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The cycle class of the supersingular locus of principally polarized abelian varieties

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Abstract

We prove a formula for the cycle class of the supersingular locus in the Chow ring with rational coefficients of the moduli space of principally polarized abelian varieties of dimension g in characteristic p. This formula determines this class as a monomial in the Chern classes of the Hodge bundle up to a factor that is a polynomial in p. This factor is known for $g \leq 3$. We also determine the factor for $g = 4$.

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Contents

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1 Introduction

An abelian variety over a field k of characteristic $p > 0$ is called supersingular if it is isogenous to a product of supersingular elliptic curves over the algebraic closure of k. Equivalently, by [25, Thm. 4.2], if its formal isogeny type has a Newton polygon with all slopes equal to $1/2$. Recall that the Newton polygon of an abelian variety starts at $(0, 0)$, ends at $(2g, g)$ and is lower convex and satisfies a symmetry condition. The two extreme cases are the Newton polygon with slopes 0 and 1 and break point $(g, 0)$ (the ordinary case) and the Newton polygon with slope $1/2$ (the supersingular case).

Let $\mathcal{A}_g \otimes \mathbb{F}_p$ be the moduli space of principally polarized abelian varieties of dimension g in characteristic $p > 0$. The supersingular locus S_g is defined as the closed subset of principally polarized abelian varieties of $\mathcal{A}_g \otimes \mathbb{F}_p$ that are supersingular. This locus can be considered as the most degenerate stratum in the Newton polygon stratification on $\mathcal{A}_g \otimes \mathbb{F}_p$. Its dimension is known by Li and Oort to be $[g^2/4]$ and also the number of irreducible components is known, see below. Besides the Newton polygon stratification there is another stratification on $\mathcal{A}_g \otimes \mathbb{F}_p$, the Ekedahl-Oort stratification. While the cycle classes of the Ekedahl-Oort stratification on $\mathcal{A}_g \otimes \mathbb{F}_p$ are known, the cycle classes of the Newton polgon strata in general are not. For $g = 1$ and $g = 2$ the supersingular locus is a (closed) stratum of the Ekedahl-Oort stratification and thus the class is known. For $g = 3$ the supersingular locus is not a (closed) stratum of the Ekedahl-Oort stratification, but its cycle class was determined in joint work of the first author with Ekedahl, and the result was presented in [10].

In this paper we will prove a formula for the cycle class of the supersingular locus in the Chow ring with rational coefficients of a Faltings-Chai compactification $\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$. This formula determines this class as a monomial in the Chern classes of the Hodge bundle up to a factor that is a polynomial in p. This shows that this class lies in the tautological ring, a subring of the rational Chow ring of the moduli space $\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$, and is given by a beautiful formula that generalizes Deuring's famous formula for the number of supersingular elliptic curves.

Theorem 1.1 *The cycle class of the supersingular locus S_g in the Chow ring with rational coefficients of a Faltings-Chai compactification $\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$ of the moduli space $\mathcal{A}_g \otimes \mathbb{F}_p$ lies in the tautological ring. More precisely, it is of the form*

$$[S_g] = f_g(p) \begin{cases} \lambda_g \lambda_{g-2} \cdots \lambda_2 & g \text{ even}, \\ \lambda_g \lambda_{g-2} \cdots \lambda_1 & g \text{ odd}, \end{cases}$$

where $f_g(p)$ is a polynomial in p with rational coefficients and λ_i is the ith Chern class of the Hodge bundle on $\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$.

The method for proving this rests upon a way to translate conditions on the supersingularity of Dieudonné modules into degeneracy conditions of morphisms of bundles made from the Hodge bundle. We hope that this method can be used to show that all the cycle classes of the Newton polygon stratification lie in the tautological ring. In principle, the method should lead to the determination of the missing factor, but the details of nailing this down seem formidable.

We also determine the factor $f_g(p)$ in the formula for the cycle class of the supersingular locus for $g = 4$ by different methods. This determination builds upon the method used for the case of $g = 3$ and calculates the degree of a Chern class of the Hodge bundle on a component of the supersingular locus. For this we construct an explicit smooth model of each irreducible component of S_4. In addition, for completeness we give the proof for the class for $g = 3$ that was not published in [10]. Including the well-known results for $g = 1$ and $g = 2$ we arrive at the following theorem.

Theorem 1.2 *The cycle class of the supersingular locus S_g in the Chow ring with rational coefficients of a Faltings-Chai compactification $\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$ of the moduli space $\mathcal{A}_g \otimes \mathbb{F}_p$ for $g \leq 4$ is given by*

$$[S_g] = \begin{cases} (p-1)\,\lambda_1 & g{=}1 \\ (p-1)(p^2-1)\,\lambda_2 & g{=}2 \\ (p-1)^2(p^3-1)(p^4-1)\,\lambda_3\lambda_1 & g{=}3 \\ (p-1)^3(p^3-1)(p^4-1)(p^6-1)\,\lambda_4\lambda_2 & g{=}4. \end{cases}$$

We also discuss for $g = 3$ and $g = 4$ the loci in the supersingular locus where the a-number is at least 2.

2 The moduli space $\mathcal{A}_g \otimes \mathbb{F}_p$

By $\mathcal{A}_g$ we denote the moduli stack of principally polarized abelian varieties of dimension g and by $\pi : \mathcal{X}_g \to \mathcal{A}_g$ the universal abelian variety over $\mathcal{A}_g$. It is a Deligne-Mumford stack defined over $\mathbb{Z}$. The moduli space $\mathcal{A}_g$ carries a natural vector bundle $\mathbb{E}$ of rank g, the Hodge bundle, defined as $\pi_*\Omega^1_{\mathcal{X}_g/\mathcal{A}_g}$. We denote by $\tilde{\mathcal{A}}_g$ a Faltings-Chai compactification of $\mathcal{A}_g$ as defined and treated in [6]. The Hodge bundle extends to $\tilde{\mathcal{A}}_g$ and will again be denoted by $\mathbb{E}$.

In the rest of this paper we consider the moduli stack $\mathcal{A}_g \otimes \mathbb{F}_p$ in characteristic $p > 0$. Let $\mathrm{CH}^*_{\mathbb{Q}}(\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p)$ be the Chow ring with rational coefficients of $\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$. We set $\lambda_i = c_i(\mathbb{E}) \in \mathrm{CH}^i_{\mathbb{Q}}(\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p)$ for the ith Chern class of $\mathbb{E}$ for $i = 1, \ldots, g$, see [7, Ch. 3]. These classes satisfy the relation

$$(1 + \lambda_1 + \cdots + \lambda_g)(1 - \lambda_1 + \cdots + (-1)^g \lambda_g) = 1$$

and these classes generate a subring R_g of the Chow ring $\mathrm{CH}^*_{\mathbb{Q}}(\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p)$ called the tautological ring, see [5, 10]. For $0 \leq n \leq g(g+1)/2$ the graded part of R_g of degree n has a basis $\lambda_1^{e_1} \cdots \lambda_g^{e_g}$ with $0 \leq e_i \leq 1$ and $\sum_i e_i i = n$. The ring R_g is a Gorenstein ring with socle generated by $\lambda_1\lambda_2\cdots\lambda_g$. We will denote the degree of this 0-cycle by

$$v(g) = \deg \lambda_1\lambda_2\cdots\lambda_g\,,$$

the Hirzebruch proportionality constant, and we have

$$v(g) = (-1)^{g(g+1)/2}2^{-g}\zeta(-1)\zeta(-3)\cdots\zeta(1-2g),$$

where $\zeta(s)$ is the Riemann zeta function, see [10]. We give a little table with relevant values:

g	0	1	2	3	4
$v(g)$	1	1/24	1/5760	1/2903040	1/1393459200

The tautological ring of $\mathcal{A}_g \otimes \mathbb{F}_p$ is the quotient $R_g/(\lambda_g) \cong R_{g-1}$.

The moduli space $\mathcal{A}_g \otimes \mathbb{F}_p$ carries two important stratifications, the Ekedahl-Oort stratification and the Newton polygon stratification, see [26] and [27]. The strata of the Ekedahl-Oort stratification $\mathcal{V}_\mu$ are indexed by Young diagrams or tuples $\mu = [\mu_1, \ldots, \mu_r]$ of integers with $0 \leq r \leq g$ and $\mu_i > \mu_{i+1}$, according to the usage of [4, 10]. The largest open stratum $\mathcal{V}_{[\emptyset]}$ is the locus of ordinary abelian varieties. The codimension of $\mathcal{V}_\mu$ is $\sum_i \mu_i$. The stratification can be extended to $\tilde{\mathcal{A}}_g$.

By [4, 10] we can calculate the cycle classes of the closed Ekedahl-Oort strata in $\mathcal{A}_g \otimes \mathbb{F}_p$ and $\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$. For example the cycle class of the locus of abelian varieties with p-rank $\leq f$ (corresponding to $\mu = [g-f]$) is

$$[\overline{\mathcal{V}}_{[g-f]}] = (p-1)(p^2-1)\cdots(p^{g-f}-1)\lambda_{g-f} \tag{1}$$

and the cycle class of the smallest stratum, the locus of superspecial abelian varieties (corresponding to $\mu = [g, g-1, \ldots, 1]$) is

$$[\mathcal{V}_{[g,g-1,\ldots,1]}] = (p-1)(p^2+1)\cdots(p^g+(-1)^g)\lambda_1\lambda_2\cdots\lambda_g\,.$$

This formula implies as a special case a result of Ekedahl [3], namely that

$$\sum \frac{1}{\#\mathrm{Aut}(X)} = (p-1)(p^2+1)\cdots(p^g+(-1)^g)\,v(g)\,, \tag{2}$$

where the sum is over the isomorphism classes of principally polarized superspecial abelian varieties over $\overline{\mathbb{F}}_p$ and $v(g)$ the proportionality constant defined above. A formula for the actual number of isomorphism classes of superspecial abelian varieties with a level $n \geq 3$ structure is obtained by multiplying the formula for the degree of $\mathcal{V}_{[g,g-1,\ldots,1]}$ by the degree of the natural map $\mathcal{A}_g[n] \to \mathcal{A}_g$ (as stacks) with $\mathcal{A}_g[n]$ the moduli space of principally polarized abelian varieties with a level n structure.

