otimes \pi^* \det(\mathbb{E})^{-1} \otimes \pi^* \omega_B^{-1} \\ =& \omega_\pi \otimes \mathcal{O}(-\Pi_2 - \Pi_3) \otimes \phi^* \mathcal{O}(3) \otimes \pi^* \det(\mathbb{E})^{-1}\,.\end{aligned}$$

On the other hand, we have $\phi^*\mathcal{O}(Q) = \phi^*\mathcal{O}(2) \otimes \mathcal{O}(\pi^* A)$ and $\phi^*\mathcal{O}(1) = N$, hence we get

$$\mathcal{O}(u^* A) = N \otimes \omega_\pi \otimes \mathcal{O}(-\Pi_2 - \Pi_3) \otimes \pi^* \det(\mathbb{E})^{-1}\,. \tag{11}$$

By Lemma 13.2 we have $N = \omega_\pi^{-1} \otimes \mathcal{O}(\Pi_2 + \Pi_3) \otimes O(U)$. Substituting this in (11) we get the desired result. □

The effective divisor Q yields a modular form and Lemma 13.3 gives its weight.

Corollary 13.4. *The effective divisor Q on* $\mathbb{P}(\mathbb{E})$ *defines a modular form* $\chi_{2,0,4}$ *on* $\tilde{\mathcal{H}}_{3,2}$ *of weight (2,0,4), that is, a nonzero section of* $\mathrm{Sym}^2(\mathbb{E}) \otimes \det(\mathbb{E})^4$.

Since the divisor $\Delta_2 + \Delta_3 + \Delta_4$ is not a pull back from the moduli space $\overline{\mathcal{H}}_3$, the modular form does not descend to $\overline{\mathcal{H}}_3$. Recall that the modular form $\chi_{4,0,8}$ restricted to the hyperelliptic locus was associated to a divisor D that equals $2\,Q$.

Remark 13.5. In the same vein as above, we can determine in an alternative way the result of Proposition 2.1 on class of the closure $\overline{D}$ of the ramification divisor D of the universal genus 2 curve. By the theory of admissible covers, there is a natural map $\overline{\mathcal{H}}_{2,2} \to \overline{\mathcal{M}}_2$ with the property that the pull back of the Hodge bundle on $\overline{\mathcal{M}}_2$ is the Hodge bundle on $\overline{\mathcal{H}}_{2,2}$ associated to the corresponding family of admissible covers. Hence the pull back of the $\mathcal{O}(1)$ of the bundle $\mathbb{P}(\mathbb{E}) \to \overline{\mathcal{M}}_2$ equals the $\mathcal{O}(1)$ of the bundle $\mathbb{P}(\mathbb{E}) \to \overline{\mathcal{H}}_{2,2}$. Let $\Sigma = \phi_*(\sum_{k=1}^6 \tilde{S}_k)$, with $\phi : \mathbb{P} \to \mathbb{P}(\mathbb{E})$ the map defined in Section 11. By geometry, the pull back of $\overline{D}$ to the bundle $\mathbb{P}(\mathbb{E})$ over $\overline{\mathcal{H}}_{2,2}$ equals Σ. By Remark 11.4, we have $\phi^*\Sigma = \tilde{S} + 2\,\Pi_2 + 6\,\Pi_3$. By using the formulas of Section 12, we have for $g = 2$:

$$\phi^*\mathcal{O}(6) = \tilde{M} - 6\,R = \tilde{S} + \frac{12}{5}\Pi_2 + \frac{2}{5}\Pi_2^c + \frac{24}{5}\Pi_3 - \frac{3}{5}R\,.$$

We now write $[\overline{D}] = \mathcal{O}(6) + u^*(a\,\delta_0 + b\,\delta_1)$. By pulling back to $\tilde{\mathbb{P}}$ and using the above formulas, we get (we refer to the diagram in Section 11 for notation)

$$\tilde{S} + \frac{12}{5}\Pi_2 + \frac{2}{5}\Pi_2^c + \frac{24}{5}\Pi_3 - \frac{3}{5}R + \pi^*(2a\,\Delta_0 + b\,\Delta_3) = \tilde{S} + 2\,\Pi_2 + 6\,\Pi_3\,.$$

This implies

$$\pi^*(2a\,\Delta_0 + b\,\Delta_3) = -\frac{2}{5}(\Pi_2 + \Pi_2^c) + \frac{3}{5}(2\,\Pi_3 + R) = \pi^*\left(-\frac{2}{5}\Delta_2 + \frac{3}{5}\Delta_3\right),$$

hence $a = -1/5$ and $b = 3/5$ and the result follows by using the formula $10\,\lambda = \delta_0 + 2\,\delta_1$.

14 | COMPARISON WITH THE HODGE BUNDLE

We know by Lemma 10.3 that the line bundle $N = \mathcal{O}_{\tilde{\mathbb{P}}}(D_k + E_k - R)$ on $\tilde{\mathbb{P}}$ over $\widetilde{\mathcal{H}}_{3,2}$ has the property that $\pi_*(N) \cong \mathbb{E}$ up to codimension 2. We now deal with the push forward of the tensor powers of N.

Lemma 14.1. *We have for* $m \in \mathbb{Z}_{\geq 1}$

$$c_1(\pi_*(N^{\otimes m})) = \frac{2\,m^2+m}{14}\,\Delta_2 + \frac{5\,m^2-m}{14}(\Delta_3+\Delta_4)\,.$$

Proof. We apply Grothendieck–Riemann–Roch to π and $N^{\otimes m}$ as in (10) in the proof of Proposition 10.2. Recall that N corresponds to the divisor (class) $D_k + E_k - R$. We use that $R^1\pi_* N^{\otimes m} = 0$ for all m and find

$$c_1(\pi_*(N^{\otimes m})) = \frac{1}{2}\pi_*\left(m^2(D_k+E_k-R)^2 - m\,\omega_\pi\cdot(D_k+E_k-R)\right)$$

and using the relations (8) and (9) of the proof of Proposition 10.2, we get

$$c_1(\pi_*(N^{\otimes m})) = \frac{2\,m^2+m}{14}\,\Delta_2 + \frac{5\,m^2-m}{14}(\Delta_3+\Delta_4)$$

as required. □

Proposition 14.2. *On B we have the exact sequence*

$$0 \to \mathrm{Sym}^{m-2}(\mathbb{E})\otimes\mathcal{O}(-A) \to \mathrm{Sym}^m(\mathbb{E}) \to \pi_*(N^{\otimes m}) \to 0,$$

with $A = 4\lambda - (\Delta_2+\Delta_3+\Delta_4)$.