3 Irreducible components of the supersingular locus

The number of irreducible components of the supersingular locus S_g in $\mathcal{A}_g \otimes \mathbb{F}_p$ was determined by Deuring for $g = 1$, by Katsura and Oort for $g = 2$ ([19, 20]) and in general by Li and Oort for $g \geq 3$, [22, 4.9]. The actual number of irreducible components in $\mathcal{A}_g \otimes \overline{\mathbb{F}}_p$ is given by a class number $h_p(g)$ for g odd and a similar class number $h'_p(g)$ for g even. Here $h_p(g)$ (resp. $h'_p(g)$) is the class number of the principal (resp. non-principal) genus in the hermitian space B^g, with B the definite quaternion algebra over $\mathbb{Q}$ ramified exactly at p and ∞. These class numbers are difficult to deal with, see for example [16, p. 147], and one gets better and more useful formulas by counting in a stacky way, that is, taking into account weights equal to the inverse of the order of the automorphism groups of the objects that one counts. For example, for $g = 1$ the class number of the quaternion algebra B over $\mathbb{Q}$ split outside p and ∞, is given by

$$h_p(1) = \frac{p-1}{12} + \left(1 - \left(\frac{-3}{p}\right)\right)\frac{1}{3} + \left(1 - \left(\frac{-4}{p}\right)\right)\frac{1}{4},$$

with the Legendre symbols. But a stacky interpretation of this number leads to the much more elegant formula

$$\sum \frac{1}{\#\mathrm{Aut}(E)} = \frac{p-1}{24}$$

with the summation over all isomorphism classes of supersingular elliptic curves defined over $\overline{\mathbb{F}}_p$.

We will denote by N_g the number of irreducible components of the supersingular locus in the stacky sense, that is, where each irreducible component is counted with a certain weight w related to the automorphism group as explained below.

This number N_g has the property that the number $N_g[n]$ of irreducible components of the supersingular locus on the moduli space $\mathcal{A}_g[n]$ with a level $n \geq 3$ structure with p prime to n equals

$$N_g[n] = N_g \cdot \deg(\mathcal{A}_g[n] \to \mathcal{A}_g)\,.$$

An irreducible component of the supersingular locus of $\mathcal{A}_g[n]$ is given by a triple (E^g, η, ν) with E a supersingular elliptic curve, η a polarization with kernel equal to the kernel $E^g[F^{g-1}]$ of F^{g-1} with F Frobenius and ν a level n structure, see [22] and the next section. Since p does not divide n, a level n structure on E^g does not interfere with the inseparable isogenies $E^g \to Y_0$ that give rise to the objects of an irreducible component and descends to a level n structure on Y_0. So we count such an irreducible component of the supersingular locus of $\mathcal{A}_g$ with weight $w = 1/\#\mathrm{Aut}(E^g, \eta)$.

Proposition 3.1 *The number N_g of irreducible components of the supersingular locus in $\mathcal{A}_g \otimes \overline{\mathbb{F}}_p$ (in the stacky sense) is*

$$\begin{cases} (p-1)(p^2+1)(p^3-1)\cdots(p^g-1)\,v(g) & \textit{for } g \textit{ odd},\\ (p^2-1)(p^6-1)\cdots(p^{2g-2}-1)\,v(g) & \textit{for } g \textit{ even}. \end{cases}$$

The stacky interpretation that we use reduces to the mass of the principal (resp. non-principal genus) and can be deduced from [3] or [13–15]. One finds this mass formula also in [9, p. 123] and in [28].

For odd g the irreducible components of S_g are in bijective correspondence with the isomorphism classes of superspecial principally polarized abelian varieties of dimension g, hence the formula for N_g follows immediately from Ekedahl's result (2). For even g one has a correction factor

$$\frac{(p+1)(p^3+1)\cdots(p^{g-1}+1)}{(p^2+1)(p^4+1)\cdots(p^g+1)}.$$

Here for g even the numerator can be interpreted as the number of totally isotropic subspaces of dimension $g/2$ in a g-dimensional hermitian space over $\mathbb{F}_{p^2}$ with conjugation given by Frobenius, while the denominator equals the number of totally isotropic subspaces of dimension $g/2$ in a symplectic space of dimension g over $\mathbb{F}_{p^2}$. See also the description in [28].

4 Flag type quotients

Work of Oda and Oort ([24]) makes it possible to parametrize the irreducible components of the supersingular locus S_g by so-called flag type quotients. For an abelian variety X over an algebraically closed field k of characteristic p we will denote the subgroup scheme $\ker(F) \cap \ker(V)$ by $A(X)$ with F and V Frobenius and Verschiebung on X. It is a subgroup scheme of order $p^{a(X)}$ with $a(X)$ the a-number of X. A supersingular abelian variety has $1 \leq a(X) \leq g$ and if $a(X) = g$ and $g \geq 2$ then X is isomorphic to the base change to k of a product E^g with E a supersingular elliptic curve defined over $\mathbb{F}_p$.

For a supersingular abelian variety X of dimension g the a-number tends to go up when one replaces X by $X/A(X)$, though it is not true that $a(X/A(X)) \geq \min(g, a(X)+1)$ as asserted in the proof of [22, 1.8 Lemma] that refers to [21]; see [17, Remark 3.17] for a counterexample. Nevertheless, by starting with $X = X_0$ and putting $X_{i+1} = X_i/A(X_i)$ one arrives after $g-1$ steps at a superspecial abelian variety X_{g-1}, that is, an abelian variety with with $a(X_{g-1}) = g$, as follows from [30, Lemma 9]. Then the kernel of the dual map is contained in $\ker(F^{g-1})$, hence one finds a homomorphism $Y \to X$ with $Y = X_{g-1}^{(p^{g-1})}$. This implies the fact that for a supersingular abelian variety X there exists a minimal isogeny $\rho : E^g \to X$ with E a supersingular elliptic curve with the property that any other homomorphism $h : Z \to X$ of a superspecial abelian variety Z factors uniquely through ρ. If $a(X) = 1$ this minimal isogeny is obtained in $g-1$ steps

$$Y_{g-1} \to Y_{g-2} \to \cdots \to Y_0 = X$$

where $Y_{g-1} = E^g \otimes \mathrm{Spec}(k)$ and $Y_i = Y_{g-1}/G_i$ for $i = 1, \ldots, g-1$ with $G_i = \ker(\rho) \cap Y_{g-1}[F^{g-1-i}]$. If $a(X) > 1$ this sequence needs not be unique. Taking into account also the polarizations leads to the definition of a (polarized) flag type quotient.

Definition 4.1 A polarized flag type quotient of dimension g is a diagram of abelian varieties and homomorphisms

$$\begin{tikzcd} Y_{g-1} \arrow[r, "\rho_{g-1}"] \arrow[d, "\eta_{g-1}"] & Y_{g-2} \arrow[r] \arrow[d, "\eta_{g-2}"] & \cdots \arrow[r, "\rho_1"] & Y_0 \arrow[d, "\eta_0"] \\ Y_{g-1}^t & Y_{g-2}^t \arrow[l] & \cdots \arrow[l] & Y_0^t \arrow[l] \end{tikzcd}$$

where Y_i^t is the dual of Y_i and the abelian variety Y_{g-1} is superspecial with η_{g-1} a polarization with kernel $Y_{g-1}[F^{g-1}]$ satisfying

(1) $\ker(\rho_i) \subset A(Y_i)$ is of order p^i;
(2) $\ker(\eta_i) \subseteq \ker(V^j \circ F^{i-j})$ for $0 \leq j \leq i/2$.

This flag type quotient is called rigid if $G_i = G_0 \cap G[F^{g-1-i}]$ with $G_0 = \ker(Y_{g-1} \to Y_0) \cap Y_{g-1}[F^{g-1}]$. The term 'rigid' refers to the fact that in this case the corresponding flag type is unique.

The main references for flag type quotients are [24] and [22, Sections 7,9.6,9.7].

5 Dieudonné modules and displays

The theory of Dieudonné modules makes it possible to describe flag type quotients in terms of Dieudonné modules.

Here k will denote an algebraically closed field of characteristic p and $W = W(k)$ the ring of Witt vectors of k. We define a ring

$$A = W[F, V]/(FV - p, VF - p, Fa - a^\sigma F, aV - Va^\sigma, \forall a \in W)$$

and set $A_{1,1} := A/(F - V)$.

A polarized flag type quotient as described in Definition 4.1 corresponds to a flag of contravariant Dieudonné modules

$$M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{g-1}$$

with dual modules M_i^t satisfying

(1) $M_{g-1} = A_{1,1}^g$ provided with a quasi-polarization

$$\langle \, , \, \rangle : M_{g-1} \otimes_W M_{g-1}^t \to Q(W) \, ,$$

with $Q(W)$ the field of quotients of W, that induces an identification $M_{g-1}^t = F^{g-1} M_{g-1}$;

(2) $(F, V)M_i \subset M_{i-1}$ and $\dim(M_i/M_{i-1}) = i$ for $i = 0, \ldots, g-1$;

(3) $(F,V)^i M_i \subset M_i^t$ for $i = 0, \ldots, g-1$.

We call such a flag a polarized Dieudonné flag of length g. It is called rigid if $M_i = M_0 + F^{g-1-i} M_{g-1}$ for $i = 0, \ldots, g-1$. We observe that rigidity implies

$$M_i = M_m + F^{g-1-i} M_{g-1} \quad \text{for } m < i \leq (g-1)\,.$$

We can translate rigid polarized flag type quotients in terms of displays, replacing Dieudonné modules by displays. We recall the definition of displays (cf. [29, Section 1]). Let R be a commutative unitary ring of characteristic p. Let $W(R)$ be the ring of Witt vectors. Let $\mathfrak{f} : W(R) \to W(R)$ be Frobenius and $\mathfrak{v} : W(R) \to W(R)$ Verschiebung. Set $I_R = \mathfrak{v}(W(R))$. A *display over* R is a quadruple (P, Q, F, V^{-1}) consisting of a finitely generated projective $W(R)$-module P, a $W(R)$-submodule Q of P and homomorphisms $F : P^{(p)} \to P$ and $V^{-1} : Q^{(p)} \to P$, where $M^{(p)} := W(R) \otimes_{\mathfrak{f}, W(R)} M$, with the properties:

(i) $I_R P \subset Q \subset P$ and there exists a decomposition of P into a direct sum of $W(R)$-modules $P = L \oplus T$, such that $Q = L \oplus I_R T$;
(ii) V^{-1} is an epimorphism;
(iii) For $x \in P$ and $w \in W(R)$ we have $V^{-1}(1 \otimes \mathfrak{v}(w)x) = wFx$.

By [29, Lemma 9], we have an isomorphism

$$V^{-1} \oplus F : (L \oplus T)^{(p)} \to P. \tag{3}$$

The matrix (with respect to a basis of P) associated to this isomorphism is a generalization of the classical display ([23]).

Remark 5.1 If R is a perfect field, then P is the usual Dieudonné module, $I_R = p\,W(R)$ and Q is the V-image VP, so Q is determined by the Dieudonné module P. But if R is not a perfect field, then Q is not determined by P together with F, V; conversely P is determined by the V^{-1}-image of $Q^{(p)}$.

By a result of Li-Oort [22, 3.7] the moduli space of polarized Dieudonné flags of length g exists and is projective. Moreover, by [22, 3.7] the moduli of rigid polarized Dieudonné flags of length g exists and is quasi-projective, and by [22, 7.6] it is nonsingular.

6 The cycle class of the supersingular locus

In this section we will show that the cycle class of the supersingular locus S_g in $\mathcal{A}_g \otimes \mathbb{F}_p$ lies in the tautological ring R_g generated by the Chern classes λ_i $(i = 1, \ldots, g)$ of the Hodge bundle $\mathbb{E}$ on a Faltings-Chai compactification of $\mathcal{A}_g \otimes \mathbb{F}_p$ and give a formula for it that fixes the class up to a multiplicative constant.

Here the cycle class is taken in the Chow ring with rational coefficients of a Faltings-Chai compactification $\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$ of $\mathcal{A}_g \otimes \mathbb{F}_p$.

Theorem 6.1 *The cycle class of the supersingular locus on $\mathcal{A}_g \otimes \mathbb{F}_p$ in $\mathrm{CH}^*_{\mathbb{Q}}(\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p)$ is a non-zero multiple of $\lambda_g\lambda_{g-2}\cdots\lambda_1$ if g is odd and of $\lambda_g\lambda_{g-2}\cdots\lambda_2$ if g is even. The multiple is a polynomial in p with rational coefficients.*

Before we give the details of the proof we describe the set-up. For the proof we will use the presentation of Frobenius on the covariant Dieudonné module M of p-rank 0 and a-number 1 with a principal quasi-polarization $\langle\ ,\ \rangle$ as given by Oort in [27]. His description of the display of such a module M is as follows. With W the Witt ring of k, an algebraically closed field of characteristic $p > 0$, there exists a W-basis $e_1, \ldots, e_g, e_{g+1}, \ldots, e_{2g}$ that is symplectic (meaning that $\langle e_i, e_j\rangle = 0$ for $i, j \leq g$ and for $i, j > g$ and $\langle e_i, e_{g+j}\rangle = \delta_{ij}$ for $1 \leq i, j \leq g$) such that Frobenius is given by the formulas

$$
\begin{aligned}
F e_j &= \sum_{i=1}^{2g} \gamma_{ij} e_i, \quad (1 \leq j \leq g)\,,\\
e_j &= V(\sum_{i=1}^{2g} \gamma_{ij} e_i)\,, \quad (g+1 \leq j \leq 2g)\,,
\end{aligned}
$$

where $\gamma = (\gamma_{ij})$ is a W-valued $2g \times 2g$ matrix which is symplectic in the sense that

$$
\gamma \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} \gamma^t = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}.
$$

We write γ as a matrix of $g \times g$ blocks

$$
\gamma = (\gamma_{ij}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
$$

We denote the Frobenius endomorphism of the Witt ring W by σ. Note that the σ-linear map F is given by the matrix

$$
\begin{pmatrix} a & pb \\ c & pd \end{pmatrix}.
$$

Oort shows ([27, p. 191] that if M has p-rank 0 and a-number 1 we may choose the basis such that the matrix γ is of the form (called normal)

$$
a_{ij} = d_{ij} = \begin{cases} 1 & i = j+1, \\ 0 & i \neq j+1, \end{cases} \quad c_{ij} = \begin{cases} 1 & (i,j) = (1,g) \\ 0 & else \end{cases},
$$

and $b_{ig} = 0$ for $i \neq 1$. In particular, since we assume p-rank zero we have $a_{ig} = 0$ for $i = 1, \ldots, g$, see [27, page 191] after Lemma 2.2.

Lemma 6.2 *For a normal form γ we have $\gamma_{i,2g} = 0$ for $i = 2, \ldots, g$ and $\gamma_{1,j} = 0$ for $j = g+1, \ldots, 2g-1$ and $\gamma_{1,2g} = -1$. Moreover, the square matrix*

$$\tilde{\gamma} = \begin{pmatrix} \gamma_{2,g+1} & \cdots & \gamma_{2,2g-1} \\ \vdots & & \vdots \\ \gamma_{g,g+1} & \cdots & \gamma_{g,2g-1} \end{pmatrix}$$

is symmetric.

Proof We have $ab^t = ba^t$ and $b^t d = d^t b$. In view of the shape of the matrices a and d the result follows as γ is symplectic. □

We now change this normal form into a so-called strong normal form as follows. We can take γ_{ij} as a Teichmüller lift for $i \neq g$ and $j \neq 2g-1$, after changing the basis $\{e_i\}$ of M. Now we consider the $\gamma_{i,j}$ only for $2 \leq i < g$ and $g < j < 2g-1$, as the others are kept as Teichmüller lifts under the following operation. Let $t_{ij} := (\gamma_{i,j} - [\overline{\gamma_{i,j}}])/p$, where $[u]$ denotes the Teichmüller lift of $u \in k$. We replace e_{j+1} by $e_{j+1} + p\, t_{ij} e_i$ and e_{g+i} by $e_{g+i} + p\, t_{ij} e_{j+1-g}$. After the change this new basis is still symplectic and the new $\gamma_{i,j}$ becomes the Teichmüller lift $[\overline{\gamma_{ij}}]$ and the new $\gamma_{i+1,j+1}$ becomes $\gamma_{i+1,j+1} + t_{ij}^{\sigma}$; by symmetry (Lemma 6.2), similar things hold for $\gamma_{j+1-g,g+i-1}$ and $\gamma_{j+2-g,g+i}$; at the same time the other new $\gamma_{i'j'}$ do not change.

By carrying out this operation going from lower $i+j$ to higher, we get the desired the basis. We call such (γ_{ij}) *a strong normal form*.

Given such a basis in strong normal form, we have according to [27, Lemma 2.6] that there exists an element $P \in A$ such that

$$F^{2g} e_1 = P e_1 \quad \text{with} \quad P = \sum_{i=1}^{g} \sum_{j=g}^{2g} p^{j-g} \gamma_{ij}^{\sigma^{2g-j}} F^{2g+i-j-1}, \tag{4}$$

with $Fx = x^{\sigma} F$ for $x \in W$ and repeated application of F is in the σ-linear sense (cf. [27, p. 195]).

Remark 6.3 We know that the Ekedahl-Oort stratum $\mathcal{V}_{\mu}$ with $\mu = [g, 1]$ corresponding to p-rank 0 and a-number 2 has codimension 1 in the p-rank 0 locus V_0, hence the generic point of every irreducible component of V_0 has $a = 1$. Moreover, by the results of Li-Oort [22] we know that each irreducible component of the supersingular locus S_g has an open dense subset where the a-number equals 1.

One can read off supersingularity from the matrix $\tilde{\gamma}$ in strong normal form using Oort's result on the action of F on e_1 given in (4), see [27, Cor. 2.8].