Proof. By Lemma 13.3, we have on $\mathbb{P}(\mathbb{E})$ the exact sequence

$$0 \to \mathcal{O}(m-2)\otimes u^*\mathcal{O}(-A) \to \mathcal{O}(m) \to \mathcal{O}(m)_{|Q} \to 0.$$

Applying u_* and observing that $R^1u_*\mathcal{O}(m-2)$ vanishes gives the result. □

A section of $\mathrm{Sym}^j(\mathbb{E})\otimes\det(\mathbb{E})^k$ over $\mathcal{H}_3$ pulls back to the stack $[W^0_{8,-2}/(\mathrm{GL}(W)/(\pm 1_W))]$ as a section of $\mathrm{Sym}^j(\mathrm{Sym}^2(W))\otimes\det(W)^{k/2}$ for even k. We have an isotypical decomposition

$$\mathrm{Sym}^j(\mathrm{Sym}^2(W)) = \oplus_{n=0}^{\lfloor j/2\rfloor}\mathrm{Sym}^{2j-4n}(W)\otimes\det(W)^{2n}\,,$$

where we assume here and in the rest of this section that the characteristic is 0 or not 2 and high enough for this identity to hold (or use divided powers as in [1, 3.1]). A section of $\mathrm{Sym}^j(\mathbb{E})\otimes\det(\mathbb{E})^k$ over $\mathcal{M}_3^{nh}$ pulls back to $[V_{4,0,-1}/\mathrm{GL}(V)]$, where we now write V for the standard space of dimension 3. An identification $V \cong \mathrm{Sym}^2(W)$ corresponds to an embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ with image a smooth quadric. If we view V with basis x, y, z, the kernel of the projection

$$\mathrm{Sym}^j(V) = \mathrm{Sym}^j(\mathrm{Sym}^2(W)) \to \mathrm{Sym}^{2j}(W)$$

consists of the polynomials of degree j in x, y, z that vanish on the quadric. Thus, in view of the isotypical decomposition above, the exact sequence

$$0 \to \mathrm{Sym}^{m-2}(\mathbb{E})\otimes\mathcal{O}(-A) \to \mathrm{Sym}^m(\mathbb{E}) \to \pi_*(N^{\otimes m}) \to 0$$

corresponds to (the pull back to $W^0_{8,-2}$ of) an exact sequence

$$0 \to \left(\mathrm{Sym}^{m-2}(\mathrm{Sym}^2 W)\right)\otimes\det(W)^2 \to \mathrm{Sym}^m(\mathrm{Sym}^2 W) \to \mathrm{Sym}^{2m}(W) \to 0\,.$$

The section $\chi_{4,0,8}$ of $\mathrm{Sym}^4(\mathbb{E}) \otimes \det(\mathbb{E})^8$ restricted to the hyperelliptic locus allows three projections according to the decomposition

$$\mathrm{Sym}^4(\mathrm{Sym}^2 W) \otimes \det(W)^{24} = W_{8,24} \oplus W_{4,26} \oplus W_{0,28} \,. \tag{12}$$

Lemma 14.3. *The projections to the three summands in (12) of the pull back of $\chi_{4,0,8}$ to $\mathcal{H}_{3,2}$ define modular forms on $\overline{\mathcal{H}}_{3,2}$ of weights (4,0,8), $(2,0,4)$, and (0,0,14) and these are up to a scalar given by the covariants $f_{8,-2}\,\mathfrak{d}$, $f_{4,-1}\,\mathfrak{d}$ and the discriminant $\mathfrak{d}$.*

Proof. The identification of $\mathbb{E}$ with $\mathrm{Sym}^2(W)$ corresponds to the embedding of $\mathbb{P}^1$ as a conic C in $\mathbb{P}^2$. A ternary quartic Q contains C either 0, 1, or 2 times, say $Q = mC + R$ with $0 \le m \le 2$. The three projections correspond to $R \cap C$ and give the universal binary octic, the universal binary quartic, and 1 up to twisting. The first projection was identified in Proposition 6.1. The argument for the second is similar, while the third descends to $\overline{\mathcal{H}}_3$ and does not vanish on $\mathcal{H}_3$. Therefore, it must be a multiple of the discriminant. Taking into account the action of $\mathrm{GL}_2/\pm 1_W$, we get the indicated weights (namely $2(14+\epsilon)$ with $\epsilon = -2, -1, 0$). □

15 | MORE MODULAR FORMS FOR GENUS 3

We will use more effective divisors on projectivized Hodge bundles to produce more modular forms. Note that the connection between divisors on projectivized Hodge bundles and modular forms can also be used in the other direction: obtaining results on cycle classes using modular forms. We give a few examples. To a canonical quartic plane curve C, we can associate a curve $\check{S}$ in the dual plane of lines intersecting C equianharmonically. It corresponds to a contravariant (concomitant) σ of the ternary quartic given by Salmon in [25, p. 264] and it is defined by an equivariant GL(3) embedding $W[4,4,0] \hookrightarrow \mathrm{Sym}^2(\mathrm{Sym}^4(W))$. It gives rise to a divisor in $\mathbb{P}(\mathbb{E}^\vee)$ and a modular form $\chi_{0,4,16}$ of weight (0,4,16). We refer to [5, p. 54] for the relation between invariant theory of ternary quartics and modular forms. The Siegel modular form $\chi_{0,4,16}$ vanishes with order 2 at infinity and order 4 along the locus $\mathcal{A}_{2,1}$ of decomposable abelian threefolds. With $\check{u} : \mathbb{P}(\mathbb{E}^\vee) \to \overline{\mathcal{M}}_3$, the projection we have $\check{u}_*(\mathcal{O}_{\mathbb{P}(\mathbb{E}^\vee)}(1)) = \mathbb{E}^\vee \cong \wedge^2\mathbb{E} \otimes \det(\mathbb{E})^{-1}$, and we thus find an effective divisor on $\mathbb{P}(\mathbb{E}^\vee)$ over $\tilde{\mathcal{A}}_3$ with class $[\check{S}] = [\mathcal{O}_{\mathbb{P}(\mathbb{E}^\vee)}(4)] + 20\,\lambda - 2\,\delta$ and it vanishes with multiplicity 4 along $\mathcal{A}_{2,1}$. We thus find on $\mathbb{P}(\mathbb{E}^\vee)$ over $\overline{\mathcal{M}}_3$ a relation

$$[\check{S}] = [\mathcal{O}_{\mathbb{P}(\mathbb{E}^\vee)}(4)] + 20\,\lambda - 2\,\delta_0 - 4\,\delta_1 \,,$$

where we identify λ and δ_i with their pull backs to $\mathbb{P}(\mathbb{E}^\vee)$. Similarly, in the dual plane, we have the sextic $\check{T}$ of lines intersecting the quartic curve in a quadruple of points with j-invariant 1728. The corresponding concomitant τ corresponds to $W[6,6,0] \hookrightarrow \mathrm{Sym}^3(\mathrm{Sym}^4(W))$ and defines a modular form of weight (0,6,24) vanishing with multiplicity 3 at infinity and multiplicity 6 along $\mathcal{A}_{2,1}$. We thus get a cycle relation

$$[\check{T}] = [\mathcal{O}_{\mathbb{P}(\mathbb{E}^\vee)}(6)] + 30\,\lambda - 3\,\delta_0 - 6\,\delta_1 \,.$$

The concomitant $\sigma^3 - 27\,\tau^2$ vanishes on the locus of double conics and the corresponding modular form of weight (0,12,48) is divisible by χ_{18}^2 as can be checked using the methods of [5]. Dividing by χ_{18}^2 gives a cusp form of weight (0,12,12) vanishing with multiplicity 2 at infinity and multiplicity 3 along $\mathcal{A}_{2,1}$. It is classically known (see, e.g., [3, p. 43]) that this concomitant defines the dual curve $\check{C}$ to the canonical image C in $\mathbb{P}(\mathbb{E})$. We thus find an effective divisor in $\mathbb{P}(\mathbb{E}^\vee)$ containing the closure of the dual curve with class

$$12\,[\mathcal{O}_{\mathbb{P}(\mathbb{E}^\vee)}(1)] + 24\,\lambda - 2\,\delta_0 - 3\,\delta_1 \,.$$