Corollary 6.4 *Let γ be the matrix in strong normal form for the module M. Then the module M is supersingular if $\gamma_{ij} \equiv 0 \pmod{p}$ for $2 \leq i \leq g-1$, $g+1 \leq j \leq 2g-2$ with $i+j \leq 2g$. Equivalently, since γ_{ij} is a Teichmüller lift, if $\gamma_{ij} = 0$ for $2 \leq i \leq g-1$, $g+1 \leq j \leq 2g-2$ with $i+j \leq 2g$.*

Note that because of the symmetry this gives a priori

$$\sum_{j=1}^{\lfloor g/2\rfloor}(g-2j)=\frac{g(g-1)}{2}-\left[\frac{g^2}{4}\right]=\dim V_0-\dim S_g$$

conditions for supersingularity, where V_0 is the p-rank zero locus.

We now begin the proof of Theorem 6.1.

Proof The strategy is now to impose consecutively conditions that together imply supersingularity by Corollary 6.4, where we assume that γ is in strong normal form; we begin by requiring the vanishing modulo p of the column of entries that is the transpose of

$$(\gamma_{2,g+1},\ldots,\gamma_{g-1,g+1}),$$

and continue by requiring the vanishing modulo p of the column of entries whose transpose is

$$(\gamma_{3,g+2},\ldots,\gamma_{g-2,g+2}),$$

and so on, till finally the column with transpose $(\gamma_{g/2,3g/2-1},\gamma_{g/2+1,3g/2-1})$ of length 2 for g even or the vanishing of the single entry $\gamma_{(g+1)/2,(3g-1)/2}$ for g odd.

For example, for $g=5$ we require the vanishing modulo p of the red entries in the symmetric matrix

$$\tilde{\gamma}=\begin{pmatrix}\gamma_{26} & \gamma_{27} & \gamma_{28} & \gamma_{29}\\ \gamma_{36} & \gamma_{37} & \gamma_{38} & \gamma_{39}\\ \gamma_{46} & \gamma_{47} & \gamma_{48} & \gamma_{49}\\ \gamma_{56} & \gamma_{57} & \gamma_{58} & \gamma_{59}\end{pmatrix}$$

giving 4 conditions.

In terms of displays, we have an $\mathfrak{f}$-linear map $V^{-1}\oplus F: M=L\oplus T\to M$, see (3). We write F/p for the composition

$$VM/pM\to M/pM\xrightarrow{V^{-1}\oplus F}M/pM\to M/VM\,.$$

This map is given by the square matrix $(\gamma_{ij})_{1\leq i\leq g,g+1\leq j\leq 2g}$. Then by the vanishing indicated in Lemma 6.2 we may restrict to submodules of rank $g-1$ generated by $g-1$ consecutive generators in VM/pM and M/VM:

$$G=\langle e_{g+1},e_{g+2},\ldots,e_{2g-1}\rangle\longrightarrow H=\langle e_2,e_3,\ldots,e_g\rangle\,.$$

We have increasing filtrations for $i=1,\ldots,g-1$ of G and H given by

$$G_i=\langle e_{g+1},e_{g+2},\ldots,e_{g+i}\rangle\quad\text{and}\quad H_i=\langle e_2,e_3,\ldots,e_{i+1}\rangle\,.$$

That the p-rank is zero means that the image of G_{g-1} is in H_{g-1}. If we identify $\mathrm{Lie}(X)$ with VM/pM for the abelian variety X corresponding to the dual of M (cf. [1, 4.3.12] and [22, 5.4, 7.4]), we can view the induced map $F/p: G_{g-1}\to H_{g-1}$ as a symmetric morphism between vector bundles of rank $g-1$ made from the Hodge bundle and its dual by Frobenius twists. Since we wish to have the filtrations we will have to work on a cover of the p-rank zero locus V_0.

We now consider $G\langle 1\rangle$, the module generated by e_{g+1}. We require that it maps to zero modulo p under $F/p : G\langle 1\rangle \to H\langle 1\rangle$ with the module $H\langle i\rangle$ generated by $e_{i+1}, \ldots, e_{g-i}$. We can view the semi-linear map $G\langle 1\rangle \to H\langle 1\rangle$ defined by F/p modulo p as a morphism of a line bundle to a vector bundle of rank $g-2$, where these bundles are made from the Hodge bundle by truncations and Frobenius twists. We consider the locus where this morphism vanishes. The vanishing of this morphism corresponds to the vanishing modulo p of the vector $(\gamma_{2,g+1}, \ldots, \gamma_{g-1,g+1})$.

If this morphism vanishes then by the symmetry $\gamma_{2,g+2}$ vanishes modulo p and we can consider a morphism $G\langle 2\rangle \to H\langle 2\rangle$ induced by F/p with $G\langle j\rangle = G_j/G_{j-1}$ generated by e_{g+j} and require its vanishing modulo p. By induction, assuming the vanishing modulo p of the semi-linear morphism

$$G\langle j\rangle \longrightarrow H\langle j\rangle \tag{5}$$

for $j = 1, \ldots, s$, we get a next morphism $G\langle s+1\rangle \to H\langle s+1\rangle$. We require inductively that these morphisms vanish for $j = 1, \ldots, [(g-1)/2]$ on an appropriate covering space of V_0 where we have the filtrations. Supersingularity follows if the conditions that the induced map $G\langle j\rangle \to H\langle j\rangle$ is zero are satisfied successively for $j = 1, \ldots, [(g-1)/2]$.

The locus where the morphism (5) vanishes has cycle class expressed in the Chern classes of $G\langle j\rangle$ and $H\langle j\rangle$; for example for $j = 1$ the cycle class is the $(g-2)$th Chern class of the dual of $G\langle 1\rangle \otimes (H\langle 1\rangle)^\vee$.

We now work on the space of flags $\mathfrak{F}$ on the cohomology H^1_{dR} of the universal principally polarized abelian variety as introduced in [4, Section 3]. The de Rham cohomology sheaf $\mathcal{H}^1_{\mathrm{dR}}(X/S)$ for a principally abelian variety $X \to S$ is a locally free sheaf $\mathbb{H}$ of rank $2g$ on S fitting in an exact sequence

$$0 \to \mathbb{E} \to \mathbb{H} \to \mathbb{E}^\vee \to 0\,.$$

The flags in question are complete symplectic flags on $\mathbb{H}$ extending flags $\mathbb{E}(i)$ on the Hodge bundle with $\mathrm{rank}(\mathbb{E}(i)) = i$ for $i = 1, \ldots, g$. These flags on the de Rham cohomology sheaf $\mathbb{H}$ satisfy $\mathbb{E}(g+i) = \mathbb{E}(g-i)^\perp$ and thus are determined by the flag on $\mathbb{E}$. This flag space is a stratified space with strata indexed by elements of the Weyl group of the symplectic group. The stratum corresponding to the longest so-called final element (or Kostant element) of the Weyl group (see [4, Section 3]) parametrizes flags compatible with the action of V and F. Its closure contains the final stratum lying over the p-rank zero locus V_0.

Thus we work on the closure of the final stratum $\mathfrak{F}_w$ of $\mathfrak{F}$ corresponding to p-rank zero. This stratum allows a morphism that is generically finite to V_0. The symplectic flags over a generic point of V_0 are compatible with the action of V and F and also compatible with the filtration defined by the basis used in the description by Oort of the display given above.

We can view the induced map $F/p : G_{g-1} \to H_{g-1}$ as a symmetric morphism between modules of rank $g-1$ that induces a morphism of vector bundles $G\langle 1\rangle \to H\langle 1\rangle$ on $\mathfrak{F}_w$. The vector bundles induced by G and H have filtrations whose graded quotients are Frobenius twists of of the graded quotients of the Hodge bundle $\mathbb{E}(i)/\mathbb{E}(i-1)$ or their duals. Therefore the Chern classes of their graded quotients are of the form $\pm p^{r_i}\ell_i$ where $\ell_i = c_1(\mathbb{E}(i)/\mathbb{E}(i-1))$ $(i = 1, \dots, g)$ are the Chern classes of the graded quotients of the Hodge bundle on the final stratum and $r_i \in \mathbb{Z}$.

The conditions on the vanishing modulo p of rows of entries can now be viewed as a degeneracy condition for a morphism between vector bundles on $\mathfrak{F}_w$.

We shall calculate the cycle class of the Zariski closure of the degeneracy locus of this map over the open part of V_0 where $a = 1$. Note that on the open stratum $\mathcal{F}_w$ we have $a = 1$. This Zariski closure is contained in the supersingular locus as the Newton polygon can only go up under specialization. Moreover, for $g \geq 2$ each irreducible component of S_g has an open dense set with $a = 1$, hence intersects the degeneracy locus over V_0.

We know that the codimension of the degeneracy locus equals the number of conditions imposed by Corollary 6.4 in the supersingular case, hence also for the intermediate cases defined by the vanishing of $G\langle j\rangle \to H\langle j\rangle$. The theory of degeneracy loci [8] tells us that the cycle classes of these degeneracy loci on $\mathfrak{F}_w$ are polynomials in the classes ℓ_i.

To calculate these, we begin by remarking that the cycle class of the p-rank zero locus V_0 in $\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$ is a multiple of λ_g by [4]. We carry out induction and assume that the image under the Gysin map from $\mathfrak{F}_w$ to $\mathcal{A}_g \otimes \mathbb{F}_p$ of the class of the locus over V_0 where F/p maps $G\langle s\rangle$ to zero in $H\langle s\rangle$ for $s = 1, \dots, j-1$ is a multiple of $\lambda_g \lambda_{g-2} \cdots \lambda_{g+2-2j}$.

The locus where the morphism $G\langle j\rangle \to H\langle j\rangle$ is zero has as cycle class the $(g-2j)$th Chern class of the dual of $G\langle j\rangle \otimes (H\langle j\rangle)^\vee$. With $r = g - 2j = \mathrm{rank}(H\langle j\rangle)$ this Chern class is

$$(-1)^r (c_r(H\langle j\rangle) - c_{r-1}(H\langle j\rangle) c_1(G\langle j\rangle)) .$$

In order to calculate the class of the corresponding locus on $\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$ we have to apply a Gysin map from the Chow group of $\mathfrak{F}_w$ to the Chow group of $\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$ and calculate the image of the class of the degeneracy locus.

We first look at the case $j = 1$.

Lemma 6.5 *The pushdowns to $\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$ of the classes $c_{g-2}(H\langle 1\rangle)$ and $c_{g-3}(H\langle 1\rangle)c_1(G\langle 1\rangle)$ on $\mathfrak{F}_w$ are multiples of λ_{g-2}.*

Proof The filtration on $\mathbb{E}$ is extended to the de Rham bundle by $\mathbb{E}_{g+i} = (\mathbb{E}_{g-i})^\perp$ as in [4, Section 3]. This symplectic pairing is different from the one used in the description of the display in [27]. Since we use covariant Dieudonné modules we have to take duals and Frobenius twists to relate the Chern roots of $G\langle j\rangle$ and $H\langle j\rangle$ to those of the Hodge bundle. The Chern roots of $G\langle j\rangle$ and $H\langle j\rangle$ are determined by the filtrations G_i and H_i. We write l_i for these roots, while writing ℓ_i for the roots of $\mathbb{E}$. Then the Chern roots of $H\langle 1\rangle$ given by this filtration are $l_2, \dots, l_{g-1}$ and that of $G\langle 1\rangle$ is $-l_1$.

The Chern class $c_{g-2}(H\langle 1\rangle)$ is then the $(g-2)$th elementary symmetric function in $l_2, \ldots, l_{g-1}$. The $(g-2)$th symmetric function in $l_2, \ldots, l_{g-1}$ is a Frobenius twist of the $(g-2)$th symmetric function in $l_1, \ldots, l_{g-2}$ (cf. the proof of [4, Lemma 12.3]) and is a multiple of $\lambda_{g-2}(g-2) = c_{g-2}(\mathbb{E}(g-2))$. Now by [4, Lemma 12.3] the pushdown of $\lambda_{g-2}(g-2)$ equals a non-zero multiple of λ_{g-2}. The morphism from $\mathfrak{F}$ to $\mathcal{A}_g \otimes \mathbb{F}_p$ is fibered by generically finite morphisms π_i defined by forgetting a step of the flag $\mathbb{E}(i) \subsetneq \mathbb{E}(i+1) \subsetneq \cdots \subsetneq \mathbb{E}(g)$. We have for the Chern classes $\lambda_r(i) = c_r(\mathbb{E}(i))$ of the partial flag the formula $(\pi_i)^*(\lambda_r(i+1)) = \ell_{i+1}\lambda_{r-1}(i) + \lambda_r(i)$. For the Chern roots l_i that we use here a similar formula holds. Therefore, again by [4, Lemma 12.3], the pushdown of $c_{g-3}(H\langle 1\rangle)c_1(G\langle 1\rangle)$ is also a multiple of λ_{g-2}. □

We conclude that the class of the locus where $G\langle 1\rangle \to H\langle 1\rangle$ vanishes on V_0 is a multiple of the class λ_{g-2} on V_0. Since this is a Chern class of a vector bundle on $\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$ and the class of V_0 is a multiple of λ_g we find that the class of the vanishing locus in $\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$ of this bundle morphism on V_0 is a multiple of $\lambda_g\lambda_{g-2}$.

We now carry out induction. We restrict to the locus Z where the consecutive morphisms $G\langle s\rangle \to H\langle s\rangle$ for $s = 1, \ldots, j-1$ vanish. Then the class of the locus of vanishing of (5) equals up to a sign the $(g-2j)$th Chern class of $G\langle j\rangle \otimes (H\langle j\rangle)^\vee$ and this is $c_{g-2j}(H\langle j\rangle) - c_{g-2j-1}(H\langle j\rangle)c_1(G\langle j\rangle)$. By the argument given in Lemma 6.5 the class $c_{g-2j}(H\langle j\rangle)$ is a non-zero multiple of the $(g-2j)$th elementary symmetric function in $g-2j$ consecutive classes ℓ_i. We can view this as obtained by applying a Frobenius power to $\ell_1, \ldots, \ell_{g-2j}$, or use [4, Lemma 12.3], hence this elementary symmetric function represents a multiple of $\lambda_{g-2j}(g-2j)$. The pushdown of this is a multiple of λ_{g-2j}. The argument for $c_{g-2j-1}(H\langle j\rangle)c_1(G\langle j\rangle)$ is similar, as in Lemma 6.5. Therefore the cycle class of the vanishing locus is a multiple of λ_{g-2j} on the locus Z and Z has as class a multiple of $\lambda_g\lambda_{g-2}\cdots\lambda_{2g+2-2j}$. As λ_{g-2j} is the Chern class of a vector bundle on $\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$ we find as cycle class on $\tilde{\mathcal{A}}_g \otimes \mathbb{F}_p$ a multiple of $\lambda_g\lambda_{g-2}\cdots\lambda_{2g-2j}$.

By induction we may assume that the class has as coefficient a polynomial in p with rational coefficients as this is true for the class of V_0. By the formula for the Chern class of $G\langle j\rangle \otimes (H\langle j\rangle)^\vee$ and the fact that under the Gysin map no denominators are introduced we see that the coefficient is a polynomial in p. This finishes the proof of Theorem 6.1. □

Remark 6.6 i) By analyzing more precisely the characteristic classes of the degeneracy loci in the proof, it should be possible to determine the multiple $f(p)$ as a polynomial in p, but this involves many subtleties. ii) By interpreting Newton polygon strata contained in the p-rank zero locus as degeneracy loci as done in the proof of Theorem 6.1 we saw that the cycle classes of these loci lie in the tautological ring. This suggests that all Newton polygon classes are tautological.

7 Moduli of flag type quotients for $g = 3$

In this section and the next we calculate the cycle class of the supersingular locus S_3. We consider an irreducible component of the space of polarized flags of Dieudonné modules for $g = 3$, defined by the choice of a quasi-polarization on $A_{1,1}^3$. This space is the Zariski closure of the moduli of rigid polarized Dieudonné flags. A description was given in [22, p. 58]. Thus we look at polarized flags $(E^3, \eta) = (Y_2, \eta) \xrightarrow{\rho_2} (Y_1, \eta_1) \xrightarrow{\rho_1} (Y_0, \eta_0)$ corresponding to a polarized flag of Dieudonné modules

$$M_0 \subset M_1 \subset M_2 = A_{1,1}^3 = A\langle x, y, z\rangle$$

with the quasi-polarization given by

$$\langle x, Fx\rangle = \langle y, Fy\rangle = \langle z, Fz\rangle = 1/p\,.$$

Since $FM_2 \subset M_1$ with $\dim(M_1/FM_2) = 1$ the module M_1 is determined by a 1-dimensional subspace of M_2/FM_2, say generated by a vector $v = ax + by + cz$. The condition $(F, V)M_1 \subset M_1^t$ requires $\langle v, Fv\rangle \in W$, that is, if we view the coefficients a, b, c as elements of k, the condition $(F, V)M_1 \subset M_1^t$ is satisfied if and only if

$$a^{p+1} + b^{p+1} + c^{p+1} = 0\,.$$

Thus the moduli space $\mathcal{F}_1$ of truncated flags $M_1 \subset M_2$ can be identified with a Fermat curve $\mathcal{X}_{p+1} \subset \mathbb{P}^2 = \mathrm{Gr}(1, 3)$ (when using Dieudonné modules). The module M_0 is determined by a 2-dimensional subspace $M_0/FM_1 \subset M_1/FM_1$. Assuming rigidity, we see that it is spanned by two vectors

$$w_1 = v_0, \qquad w_2 = \alpha Fx + \beta Fy + \gamma Fz\,,$$

and the condition $M_0 \subseteq M_0^t$ gives $a\alpha + b\beta + c\gamma = 0$. This implies that M_1/M_0 defines a sheaf isomorphic to $\mathcal{O}_{\mathcal{F}_1}(1)$. Moreover, the degree p^2 homomorphism

$$\eta_1 : Y_1 \to Y_0 \xrightarrow{\sim} Y_0^t \to Y_1^t$$

shows that M_1/M_1^t is self dual, and it defines a locally free sheaf isomorphic to $\mathcal{O}_{\mathcal{F}_1}(1) \oplus \mathcal{O}_{\mathcal{F}_1}(-1)$.

This implies that the moduli space of rigid polarized Dieudonné flags with given quasi-polarization η admits a structure

$$\mathcal{F}_0^0 \to \mathcal{F}_1 \to \mathcal{F}_2 = \text{point}$$

with $\mathcal{F}_0^0$ the open dense part of the $\mathbb{P}^1$-bundle $\mathcal{F}_0 = \mathbb{P}(\mathcal{O}_{\mathcal{F}_1}(1) \oplus \mathcal{O}_{\mathcal{F}_1}(-1))$ that is the complement of the unique section with negative self-intersection number. The Zariski closure is obtained by taking the full $\mathbb{P}^1$-bundle $\mathcal{F}_0$.

The morphism $\mathcal{F}_0 \to S_3 \subset \mathcal{A}_3 \otimes \mathbb{F}_p$ is of finite degree onto its image, and the image forms an irreducible component of S_3. The degree equals $\#\mathrm{Aut}(E^3, \eta)/\{\pm 1\}$, but we may consider instead of $\mathcal{F}_0$ the stack by dividing $\mathcal{F}_0$ through the action of $\mathrm{Aut}(E^3, \eta)$ and then have a morphism of degree 1. This is what we shall do. The natural morphism to $\mathcal{A}_3 \otimes \mathbb{F}_p$ contracts the section.