This effective divisor class can also be given by the cycle

$$B = \{(C,\eta) \in \mathbb{P}(\mathbb{E}^\vee) : \ \mathrm{div}(\eta) \text{ has a point of multiplicity } 2\}$$

over $\mathcal{M}_3$ and Korotkin and Zograf in [22, Theorem 1] determined the class of its closure $\overline{B}$

$$[\overline{B}] = 12\,[\mathcal{O}_{\mathbb{P}(\mathbb{E}^\vee)}(1)] + 24\,\lambda - 2\,\delta_0 - 3\,\delta_1\,.$$

Another example of an effective divisor for genus 3 is provided by the Weierstrass divisor W with class

$$[\overline{W}] = 24\,[\mathcal{O}_{\mathbb{P}(\mathbb{E}^\vee)}(1)] + 68\,\lambda - 6\,\delta_0 - 12\,\delta_1$$

as given by Gheorghita in [12]. Here, we get a section of

$$\mathrm{Sym}^{24}(\wedge^2\mathbb{E}) \otimes \det(\mathbb{E})^{44}(-6\,\delta_0 - 12\,\delta_1).$$

This gives a Teichmüller modular form of weight (0,24,44) vanishing with multiplicity 6 at the cusp. It descends to a Siegel modular form.

Corollary 15.1. *The dual of the canonical curve defines a Siegel modular cusp form of degree 3 of weight (0,12,12) vanishing with multiplicity 2 at infinity. The Weierstrass divisor defines a cusp form of weight (0,24,44) vanishing with multiplicity 6 at infinity.*

16 | THE HYPERTANGENT DIVISOR

A generic canonically embedded curve C of genus 3 has 24 (Weierstrass) points where the tangent line intersects C with multiplicity 3. The union of these 24 lines forms a divisor in $\mathbb{P}^2$. Taking the closure of this divisor for the universal curve over $\mathcal{M}_3$ defines a divisor H in $\mathbb{P}(\mathbb{E})$ over $\overline{\mathcal{M}}_3$, which we call the hypertangent line divisor. We calculate the class of this divisor over $\overline{\mathcal{M}}_3$ and also calculate the class of a corresponding divisor over $\overline{\mathcal{H}}_{3,2}$.

The calculation over $\overline{\mathcal{M}}_3$ uses the divisors $\check{S}$ and $\check{T}$ in $\mathbb{P}(\mathbb{E}^\vee)$ over $\overline{\mathcal{M}}_3$ as defined in the preceding section. It is a classical result that the intersection $\check{S} \cdot \check{T}$ in the generic fiber is the 0-cycle consisting of the 24 points defining the 24 hyperflexes of the generic curve C, see [25]. We consider the incidence variety

$$I = \{(p,\ell) \in \mathbb{P}(\mathbb{E}) \times_{\overline{\mathcal{M}}_3} \mathbb{P}(\mathbb{E}^\vee) \,:\, p \in \ell\}.$$

Let $\rho : I \to \mathbb{P}(\mathbb{E})$ and $\check{\rho} : I \to \mathbb{P}(\mathbb{E}^\vee)$ be the two projections fitting in the commutative diagram

$$\begin{array}{ccc} I & \xrightarrow{\check{\rho}} & \mathbb{P}(\mathbb{E}^\vee) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \check{u}} \\ \mathbb{P}(\mathbb{E}) & \xrightarrow{u} & \overline{\mathcal{M}}_3 \end{array}$$

We have the tautological sequence on $\mathbb{P}(\mathbb{E})$

$$0 \to F \to u^*(\mathbb{E}) \to \mathcal{O}_{\mathbb{P}(\mathbb{E})}(1) \to 0$$

and a similar one on $\mathbb{P}(\mathbb{E}^\vee)$

$$0 \to \check{F} \to \check{u}^*(\mathbb{E}^\vee) \to \mathcal{O}_{\mathbb{P}(\mathbb{E}^\vee)}(1) \to 0\,.$$

Now note that I can be identified with the $\mathbb{P}^1$-bundle $\mathbb{P}(F^\vee)$ on $\mathbb{P}(\mathbb{E})$, but also with the $\mathbb{P}^1$-bundle $\mathbb{P}(\check{F}^\vee)$ on $\mathbb{P}(\mathbb{E}^\vee)$.

The tautological inclusion $F \to u^*\mathbb{E}$ induces a surjection $u^*\mathbb{E}^\vee \to F^\vee$ and this gives an inclusion $\mathbb{P}(F^\vee) \to \mathbb{P}(u^*\mathbb{E}^\vee)$ of projective bundles over $\mathbb{P}(\mathbb{E})$, which composed with natural map $\mathbb{P}(u^*\mathbb{E}^\vee) \to \mathbb{P}(\mathbb{E}^\vee)$ gives the map $\check{\rho} : I = \mathbb{P}(F^\vee) \to$

$\mathbb{P}(\mathbb{E}^\vee)$. This implies

$$\mathcal{O}_{\mathbb{P}(F^\vee)}(1) = \check{\rho}^* \mathcal{O}_{\mathbb{P}(\mathbb{E}^\vee)}(1) \quad \text{and similarly} \quad \mathcal{O}_{\mathbb{P}(\check{F}^\vee)}(1) = \rho^* \mathcal{O}_{\mathbb{P}(\mathbb{E})}(1)\,. \tag{13}$$

With

$$f = c_1(F), \quad \check{f} = c_1(\check{F}), \quad h = c_1(\mathcal{O}_{\mathbb{P}(\mathbb{E})}(1)), \quad \check{h} = c_1(\mathcal{O}_{\mathbb{P}(\mathbb{E}^\vee)}(1)),$$

this gives the identities of pull backs of the first Chern classes $c_1(\mathbb{E}) = -c_1(\mathbb{E}^\vee) = \lambda$

$$\rho^*(f) + \rho^*(h) = \rho^* u^*(\lambda) = \check{\rho}^* \check{u}^*(\lambda) = -\check{\rho}^*(\check{f}) - \check{\rho}^*(\check{h})\,.$$

Since $I = \mathbb{P}(F^\vee)$ over $\mathbb{P}(\mathbb{E})$ and $\check{\rho}^* \check{h} = c_1(\mathcal{O}_{\mathbb{P}(F^\vee)}(1))$, the Chern classes of $F^\vee$ and the first Chern class of the tautological line bundle satisfy the relation

$$\check{\rho}^* \check{h}^2 + \rho^*(f)\, \check{\rho}^*(\check{h}) + \rho^*(c_2(F)) = 0\,. \tag{14}$$

Corollary 16.1. *Under the map $\rho_* \check{\rho}^*$ we have*

$$\check{h}^2 \mapsto h - u^*(\lambda), \quad \check{h}\, \check{u}^*(\xi) \mapsto u^*(\xi), \quad \check{u}^*(\eta) \mapsto 0$$

for $\xi \in \mathrm{CH}^1(\overline{\mathcal{M}}_3)$ and $\eta \in \mathrm{CH}^2(\overline{\mathcal{M}}_3)$.