8 The cycle class of S_3

Here we give the proof of the formula for the cycle class of S_3 stated in [10, Thm. 11.3]. The first author learned from Ekedahl at that time how to calculate the Hodge bundle for flag type quotients. Ekedahl employed this in [2, Cor. 3.4]. This idea was used in [10] to calculate the cycle class of S_3. As done at the time of [10], here we will not use the results of Section 6.

The Chow rings with rational coefficients of $\mathcal{A}_3 \otimes \mathbb{F}_p$ and $\tilde{\mathcal{A}}_3 \otimes \mathbb{F}_p$ are known by [11]. The ring $\mathrm{CH}^*_{\mathbb{Q}}(\tilde{\mathcal{A}}_3 \otimes \mathbb{F}_p)$ is generated by the Chern classes of the Hodge bundle and boundary classes σ_1 and σ_2. A priori the class of S_3 is a linear combination of the generators of $\mathrm{CH}^4_{\mathbb{Q}}(\tilde{\mathcal{A}}_3 \otimes \mathbb{F}_p)$, viz. λ_1^4, $\lambda_1^3\sigma_1$, $\lambda_1^2\sigma_1^2$ and $\lambda_1\sigma_1\sigma_2$, see [11]. But since $S_3 \cdot \sigma_1^2 = 0 = S_3 \cdot \sigma_2$ we see from the multiplication table 3f in [11, p. 765] that the class of S_3 is a multiple of $\lambda_1^4 = 8\,\lambda_1\lambda_3$. Alternatively, this follows from the fact that S_3 is contained in V_0, the p-rank 0 locus, whose class is a multiple of λ_3.

Theorem 8.1 *The class of the supersingular locus for genus* 3 *in the Chow ring with rational coefficients of a Faltings-Chai compactification of $\mathcal{A}_3 \otimes \mathbb{F}_p$ is given by*

$$[S_3] = (p-1)^2(p^3-1)(p^4-1)\,\lambda_1\lambda_3\,.$$

Proof The class $[S_3]$ is a multiple of $\lambda_1\lambda_3$ and the multiple can be determined by calculating the intersection number with λ_2. Using the flag type quotients we see above that an irreducible component of the supersingular locus S_3 in $\mathcal{A}_3 \otimes \overline{\mathbb{F}}_p$ is the image of a surface $\mathcal{F}_0$ under a map $\mathcal{F}_0 \to \mathcal{A}_3 \otimes \overline{\mathbb{F}}_p$ of degree $\#\mathrm{Aut}(E^3, \eta)/\{\pm 1\}$ (or degree 1 if we consider the corresponding stack) and $\mathcal{F}_0$ is of the form

$$\mathcal{F}_0 \xrightarrow{\pi_0} \mathcal{F}_1 \xrightarrow{\pi_1} \mathcal{F}_2 = \text{point}\,,$$

where $\mathcal{F}_i$ parametrizes partial flag type quotients $Y_2 \to \cdots \to Y_i$. More precisely, a component of S_3 is the image under a morphism of a $\mathbb{P}^1$-bundle $\mathcal{B} = \mathcal{F}_0$ over the Fermat curve $\mathcal{F}_1 = \mathcal{X}_{p+1}$ of degree $p+1$ in $\mathbb{P}^2$ that blows down the unique section S with negative self-intersection number of the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(-1))$ over $\mathcal{X}_{p+1}$. A point of $\mathcal{F}_1$ corresponds to the choice of a subgroup scheme α_p^2 in $E^3[F]$.

If we use contravariant Dieudonné modules over a geometric point of $\mathcal{F}_i$ we have for $i = 0$ and $i = 1$ an exact squence

$$0 \to pM_{i+1}/pM_i \to VM_i/pM_i \to VM_{i+1}/pM_{i+1} \to VM_{i+1}/VM_i \to 0.$$

Over $\mathcal{F}_i$, we can identify $\mathrm{Lie}(Y_i)^\vee$ with VM_i/pM_i (cf. [1, 4.3.12] and [22, 5.4, 7.4]), more precisely with $\mathrm{Q}_i/I_{\mathcal{O}_{\mathcal{F}_i}}\mathrm{P}_i$, where $(\mathrm{P}_i, \mathrm{Q}_i, F, V^{-1})$ is the display associated to Y_i. (Note that Q_i and $I_{\mathcal{O}_{\mathcal{F}_i}}\mathrm{P}_i$ become VM_i and pM_i respectively if we pull them back to the spectrum of a perfect field.) By the exact sequence we have in the Grothendieck group $K_0(\mathcal{F}_i)$ the relation

$$\mathrm{Lie}(Y_i)^\vee = \mathrm{Lie}(Y_{i+1})^\vee - Q_i + Q_i^{(p)}$$

with Q_i the locally free $\mathcal{O}_{\mathcal{F}_1}$-module defined by VM_{i+1}/VM_i. Here $\mathrm{Lie}(Y_{i+1})$ denotes the pull back under π_i. We pull back the relation

$$\mathrm{Lie}(Y_1)^\vee = \mathrm{Lie}(Y_2)^\vee - Q_1 + Q_1^{(p)}$$

under π_0 to $K_0(\mathcal{F}_0)$ and then find in $K_0(\mathcal{F}_0)$ suppressing the π_0^*

$$\mathbb{E} = \mathrm{Lie}(Y_0)^\vee = [3] - Q_1 + Q_1^{(p)} - Q_0 + Q_0^{(p)},$$

where the [3] stands for the class of the trivial rank 3 bundle $\pi_0^*\pi_1^*(\mathrm{Lie}(Y_2))^\vee$. From the short exact sequence

$$0 \to VM_1/pM_1 \to VM_2/pM_2 \to VM_2/VM_1 \to 0$$

we get the exact sequence of vector bundles

$$0 \to U_1 \to \pi_1^*(\mathrm{Lie}(Y_2)^\vee) \to Q_1 \to 0$$

with $\mathrm{rank}(Q_1) = 2$ that comes from the universal tautological exact sequence of bundles on the Grassmannian. Here U_1 has rank 1 and $\pi_1^*\mathrm{Lie}(Y_2)$ is trivial. This implies that $[Q_1] = [3] - [U_1]$ in the Grothendieck group of vector bundles and so the total Chern class of $\mathrm{Lie}(Y_0)^\vee$ is given by

$$(1-\ell_1)(1-p\ell_1)^{-1}(1+\ell_0)^{-1}(1+p\ell_0),$$

where $\ell_i = c_1(Q_i)$. Now ℓ_1 lives on the curve $\mathcal{F}_1 = \mathcal{X}_{p+1}$, so $\ell_1^2 = 0$. This gives for the classes λ_1 and λ_2 the relations in $\mathrm{CH}^*_{\mathbb{Q}}(\mathcal{F}_0)$

$$\lambda_1 = (p-1)(\ell_0+\ell_1), \quad \lambda_2 = (p-1)^2\ell_0\ell_1 - (p-1)\ell_0^2.$$

The identity $\lambda_1^2 = 2\lambda_2$ that holds in the tautological ring R_3 implies that $(p-1)^2(\ell_0^2 - \ell_1^2) = 0$, hence $\ell_0^2 = 0$. Since $\deg(\ell_1) = p+1$ on $\mathcal{F}_1$ and ℓ_0 represents $\mathcal{O}(1)$ on the fibres of $\mathcal{F}_0 \to \mathcal{F}_1$ we find $\deg(\ell_0\ell_1) = p+1$. We thus find that $\deg(\lambda_2) = (p+1)(p-1)^2$ on each irreducible component of S_3. We get

$$\begin{aligned}\deg(\lambda_2[S_3]) &= (p+1)(p-1)^2\, N_3 \\ &= (p+1)(p-1)^2(p-1)(p^2+1)(p^3-1)\, v(3).\end{aligned}$$

On the other hand, $\deg(\lambda_1\lambda_2\lambda_3) = v(3)$ and this implies the result. □

The morphism $\pi_0 : \mathcal{F}_0 \to \mathcal{F}_1$ is a $\mathbb{P}^1$-bundle over a Fermat curve of degree $p+1$ with a section with image S. The Picard group of $\mathcal{F}_0$ is generated by the pullback under π_0 of the Picard group of $\mathcal{F}_1$ and by the class of the section S.

Proposition 8.2 *We have* $[S] = \ell_0 - \ell_1$ *and* $S^2 = -2(p+1)$.

Proof Let X be a fibre of π_0. We have $XS = 1$ and $(S - \ell_0)X = 0$, hence $S - \ell_0 = \pi_0^*(D)$ with D a divisor class on $\mathcal{F}_1$. This gives $(S - \ell_0)^2 = 0$. The identity $\lambda_1^2 = 2\lambda_2$ implies $\ell_0^2 = 0$ and thus $S^2 - 2\ell_0 S = 0$. Now we use the fact that S is contracted under the map of $\mathcal{F}_0$ to $\mathcal{A}_3 \otimes \mathbb{F}_p$. This implies that λ_1 restricted to S vanishes, hence $(\ell_0 + \ell_1)S = 0$. We thus get $S^2 = 2\ell_0 S = -2\ell_1 S$ and on the other hand $S^2 = \ell_0 S + \pi_0^*(D)S = -\ell_1 S + \pi_0^*(D)S$, hence $\pi_0^*(D) = -\ell_1$ and $S = \ell_0 - \ell_1$. The fact that $\ell_0\ell_1 = p + 1$ and $\ell_0^2 = \ell_1^2 = 0$ implies $S^2 = -2(p + 1)$. □

9 Loci for $g = 3$ defined by conditions on the a-number

We now discuss subloci of S_3 defined by the inequality $a \geq 2$. Here a indicates the a-number of an abelian variety. Let J with $\#J = N_3$ be the set of irreducible components of S_3 (where we count in the stacky way). Each irreducible component of S_3 is the image under a morphism of a $\mathbb{P}^1$-bundle $\mathcal{F}_0 \to \mathcal{F}_1$ that blows down a section. The curve $\mathcal{F}_1$ has $p^3 + 1$ points rational over $\mathbb{F}_{p^2}$ and $\#\mathcal{F}_0(\mathbb{F}_{p^2}) = (p^3 + 1)(p^2 + 1)$ and each point of $\mathcal{F}_0(\mathbb{F}_{p^2})$ defines a superspecial abelian variety. Let $\sqcup_{j\in J}\mathcal{F}_0^j$ be the disjoint union of the smooth models of the irreducible components of S_3. Under the natural morphism

$$m : \sqcup_{j\in J}\mathcal{F}_0^j \longrightarrow S_3 \subset \mathcal{A}_3 \otimes \mathbb{F}_p\,.$$

the $N_3(p^3+1)(p^2+1)$ superspecial points of $\sqcup_{j\in J}\mathcal{F}_0^j$ map to N_3 superspecial points of S_3. Thus each superspecial point of S_3 is the image of $(p^3+1)(p^2+1)$ points and this multiplicity can be explained as follows. On each surface $\mathcal{F}_0^j$ a section is contracted giving a factor $p^3 + 1$, while the image of an $\mathbb{F}_{p^2}$-rational fibre of $\mathcal{F}_0^j \to \mathcal{F}_1^j$ lies on the image of $p^2 + 1$ surfaces $\mathcal{F}_0^j$. This can be checked by using Ekedahl-Oort strata and their classes as follows.

Each $\mathbb{F}_{p^2}$-rational point of $\mathcal{F}_1^j$ determines a fibre in the $\mathbb{P}^1$-bundle $\mathcal{F}_0^j \to \mathcal{F}_1^j$ and the image under m of such a fibre provides a component of the Ekedahl-Oort locus $\mathcal{V}_{[3,2]}$. This locus $\mathcal{V}_{[3,2]}$ consists of a finite union of $\mathbb{P}^1$s. By [4] we know the class of this locus:

$$[\overline{\mathcal{V}}_{[3,2]}] = (p-1)^2(p^6-1)\,\lambda_2\lambda_3\,.$$

Since the degree of the determinant λ_1 of the Hodge bundle restricted to such a $\mathbb{P}^1$ is $p - 1$, we find that $\overline{\mathcal{V}}_{[3,2]}$ has

$$m_{3,2} = \frac{\deg([\overline{\mathcal{V}}_{[3,2]}]\,\lambda_1)}{p-1} = (p-1)(p^6-1)\,v(3)$$

irreducible components, each a copy of $\mathbb{P}^1$. Here we count in the stacky sense. Each such component contributes $p^2 + 1$ superspecial points and we see from

$$m_{3,2}\,(p^2+1) = \deg(\overline{\mathcal{V}}_{[3,2,1]})\,(p^3+1)$$

that this fits with the fact that through a superspecial point there pass p^3+1 components of $\overline{\mathcal{V}}_{[3,2]}$. In fact, under the map $\mathcal{F}_0^j \to \mathcal{A}_3 \otimes \mathbb{F}_p$ a section is blown down and this section intersects the p^3+1 fibres of $\mathcal{F}_0^j \to \mathcal{F}_1^j$ over $\mathcal{F}_1^j(\mathbb{F}_{p^2})$.

We can also check that each such fibre lies on p^2+1 irreducible components of S_3; hence we find for the number of superspecial points

$$N_3\,(p^3+1)(p^2+1) = \deg(\overline{\mathcal{V}}_{[3,2,1]})(p^2+1)(p^3+1)$$

in agreement with the fact that $\mathcal{V}_{[3,2,1]}$ is the superspecial locus and that N_g equals the degree of the superspecial locus for odd g.

10 Moduli of flag type quotients for $g = 4$

In this section we construct a smooth model for each irreducible component of the supersingular locus S_4. The model is obtained by taking the Zariski closure of the moduli of rigid flag type quotients for $g = 4$ and by showing that this moduli space is smooth.

We consider the space $\mathcal{M} = \mathcal{M}_\eta$ of polarized flags of contravariant Dieudonné modules

$$M_0 \subset M_1 \subset M_2 \subset M_3$$

satisfying

(1) $M_3 = A_{1,1}^4$ provided with η, a fixed quasi-polarization $\langle\, ,\, \rangle$ that induces an identification $M_3^t = F^3 M_3$;
(2) $(F, V)M_i \subset M_{i-1}$ and $\dim(M_i/M_{i-1}) = i$;
(3) $(F, V)^i M_i \subset M_i^t$.

We say that it is *rigid* if $M_i = M_0 + F^{3-i} M_3$ for $i = 0, \ldots, 3$.

Theorem 10.1 *The Zariski closure $\mathcal{F}_0$ of the moduli space of rigid polarized Dieudonné flags of length* 4 *with given quasi-polarization on M_3 inside $\mathcal{M}$ is non-singular.*

Proof By [20, 6.1] we can choose generating elements x_1, x_2, x_3, x_4 of M_3 in the skeleton $\tilde{M}_3 = \{m \in M_3 : (F - V)m = 0\}$ of M_3 such that the pairing defined by η satisfies

$$\langle x_i, F^4 x_j \rangle = \delta_{i,5-j} \qquad \text{and} \qquad \langle x_i, F^3 x_j \rangle = 0$$

for $1 \le i \le j \le 4$. For a rigid polarized Dieudonné flag M the module M_2 is generated by $F M_3$ and a vector

$$v_0 = \sum_{i=1}^{4} a_i x_i \; \in M_3/F M_3$$

with the condition $\langle v_0, F^2 v_0\rangle \in W$. Viewing the coefficients a_i as lying in k, this amounts to the equation

$$f := a_1 a_4^{p^2} - a_1^{p^2} a_4 + a_2 a_3^{p^2} - a_2^{p^2} a_3 = 0\,.$$

This defines a smooth surface $\mathcal{F}_2$ in $\mathbb{P}^3$. This surface was studied in detail by Katsura [18]. Locally on this surface we may assume without loss of generality that $a_1 \neq 0$ and that $a_1 = 1$. Now M_1 is generated by FM_2 and a 2-dimensional subspace M_1/FM_2 in M_2/FM_2. Since $a_1 = 1$ we can assume that this 2-dimensional subspace is generated by non-zero elements v and w with

$$v = a_5 v_0 + a_6 F x_2 + a_7 F x_3 + a_8 F x_4, \quad w = a_9 F x_2 + a_{10} F x_3 + a_{11} F x_4\,. \tag{6}$$

We then have the conditions

$$\langle v, Fv\rangle \in W, \quad \langle v, Fw\rangle \in W, \quad \langle Fv, w\rangle \in W\,. \tag{7}$$

Viewing the coefficients as elements of k we find three equations all divisible by a_5. But $a_5 = 0$ yields a flag that is not rigid; indeed,

$$M_1 + FM_3 = (F, V)M_2 + Aw + FM_3 = (F, V)(Av + FM_3) + Aw + FM_3 \subset FM_3$$

but $M_2 \not\subset FM_3$, hence $M_2 \neq M_1 + FM_3$, contradicting rigidity. Removing the factor a_5 from the equations (7) by considering $\langle v, Fv\rangle/a_5$, $\langle v, Fw\rangle/a_5$ and $\langle Fv, w\rangle/a_5^p$, we get the equations

$$\begin{aligned} g_1 &:= a_1 a_8^p - a_1^p a_5^{p-1} a_8 + a_2 a_7^p - a_2^p a_5^{p-1} a_7 + a_3^p a_5^{p-1} a_6 - a_3 a_6^p = 0,\\ g_2 &:= a_1 a_{11}^p + a_2 a_{10}^p - a_3 a_9^p = 0\,,\\ g_3 &:= a_1^p a_{11} + a_2^p a_{10} - a_3^p a_9 = 0\,. \end{aligned} \tag{8}$$

Remark 10.2 The reader may verify that if the point $(1 : a_2 : a_3 : a_4) \in \mathcal{F}_2(k)$ is not rational over $\mathbb{F}_{p^2}$ then we may choose as w the element

$$(F - V)\, v_0\,.$$

Indeed, it satisfies $g_2 = 0$ and $g_3 = 0$ for any non-zero choice of v; namely with $a_1 = 1$ we have $a_1(a_4^{p^2} - a_4) + a_2(a_3^{p^2} - a_3) - a_3(a_2^{p^2} - a_2) = 0$ and similarly for g_3.

Now we first look at a point with $a_5 \neq 0$. If both a_9 and a_{10} vanish we have by $g_3 = 0$ that $w = 0$. So we may assume that, say, $a_9 \neq 0$ and then have $a_1 = a_5 = a_9 = 1$ and by changing v to $v - a_6 w$ we may assume $a_6 = 0$. The Jacobian matrix of the equations f, g_1, g_2, g_3 with respect to the variables a_j for $j = 2, 3, 4, 7, 8, 10, 11$ is

$$\begin{pmatrix} a_3^{p^2} & -a_2^{p^2} & -a_1^{p^2} & 0 & 0 & 0 & 0 \\ a_7^p & -a_6^p & 0 & -a_2^p a_5^{p-1} & -a_1^p a_5^{p-1} & 0 & 0 \\ a_{10}^p & -a_9^p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_2^p & a_1^p \end{pmatrix}$$

and this is of rank 4.