Proof. Using relation (14) gives

$$\rho_*(\check{\rho}^*(\check{h})^2) = -\rho_*(\rho^*(f)\check{\rho}^*(\check{h}) - \rho^*(c_2(F)) = -f = h - u^*(\lambda)\,.$$

The other properties follow from general intersection theory. □

Let now ψ be the class of the codimension 2 cocycle $\check{S} \cdot \check{T}$.

Lemma 16.2. *We have $\rho_* \check{\rho}^* \psi = 24\, h + 216\, \lambda - 24\, \delta_0 - 48\, \delta_1$.*

Proof. By the results of the preceding section, we have

$$\psi = 24\, \check{h}^2 + 240\, \check{h}\lambda - 24\, \check{h}\delta_0 - 48\, \check{h}\delta_1 + r$$

with $r \in \check{\rho}^* \mathrm{CH}^2(\overline{\mathcal{M}}_3)$. Corollary 16.1 implies the result. □

We now claim that the codimension 2 cycle $\check{S} \cdot \check{T}$ when restricted to the hyperelliptic locus is of the form $12\, \check{h}$, in other words, by (2) it contains an effective codimension 2 cycle with class

$$12\,(9\,\lambda - \delta_0 - 3\,\delta_1)\, \check{h} + \check{u}^*(\xi)$$

with ξ a codimension 2 class on $\overline{\mathcal{M}}_3$. We check this using the explicit form of the two concomitants σ and τ defining $\check{S}$ and $\check{T}$. Here, σ is a polynomial of degree 4 in $a_0, \ldots, a_{14}$ and degree 4 in the coordinates u_0, u_1, u_2 where $a_0, \ldots, a_{14}$ are the coefficients of the general ternary quartic. A calculation shows that σ restricted to the locus of double conics becomes a square q^2 with q of degree 2 in the u_i, while τ becomes a cube q^3. Hence, the cycle $S^\vee \cdot T^\vee$ restricted to the hyperelliptic locus is represented by an effective cycle representing $6\, q \sim 12\check{h}$. By Corollary 16.1 under $\rho_* \hat{\rho}^*$ this is sent to an effective cycle with class $12(9\,\lambda - \delta_0 - 3\,\delta_1)$. Since H is defined as the closure of the hypertangent divisor in the generic fiber, the class of H equals $\rho_* \check{\rho}^* \psi$ minus 12 times the class of the hyperelliptic locus; by Lemma 16.2 we get

$$24\, h + 216\, \lambda - 24\, \delta_0 - 48\, \delta_1 - 12(9\lambda - \delta_0 - 3\delta_1) = 24\, h + 108\, \lambda - 12\, \delta_0 - 12\, \delta_1\,.$$

We summarize.

Proposition 16.3. *The class $[H]$ of the hypertangent divisor H in $\mathbb{P}_{\overline{\mathcal{M}}_3}(\mathbb{E})$ equals $[\mathcal{O}_{\mathbb{P}(\mathbb{E})}(24)] + 108\,\lambda - 12\,\delta_0 - 12\,\delta_1$. It gives rise to a Siegel modular form of degree 3 and weight (24,0,108) vanishing with multiplicity 12 along the boundary.*

We now work on the Hurwitz space and define and calculate the class of a hypertangent H_h divisor there. It is defined by taking the eight tangent lines at the ramification points of the canonical image. More precisely, on $\tilde{\mathbb{P}}$ we have the line bundle N defined in (9). Recall that $\tilde{S}_k$ for $1 \leq k \leq 8$ is the pull back of the section S_k of $\pi_9 : \overline{\mathcal{M}}_{0,9} \to \overline{\mathcal{M}}_{0,8}$. Under restriction to the hyperelliptic locus, the Weierstrass points degenerate to the ramification points. We define the corresponding hypertangent divisor H_h in $\mathbb{P}(\mathbb{E})$ over $\overline{\mathcal{H}}_{3,2}$ by taking the tangents to the canonical image of the generic curve at the points of the sections $\tilde{S}_k$, $k = 1, \ldots, 8$ over $\mathcal{H}_{3,2}$ and then taking the closure over $\overline{\mathcal{H}}_{3,2}$.

We now consider the bundle $N(-2\tilde{S}_k)$ on $\tilde{\mathbb{P}}$. This line bundle is trivial on the generic fiber of $\pi : \tilde{\mathbb{P}} \to B$, so $\pi_*(N(-2\tilde{S}_k))$ is a line bundle on B.

Lemma 16.4. *We have*

$$c_1(R^1\pi_*N(-2\tilde{S}_k)) = \Delta_2(k^+) + \Delta_3(k^+), \quad \text{and} \quad c_1(\pi_*N(-2\tilde{S}_k)) = -\Delta_3(k^+) - \Delta_3 + E_k\,.$$

Proof. Recall that $N = \mathcal{O}(D_k + \pi^*(E_k) - R)$. The first statement follows by analyzing the restrictions over the boundary components. For the second, we apply Grothendieck–Riemann–Roch as in the proof of Proposition 10.2. By (7) and (8), we have

$$\begin{aligned} c_1(\pi_*N(-2\tilde{S}_k)) &= -\Delta_2(k^+) - 2\,\Delta_3(k^+) - \Delta_3 + c_1(R^1\pi_*N(-2\tilde{S}_k)) \\ &= -\Delta_3(k^+) - \Delta_3 + E_k\,. \end{aligned}$$

□

Put $\mathcal{F}_k = \pi_*(N(-2\tilde{S}_k))$. The injection $N(-2\tilde{S}_k) \hookrightarrow N$ induces an injection $\mathcal{F}_k \to \mathbb{E}$. Pulling back to $\mathbb{P}(\mathbb{E})$ via u^* and composing with the canonical surjection $u^*(\mathbb{E}) \to \mathcal{O}_{\mathbb{P}(\mathbb{E})}(1)$, we get an induced map

$$q : u^*\mathcal{F}_k \to \mathcal{O}_{\mathbb{P}(\mathbb{E})}(1)\,.$$

The degeneracy locus of q is an effective divisor F_k that is the vanishing divisor of a section of $\mathcal{O}_{\mathbb{P}(\mathbb{E})}(1) \otimes u^*\mathcal{F}_k^{-1}$. The interpretation is as follows. The map ϕ defines an embedding of the generic fiber of $\tilde{\mathbb{P}}$ into the generic fiber $\mathbb{P}(\mathbb{E})$. If we identify $H^0(\mathbb{P}^1, \mathcal{O}(2))$ with the fiber of $\mathbb{E}$ and projectivize, the divisor $p_1 + p_2 \in |\mathcal{O}(2)|$ is mapped to the line through through the points $\phi(p_1), \phi(p_2)$. We now sum these divisors F_k and get an effective divisor H_h with class

$$\begin{aligned} [H_h] = 8\,[\mathcal{O}(1)] - \sum_{k=1}^{8}[u^*\mathcal{F}_k] &= [\mathcal{O}(8)] + u^*\left(3\,\Delta_3 + 8\,\Delta_3 - \sum_{k=1}^{8} E_k\right) \\ &= [\mathcal{O}(8)] + u^*(8\,\lambda - 2\,\Delta_2 + \Delta_3)\,, \end{aligned}$$

where we use the formulas of Section 10 and Section 12.

We can now compare the class of the hyperelliptic hypertangent divisor H_h with that of the pull back of the hypertangent divisor H to the Hurwitz space. By Proposition 16.3, the pull back of H has class $[\mathcal{O}(24)] + 108\,\lambda - 24\,(\Delta_2 + \Delta_4) - 12\,\Delta_1$. Since the 24 Weierstrass points collapse with multiplicity 3 to the eight ramification points, we compare the class (of the pull back of) $[H]$ with that of $3\,[H_h]$. Substituting the formula for λ, we get

$$[H] - 3\,[H_h] = 9\,\Delta_1\,,$$

which means that the pull back of H vanishes with multiplicity 9 at the hyperelliptic boundary component Δ_1.

17 | GENUS 4

For a smooth curve C of genus 4, the natural map $\mathrm{Sym}^2(H^0(C,\omega_C)) \to H^0(C,\omega_C^{\otimes 2})$ is surjective and the kernel has dimension 1. It determines a quadric in $\mathbb{P}^3 = \mathbb{P}(H^0(C,\omega_C))$ containing the canonical curve. Over $\overline{\mathcal{M}}_4$, we find a corresponding exact sequence

$$0 \to U \to \mathrm{Sym}^2(\mathbb{E}) \to \pi_* \omega_C^{\otimes 2} \to 0\,.$$

The line bundle U has first Chern class $5\,\lambda - (13\,\lambda - \delta) = -8\,\lambda + \delta$ by Mumford's calculation of $c_1(\pi_*\omega_c^{\otimes 2})$ [23, Theorem 5.10]. In the bundle $\mathbb{P}(\mathbb{E})$ the quadric containing the canonical curve determines a divisor Q. Let $u : \mathbb{P}(\mathbb{E}) \to \overline{\mathcal{M}}_4$ be the projection.

Lemma 17.1. *The divisor class of Q satisfies:* $[Q] = [\mathcal{O}(2)] + u^*(8\,\lambda - \delta)$.

Proof. Observe that $u^*u_*\mathcal{O}(2) = u^*(\mathrm{Sym}^2(\mathbb{E}))$. The natural morphism $u^*u_*\mathcal{O}(2) \to \mathcal{O}(2)$ induces $u^*U \to \mathcal{O}(2)$. The divisor Q is the vanishing locus of this morphism, hence has class $[\mathcal{O}(2)] + u^*(8\,\lambda - \delta)$. □

Corollary 17.2. *The effective divisor Q defines a Teichmüller modular cusp form χ of genus 4 and weight (2,0,0,8).*

If we view a section of $\mathrm{Sym}^2(\mathbb{E})$ as a quadratic form on $\mathbb{E}^\vee$, we can take the discriminant, cf. [6]. Doing this with the form χ of weight (2,0,0,8) just constructed we get a scalar-valued modular form $D(\chi)$ of weight 34. This modular form vanishes on the closure of the locus of curves whose canonical model lies on a quadric cone. This locus has class $34\lambda - 4\,\delta_0 - 14\,\delta_1 - 18\,\delta_2$ by Teixidor i Bigas [26, Proposition 3.1] and equals the divisor of curves with a vanishing thetanull. The modular form $D(\chi)$ is the square root of the restriction to $\mathcal{M}_4$ of the product of the even theta characteristics on $\mathcal{A}_4$.

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ENDNOTE

[1] Alternatively one could use divided powers as in [1, 3.1]

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APPENDIX A: BASE POINT FREENESS

The relative dualizing sheaf ω_π of the universal family $\pi : \mathcal{C}_g \to \mathcal{M}_g$ of genus g smooth curves is base point free and the surjection $\pi^*\mathbb{E} \to \omega_\pi$ gives a map $\varphi : \mathcal{C}_g \to \mathbb{P}(\mathbb{E})$ over $\mathcal{M}_g$, which is generically an embedding. Let Γ be the image $\varphi(\mathcal{C}_g)$. We wish to describe the closure of the image over the generic points of the boundary components Δ_i for $i = 0, \ldots, [g/2]$. Over the general point of Δ_0, the sheaf ω_π is base point free and the map φ extends over this locus. But over the general point of Δ_i, $i \geq 1$, which represents a nodal curve of the form $C_1 \cup C_2$, with C_1, C_2 smooth curves of genus i and $g-i$ meeting at a nodal point x, the sheaf ω_π has a base point at x. We consider a family $\pi : Y \to B$ of stable curves of genus g with B the spectrum of a discrete valuation ring. We assume that the central fiber C is a nodal curve $C = C_1 \cup C_2$ of genera i and $g-i$ and smooth generic fiber. After a degree 2 base change $B' \to B$, we get an A_1-singularity, which we resolve resulting in a semistable family $\pi' : X \to B'$ with a special fiber, which is a chain of three curves $C = C_1' \cup R \cup C_2'$ with $C_1' \cong C_1$ and $C_2' \cong C_2$ smooth curves of genus i and $g-i$, and R a rational (-2)-curve. We have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\upsilon} & Y \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow[\sigma]{} & B \end{array}$$

The morphism υ is $(2:1)$ ramified at C_1, C_2. We have $\upsilon^*\omega_\pi = \omega_{\pi'}$, and $\sigma^*\mathbb{E}_B = \mathbb{E}_{B'}$ and $\upsilon^* C_j = 2C_j' + R$ for $j = 1, 2$. There is then a natural $(2:1)$ map $\mathbb{P}(\mathbb{E}_{B'}) \to \mathbb{P}(\mathbb{E}_B)$.

Now we will show that the system $\omega_{\pi'}(-R)$ defines a map $X \to \mathbb{P}(\mathbb{E}_{B'})$, which combined with the above $(2:1)$ map gives a $(2:1)$ map $\varphi' : X \to \mathbb{P}(\mathbb{E}_B)$ mapping the curves C_1' and C_2' to their canonical image and R to a double line. The reduced image of the map φ' describes the closure of D over b_0, the special point of B.

To avoid unnecessary notation, we now write $\pi : X \to B$ for the semistable family denoted by $\pi' : X \to B'$ above.

Proposition A.1. *Let ω be the relative dualizing sheaf of $\pi : X \to B$. Then, we have $\pi_*(\omega(-R)) \cong \pi_*(\omega)$ and the central fiber of $\pi_*(\omega(-R))$ is of codimension 1 in $H^0(C, \omega(-R))$ and defines a base point free linear system on C.*

Proof. We let $q = C_1 \cap R$ and $p = C_2 \cap R$. The exact sequence $0 \to \omega(-R) \to \omega \to \omega_{|R} \to 0$ induces a sequence

$$0 \to \pi_*(\omega(-R)) \to \pi_*(\omega) \xrightarrow{r} \pi_*(\omega)_{|R}$$

and the map r is zero because $\omega_{|C} = (\omega_{C_1}(q), \mathcal{O}_R, \omega_{C_2}(p))$, therefore the restrictions to C_1 (resp. C_2) must vanish at q (resp. p), hence extend by 0 on R. We thus see by the exactness that $\pi_*(\omega(-R)) \cong \pi_*(\omega)$.

Next, we observe that $\dim H^0(C, \omega(-R)) = g_1 + g_2 + 1$ with g_i the genus of C_i. This follows directly from $\omega(-R)_{|C} = (\omega_{C_1}, \mathcal{O}_R(2), \omega_{C_2})$.

We have the exact sequence

$$0 \to \omega(-R-C_1) \to \omega(-R) \to \omega(-R)_{|C_1} \to 0\,, \tag{A1}$$

where $\omega(-R)_{|C_1} \cong \omega_{C_1}$ and $\omega(-R-C_1)_{|C} = (\omega_{C_1}(q), \mathcal{O}_R(1), \omega_{C_2})$. For a section $(s_1, s, s_2) \in H^0(C, \omega(-R-C_1))$, the section s is the unique section of $\mathcal{O}_R(1)$ that vanishes at q and with $s(p) = s_2(p)$. We thus see $\dim H^0(C, \omega(-R-C_1)) = g_1 + g_2$. Therefore, $h^0(\omega(-R-C_1))$ has constant rank g_1+g_2 on the fibers of π, hence $R^1\pi_*(\omega(-R-C_1))$ is a line bundle. We conclude that the special fiber of $\pi_*(\omega(-R-C_1))$ equals $H^0(C, \omega(-R-C_1))$. But $\pi_*(\omega(-R)_{|C_1})$ is a torsion sheaf, hence the connecting homomorphism $\pi_*(\omega(-R)_{|C_1}) \to R^1\pi_*(\omega(-R-C_1))$ of (15) must be zero and we get an induced exact sequence

$$0 \to \pi_*(\omega(-R-C_1)) \xrightarrow{\iota} \pi_*(\omega(-R)) \xrightarrow{j} \pi_*(\omega(-R))_{|C_1}) \to 0\,.$$

Consider now a section σ of $\pi_*(\omega(-R-C_1))$ with restriction (s_1, s, s_2) to C. Suppose that $s \neq 0$. If we multiply σ with a local section τ of $\mathcal{O}(C_1)$ on X with divisor C_1, then $\iota(\sigma) = \sigma \cdot \tau_{|C}$ has as restriction to R a section of $\mathcal{O}_R(2)$ vanishing with multiplicity 2 at q and therefore it does not vanish anywhere else. Hence, the subspace of the special fiber V of $\pi_*(\omega(-R))$ of sections vanishing on C_1 has q as only base point on R. Furthermore, the map j is surjective, and choosing a section $s_1 \in H^0(C_1, \omega_{C_1})$ with $s_1(q) \neq 0$, we see that q is not a base point. Therefore, there are no base points on R. By the surjectivity of j, the restriction of V to C_1 is $H^0(C_1, \omega_{C_1})$ and therefore there are no base points on C_1. By symmetry, the same holds for C_2.

Similarly to (15), we have an exact sequence

$$0 \to \omega(-R-C_1-C_2) \to \omega(-R) \to \omega(-R)_{|C_1+C_2} \to 0,$$

and by a similar reasoning, we see that we get an exact sequence

$$0 \to \pi_*(\omega(-R-C_1-C_2)) \xrightarrow{\iota} \pi_*(\omega(-R)) \xrightarrow{j} \pi_*(\omega(-R))_{|C_1+C_2}) \to 0\,.$$

This implies that given $s_1 \in H^0(C_1, \omega_{C_1})$ and $s_2 \in H^0(C_2, \omega_{C_2})$, there is a unique element (s_1, s, s_2) in the special fiber V of $\pi_*(\omega(-R))$ mapping to (s_1, s_2) under j. The morphism $X \to \mathbb{P}(\pi_*(\omega(-R)))$ is given by the surjection $\pi^*\pi_*(\omega(-R)) \to \omega(-R)$. The image of the curve C in the special fiber of $\mathbb{P}(\mathbb{E})$ consists of the canonical images of C_1 and C_2, provided with images of p and q and the image of R, that is, the line connecting the images of p and q. If the genus $g(C_i) = 1$, then the image of C_i is a point. □

APPENDIX B: DIVISOR CLASSES OF GHEORGHITA–TARASCA AND KOROTKIN–SAUVAGET–ZOGRAF

Here, we apply the method employed in Section 16 to determine in a relatively straightforward way the divisor classes of two divisors in $\mathbb{P}(\mathbb{E}_k^\vee)$ with $\mathbb{E}_k = \pi_*(\omega_\pi^k)$, thus reproving a theorem of Gheorghita–Tarasca [13, Theorem 1] and a theorem of Korotkin–Sauvaget–Zograf [21, Theorem 1.12]. The first divisor is a generalization of a divisor in $\mathbb{P}(\mathbb{E}^\vee)$ considered by Gheorghita in [12]. We consider in $\mathbb{P}(\mathbb{E}_k^\vee)$ over $\mathcal{M}_g$ the divisor

$$G_k = \{(C, \omega) \in \mathbb{P}(\mathbb{E}_k^\vee) : \text{div}(\omega) \text{ contains a Weierstrass point}\}$$

and let $\overline{G}_k$ be the closure of G_k in $\mathbb{P}(\mathbb{E}_k^\vee)$ over $\overline{\mathcal{M}}_g$. We let $\check{u} : \mathbb{P}(\mathbb{E}_k^\vee) \to \overline{\mathcal{M}}_g$ be the natural morphism and $\check{h}$ the hyperplane class on $\mathbb{P}(\mathbb{E}_k^\vee)$.

Theorem B.1 (Gheorghita–Tarasca). *The class of $\overline{G}_k$ is given by*

$$\frac{1}{k}[\overline{G}_k] = g(g^2-1)\check{h} + 2(3g^2+2g+1)\check{u}^*\lambda - \binom{g+1}{2}\check{u}^*\delta_0 - \sum_{i=1}^{[g/2]} (g-i)i(g+3)\check{u}^*\delta_i\,.$$

The second divisor is the divisor Z_k in $\mathbb{P}(\mathbb{E}_k^\vee)$ over $\overline{\mathcal{M}}_g$ of regular k-differentials for $k \geq 2$ possessing a double zero.

Theorem B.2 (Korotkin–Sauvaget–Zograf). *The class of the divisor* Z_k *for* $k \geq 2$ *for* $k \geq 2$ *and* $(g,k) \neq (2,2)$ *is given by*

$$[Z_k] = (4k+2)(g-1)\,\check{h} + k(k+1)\,\check{u}^*\left(12\lambda - \sum_{i=0}^{[g/2]} \delta_i\right).$$

For the proof of both theorems, we use, as in Section 16, the incidence variety I_k between $\mathbb{P}(\mathbb{E}_k)$ and $\mathbb{P}(\mathbb{E}_k^\vee)$, which fits in the following commutative diagram:

$$\begin{array}{ccc} I_k & \xrightarrow{\ \check{\rho}\ } & \mathbb{P}(\mathbb{E}_k^\vee) \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \check{u}} \\ \mathbb{P}(\mathbb{E}_k) & \xrightarrow{\ u\ } & \overline{\mathcal{M}}_g \end{array}$$

We denote by h (resp. $\check{h}$) the first Chern class of the hyperplane line bundle on $\mathbb{P}(\mathbb{E}_k)$ (resp. $\mathbb{P}(\mathbb{E}_k^\vee)$), thus suppressing the dependence on k. As explained in Section 16, we have $I_k = \mathbb{P}(\check{F}_k^\vee)$ as a bundle over $\mathbb{P}(\mathbb{E}_k^\vee)$, with $\check{F}_k$ defined by the exact sequence on $\mathbb{P}(\mathbb{E}_k^\vee)$

$$0 \to \check{F}_k \to \check{u}^*\mathbb{E}_k^\vee \to \mathcal{O}_{\mathbb{P}(\mathbb{E}_k^\vee)}(1) \to 0.$$

Then, $\rho^* h = \mathcal{O}_{\mathbb{P}(\check{F}_k{}^\vee)}(1)$. Similarly, $I_k = \mathbb{P}(F_k^\vee)$ as a bundle over $\mathbb{P}(\mathbb{E}_k)$, with F_k the tautological rank $r-1$ bundle on $\mathbb{P}(\mathbb{E}_k)$. Then, $\check{\rho}^*(\check{h}) = \mathcal{O}_{\mathbb{P}(F_k{}^\vee)}(1)$.

We let

$$\gamma = \check{\rho}_*\rho^* : \mathrm{CH}^*_{\mathbb{Q}}(\mathbb{P}(\mathbb{E}_k)) \to \mathrm{CH}^*_{\mathbb{Q}}(\mathbb{P}(\mathbb{E}_k^\vee))$$

be the induced map.

Lemma B.3. *We have* $\gamma(h^i) = 0$ *for* $i \leq r-3$, $\gamma(h^{r-2}) = 1$ *and* $\gamma(h^{r-1}) = \check{h} + \check{u}^*\mathrm{c}_1(\mathbb{E}_k)$.

Proof. For dimension reasons $\gamma(h^i) = 0$ for $i \leq r-3$. Moreover, $\gamma(h^{r-2}) = 1$ by construction. Applying $\check{\rho}_*$ to the Chern class relation $\sum_{i=0}^{r-1}(-1)^{r-1-i}(\rho^*h)^i\check{\rho}^*c_{r-1-i}(\check{F}_k^\vee) = 0$, we get $\gamma(h^{r-1}) = c_1(\check{F}_k^\vee)$ and this equals $\check{h} + \check{u}^*\mathrm{c}_1(\mathbb{E}_k)$ by the exact sequence. □

Proof of Theorem B.1. Let W be the Weierstrass divisor on $\overline{C}_g$. This is an irreducible divisor. We denote by $\varphi_k : \overline{C}_g \to \mathbb{P}(\mathbb{E}_k)$ the morphism defined in Section 3. For $k \geq 2$, we have $G_k = \rho_*\rho^*(\varphi_k(W))$ over $\mathcal{M}_g$ and $\rho_*\rho^*$ sends an irreducible divisor over $\overline{\mathcal{M}}_g$ to an irreducible divisor. Therefore, we have $[\overline{G}_k] = \gamma(\varphi_{k*}[W])$. The group $\mathrm{Pic}_{\mathbb{Q}}(\overline{C}_g)$ is generated by $\omega_\pi, \pi^*\lambda$, $\pi^*\delta_0$, and γ_i (for $i = 1, \dots, g-1$) with γ_i the divisor class defined by the component of genus i lying over $\Delta_{\min(i,g-i)}$. By Cuckierman [8], the divisor class $[W]$ can be written as $w_1 - w_2$ with

$$w_1 = \binom{g+1}{2}\omega_\pi - \pi^*\lambda, \quad w_2 = \binom{g-i+1}{2}\sum_{i=1}^{g-1}\gamma_i\,.$$

Note that $\varphi_k^* h = k\,\omega_\pi$. We thus get

$$\begin{aligned} \varphi_{k*} w_1 &= \varphi_{k*}\left(\frac{g(g+1)}{2}\omega_\pi - \pi^*\lambda\right) = \varphi_{k*}\varphi_k^*\left(\frac{g(g+1)}{2k}h - u^*\lambda\right) \\ &= \left(\frac{g(g+1)}{2k}h - u^*\lambda\right)\varphi_{k*}[1] = \left(\frac{g(g+1)}{2k}h - u^*\lambda\right)\Big(\sum_{i=0}^{r-2} h^i\beta_{r-2-i}\Big) \\ &= \sum_{i=0}^{r-1} h^i u^*\left(\frac{g(g+1)}{2k}\beta_{r-1-i} - \lambda\beta_{r-2-i}\right), \end{aligned}$$

where we used (1). Lemma B.3 implies that under applying γ only the terms with h^j where $j = r-1$ or $r-2$ contribute and we get

$$\gamma(\varphi_{k_*}w_1) = \gamma(h^{r-1})\frac{g(g+1)}{2k}\beta_0 + \gamma(h^{r-2})\check{u}^*\left(\frac{g(g+1)}{2k}\beta_1 - \lambda\beta_0\right).$$

Substituting the expressions for β_0 and β_1 from Proposition 3.1 leads to

$$\gamma(\varphi_{k_*}w_1) = g(g^2-1)\,\check{h} + 2\,k\,(3g^2+2g+1)\check{u}^*\lambda - \frac{k}{2}\,g(g+1)\sum_{i=0}^{[g/2]}\check{u}^*\delta_i\,.$$

For the term $\gamma(\varphi_{k_*}w_2)$, we first observe $\gamma(\varphi_{k_*}\gamma_i) = (2i-1)k\check{u}^*\delta_i$ because the component of genus i over $\Delta_{\min(i,g-i)}$ has degree $(2i-1)k$ in $\mathbb{P}^{r-1}$ and thus maps under γ to $(2i-1)k$ times the class [1] of $(\mathbb{P}^{r-1})^\vee$ over $\Delta_{\min(i,g-i)}$. This gives

$$\begin{aligned}\gamma(\varphi_{k*}w_2) &= \frac{k}{2}\sum_{i=1}^{g-1}(g-i)(g-i+1)(2i-1)\,\check{u}^*\delta_i\\ &= \frac{k}{2}\sum_{i=1}^{[g/2]}((g-i)(g-i+1)(2i-1) + i(i+1)(2g-2i-1))\check{u}^*\delta_i\\ &= \frac{k}{2}\sum_{i=1}^{[g/2]}(2i(g-i)(g+3) - g(g+1))\check{u}^*\delta_i.\end{aligned}$$

Together this gives the correct expression for class of $[\overline{G}_k]$ as in Theorem B.1.

When $k = 1$, over $\mathcal{M}_g \cup \Delta_0$ we work as above and the coefficients of λ and δ_0 in the formula are as in the case $k \geq 2$. To find the contribution of δ_i in the formula of β_1, we work over the family over a base B, as in Appendix A, where we have the $(2:1)$ morphism $\varphi' : X \to \mathbb{P}(\mathbb{E})$ defined by the $\omega_{\pi'}(-R)$. We follow the notation of Appendix A and in the formulas, we only need to consider terms that contribute to the boundary class δ_i. By [8], the Weierstrass divisor does not pass through the node of a general element over Δ_i and thus v^*W does not contain the "exceptional" divisor R. We have by Cuckierman's formula $[v^*W] = w_1 - w_2$ with $w_1 = \binom{g+1}{2}\omega_{\pi'} - {\pi'}^*\lambda$, and as contribution to w_2 over Δ_i (for $i \leq [g/2]$), we have the expression

$$i(i+1)(2\gamma_1 + \mathfrak{r}) + (g-i)(g-i+1)(2\gamma_2 + \mathfrak{r}), \tag{B1}$$

where γ_1 (resp. γ_2) is the class of the component C_1' of genus i (resp. C_2' of genus $g-i$) over $\Delta_i \cap B$ and $\mathfrak{r}$ the class of R.

If we denote by T the closure of the reduced image of v^*W under the $(2:1)$ map $\varphi' : X' \to \mathbb{P}(\mathbb{E}_B)$, then $[G_1] = \gamma([T])$. Recall that $\varphi'^*h = \omega_{\pi'} - R$. Thus, the δ_i-contribution in $2[T]$ coming from w_1 is

$$\frac{g(g+1)}{2}\varphi'_*(\omega_{\pi'}) = \frac{g(g+1)}{2}\varphi'_*({\varphi'}^*h + \mathfrak{r}) = \frac{g(g+1)}{2}\left(h\varphi'_*[1] + \varphi'_*\mathfrak{r}\right).$$

If we apply (1), the contribution to δ_i in

$$\gamma(h{\varphi'}_*[1]) = \gamma\left(\sum_{i=0}^{g-2}h^{i+1}u^*\beta_{g-2-i}\right) = 2(g-1)(\check{h} + \check{u}^*\lambda) + \check{u}^*\beta_1$$

comes from $\check{u}^*\beta_1$ alone and equals $-4\check{u}^*\delta_i$, as δ_i appears in the formula of β_1 with coefficient -2. Since $\gamma(\varphi'_*\mathfrak{r}) = 2\,\check{u}^*\delta_i$ we get from w_1 together the contribution $-g(g+1)\check{u}^*\delta_i$. From w_2 we get by applying γ to (16), using $\gamma(\varphi'_*\gamma_1) = (2i-2)\check{u}^*\delta_i$, $\gamma(\varphi'_*\gamma_2) = (2g-2i-2)\check{u}^*\delta_i$ and $\gamma(\varphi'\mathfrak{r}) = 2\check{u}^*\delta_i$, the contribution $2(g+3)i(i-g) + g(g+1)$. Together $w_1 - w_2$ thus contributes $-2(g+3)i(i-g)$ to the coefficient of δ_i, as required. □

Proof of Theorem B.2. Here $k \geq 2$, hence we have the morphism $\varphi_k : \overline{C}_g \to \mathbb{P}(\mathbb{E}_k)$. Let $\pi_1 : \overline{C}_{g,1} \to \overline{C}_g$ be the universal curve over $\overline{C}_g$ and $s : \overline{C}_g \to \overline{C}_{g,1}$ the tautological section, the image of which we denote by S.

We claim: $\varphi_k^* F_k = \pi_{1*}(\omega_{\pi_1}^{\otimes k}(-S))$. Indeed, by our assumptions on g and k, we have $R^1\pi_{1*}(\omega_{\pi_1}^{\otimes k}(-S)) = 0$, so $\pi_{1*}(\omega_{\pi_1}^{\otimes k}(-S))$ is a vector bundle on $\overline{C}_g$. For a point $x \in \mathbb{P}(\mathbb{E}_k)$, the fiber of F_k is the hyperplane in the corresponding fiber of $\mathbb{E}_k$ representing the point x. When $x = \varphi_k(p)$ with $p \in \overline{C}_g$, then $(F_k)_x = H^0(C_p, \omega_{\pi_1}^{\otimes k}(-p))$, with C_p the corresponding fiber of π_1 over p. Hence the claim. We now have on $\overline{C}_g$ the sequence

$$0 \to \pi_{1*}(\omega_{\pi_1}^{\otimes k}(-2S)) \to \pi_{1*}(\omega_{\pi_1}^{\otimes k}(-S)) \to s^*(\omega_{\pi_1}^{\otimes k}(-S)) \to 0\,, \tag{B2}$$

with $s^*(\omega_{\pi_1}^{\otimes k}(-S)) \cong \omega_{\pi}^{k+1}$, and this sequence is exact up to codimension 2 because $R^1\pi_{1*}(\omega_{\pi_1}^{\otimes k}(-2S))$ vanishes in codimension 2 in view of the conditions on (g,k).

Let $F_k(\nu) = \pi_{1*}(\omega^{\otimes k}(-\nu S))$ for $\nu = 1,2$ with $F_k(1) \cong \varphi^* F_k$. Let $j : \mathbb{P}(F_k(1)^\vee) \to \mathbb{P}(F_k^\vee)$ be the natural map. Then, $\tilde{h} = j^*\check{\rho}^*(\check{h})$ is the class of the hyperplane line bundle on $\mathbb{P}(F_k(1)^\vee)$. The inclusion $F_k(2) \hookrightarrow F_k(1)$ induces a map $\sigma : \mathbb{P}(F_k(2)^\vee) \to \mathbb{P}(F_k(1)^\vee)$. We have the commutative diagram:

$$\begin{array}{ccccccc}
\mathbb{P}(F_k(2)^\vee) & \xrightarrow{\sigma} & \mathbb{P}(F_k(1)^\vee) & \xrightarrow{j} & I = \mathbb{P}(F_k^\vee) & \xrightarrow{\check{\rho}} & \mathbb{P}(\mathbb{E}_k^\vee) \\
 & \searrow & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \check{u}} \\
 & & \overline{C}_g & \xrightarrow{\varphi_k} & \mathbb{P}(\mathbb{E}_k) & \xrightarrow{u} & \overline{\mathcal{M}}_g
\end{array}$$

Let $\alpha = j\sigma : \mathbb{P}(F_k(2)^\vee) \to I$ and $A = \mathrm{Im}(\alpha)$ the image of this map. Then, $[Z_k] = \check{\rho}_*[A]$. By the exact sequence (17) we have $\sigma_*[\mathbb{P}(F_k(2)^\vee)] = \tilde{h} + (k+1)\rho_1^*\omega_\pi$. We observe that $j_*[1] = \rho^*[\overline{\Gamma}_k]$ and $k\,\omega_\pi = \varphi_k^* h$ and we find

$$\begin{aligned}
k[A] &= k\, j_*(j^*\check{\rho}^*\check{h}) + (k+1) j_*(\rho_1^*\phi_k^* h) = k\,\check{\rho}^*\check{h}\, j_*[1] + (k+1)\rho^*(h) j_*[1] \\
&= k\,\check{\rho}^*\check{h}\,\rho^*[\overline{\Gamma}_k] + (k+1)\rho^*(h[\overline{\Gamma}_k])\,.
\end{aligned}$$

By Proposition 3.1 and Lemma B.3, we have $\check{h}\gamma([\overline{\Gamma}_k]) = 2k(g-1)\check{h}$ and

$$\gamma(h[\overline{\Gamma}_k]) = \gamma\left(\sum_{i=0}^{r-2} h^{i+1} u^* \beta_{r-2-i}\right) = 2k(g-1)(\check{h} + \check{u}^* c_1(\mathbb{E}_k)) + \check{u}^*\beta_1 = 2k(g-1)\check{h} + k^2\check{u}^*\kappa_1$$

and thus $[Z_k] = 2(2k+1)(g-1)\check{h} + k(k+1)\check{u}^*\kappa_1$ in agreement with the formula of Theorem B.2. □