Next we look at the case where $a_5 = 0$. The vanishing of a_9 and a_{10} implies by g_3 that $a_{11} = 0$, so we may assume that $a_9 \neq 0$ or $a_{10} \neq 0$. Again without loss of generality we may assume $a_9 \neq 0$. Changing v by a multiple of w we may assume $a_6 = 0$. If now $a_7 = 0$ then g_1 forces $a_8 = 0$, hence $v = 0$. So we may assume that $a_7 \neq 0$. Then it suffices to treat the case of $a_1 = a_7 = a_9 = 1$ and $a_6 = 0$. Then the Jacobian matrix of the equations f, g_1, g_2, g_3 with respect to the variables a_j for $j = 2, 3, 4, 5, 8, 10, 11$ is

$$\begin{pmatrix} a_3^{p^2} & -a_2^{p^2} & -a_1^{p^2} & 0 & 0 & 0 & 0 \\ a_7^p & 0 & 0 & (a_1^p a_8 + a_2^p a_7) a_5^{p-2} & -a_1^p a_5^{p-1} & 0 & 0 \\ a_{10}^p & -a_9^p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_2^p & a_1^p \end{pmatrix}$$

which is of rank 4 as required. This shows that $\mathcal{F}_1$ is non-singular. Since $\mathcal{F}_0$ is a $\mathbb{P}^1$-bundle over $\mathcal{F}_1$ the result follows. □

By writing $\mathcal{F}_i$ for the Zariski closure in $\mathcal{M}$ of the moduli space of rigid polarized Dieudonné flags $M_i \subset \cdots \subset M_3$ we get a sequence

$$\mathcal{F}_0 \xrightarrow{\pi_0} \mathcal{F}_1 \xrightarrow{\pi_1} \mathcal{F}_2 \xrightarrow{\pi_2} \mathcal{F}_3 = \text{point}$$

with $\dim \mathcal{F}_i = 4 - i$ for $i = 0, 1, 2$.

We now describe the fibres of the morphism $\pi_1 : \mathcal{F}_1 \to \mathcal{F}_2$. We start by remarking that by using the symmetry of $\mathcal{F}_2$ there is no loss of generality if we look at the fibre of a point $(a_1 : a_2 : a_3 : a_4)$ of $\mathcal{F}_2$ with $a_1 = 1$. If one of a_2, a_3, a_4 lies in $\mathbb{F}_{p^2}$ then the point lies on one of the lines of $\mathcal{F}_2$. Indeed, if $a_4 \in \mathbb{F}_{p^2}$ then such a line is parametrically $(1 : t : t : a_4)$, while if, say, $a_2 \in \mathbb{F}_{p^2}$ then such a line is $(t : a_2 : 1 : t)$.

For describing the fibre over a point $(1 : a_2 : a_3 : a_4)$ we consider the equations

$$a_8^p + a_2 a_7^p - a_3 a_6^p - a_5^{p-1}(a_8 + a_2^p a_7 - a_3^p a_6) = 0\,, \tag{9}$$

and

$$a_{11}^p + a_2 a_{10}^p - a_3 a_9^p = 0, \quad a_{11} + a_2^p a_{10} - a_3^p a_9 = 0\,. \tag{10}$$

By the two equations g_2, g_3 of (10) we eliminate a_{11} and get

$$\frac{a_{10}^p}{a_9^p} = \frac{a_3^{p^2} - a_3}{a_2^{p^2} - a_2} \,. \tag{11}$$

In the neighborhood of an $\mathbb{F}_{p^2}$-valued point of $\mathcal{F}_2$, say $(1 : a_2 : a_3 : a_4)$, the expressions $a_2 - a_2^{p^2}$ and $a_3 - a_3^{p^2}$ are local coordinates. This shows that the function field of $\mathcal{F}_1$ can be generated over the function field of $\mathcal{F}_2$ by adjoining the pth root of $(a_2 - a_2^{p^2})/(a_3 - a_3^{p^2})$ as determined by (11) and then adjoining a further element via an Artin-Schreier equation (9). Hence the degree of inseparability of $\mathcal{F}_1$ over $\mathcal{F}_2$ is p.

Over an open neighborhood U of a $\mathbb{F}_{p^2}$-rational point with local coordinates $a_2 - a_2^{p^2}$ and $a_3 - a_3^{p^2}$, the equation (11) describes an inseparable cover of the blow-up of U (in $U \times \mathbb{P}^1$ with coordinates $(u : v)$ on $\mathbb{P}^1$) given by

$$u(a_2 - a_2^{p^2}) - v(a_3 - a_3^{p^2}) = 0, \quad u/v = (a_{10}/a_9)^p \,.$$

Thus we see that the morphism $\pi_1 : \mathcal{F}_1 \to \mathcal{F}_2$ factors via an inseparable cover of the blow-up $\tilde{\mathcal{F}}_2$ of $\mathcal{F}_2$ in the $\mathbb{F}_{p^2}$-rational points.

If we have a point not on a line we may assume $a_9 = 1$ and then that $a_6 = 0$. The reduced fibre is a curve in $\mathbb{P}^2$ with coordinates (a_5, a_7, a_8) given by

$$a_8^p + a_2 a_7^p - a_5^{p-1}(a_8 + a_2^p a_7) = 0 \,.$$

This is a curve with one singularity of order $p - 1$, a cusp located at $a_5 = 0$ and $a_8 + a_2^{1/p} a_7 = 0$.

Next we consider the case of a point on a line. Since the automorphism group of $\mathcal{F}_2$ acts transitively on the set of lines defined over $\mathbb{F}_{p^2}$ (by Witt's theorem, see [18]) we may assume that the line is given as $(1 : t : 0 : 0)$. The last two equations give $(t^{p^2} - t)a_{10}^p = 0$ and the first equation yields $a_8^p + t a_7^p - a_5^{p-1}(a_8 + t^p a_7) = 0$, again a curve with a cusp. So if the point is not a $\mathbb{F}_{p^2}$-valued point of $\mathcal{F}_2$ we get $a_{10}^p = 0$ and as reduced fibre again a curve with a single singularity, a cusp. If $t \in \mathbb{F}_{p^2}$, then the first equation splits as the union of p lines passing through one point.

We summarize.

Proposition 10.3 *Let $\tilde{\mathcal{F}}_2$ be the blow-up of $\mathcal{F}_2$ in all $\mathbb{F}_{p^2}$-rational points. The morphism $\pi_1 : \mathcal{F}_1 \to \mathcal{F}_2$ factors through $\mathcal{F}_1 \to \tilde{\mathcal{F}}_2 \to \mathcal{F}_2$. The morphism $\pi_1' : \mathcal{F}_1 \to \tilde{\mathcal{F}}_2$ has inseparability degree p. The reduced fibre over a non-$\mathbb{F}_{p^2}$-rational point is an irreducible curve with one singularity, a cusp singularity of order $p - 1$, while the fibre over a point on an exceptional curve is a union of p lines meeting in one point.*

11 Interpretation of the morphism $\mathcal{F}_1 \to \mathcal{F}_2$

The morphism $\pi_1 : \mathcal{F}_1 \to \mathcal{F}_2$ is inseparable and factors through the blown-up surface $\tilde{\mathcal{F}}_2$. We give an interpretation of this factorization by describing the blow-up $\tilde{\mathcal{F}}_2$ in terms of Dieudonné modules and by showing that $\mathcal{F}_1$ is realized in a natural $\mathbb{P}^2$-bundle over $\tilde{\mathcal{F}}_2$.

We begin with a moduli interpretation of the fibers of $\tilde{\mathcal{F}}_2 \to \mathcal{F}_2$.

Proposition 11.1 *The fiber of $\tilde{\mathcal{F}}_2 \to \mathcal{F}_2$ over a point $(M_2 \subset M_3) \in \mathcal{F}_2(k)$ is given by a set of lines in a 2-dimensional vector space*

$$\{L \subset V^{-1}M_2^t/FM_2 \mid \dim L = 1, \quad L \textit{ contains } (F, V)M_2 \bmod FM_2\}.$$

Proof We begin by observing two facts:

(i) $(FM_2 \subset)\ (F, V)M_2 \subset V^{-1}M_2^t$
(ii) $V^{-1}M_2^t/FM_2$ is a k-vector space of dimension two.

Indeed, (i) follows from $V(F, V)M_2 \subset (F, V)^2M_2 \subset M_2^t$. To prove (ii), consider the dual of $FM_3 \subset M_2 \subset M_3$:

$$M_3^t \subset M_2^t \subset V^{-1}M_3^t.$$

By $V^{-1}M_3^t = F^{-1}M_3^t = F^2M_3 \subset FM_2$, we have $M_2^t \subset FM_2$. This means that V (and therefore p) kills $V^{-1}M_2^t/FM_2$, whence $V^{-1}M_2^t/FM_2$ is a k-vector space. Looking at the inclusions $M_2^t \subset FM_2 \subset V^{-1}M_2^t$, we have

$$\begin{aligned}\dim V^{-1}M_2^t/FM_2 &= \dim V^{-1}M_2^t/M_2^t - \dim FM_2/M_2^t \\ &= 4 - \dim FM_2/F^2M_3 - \dim V^{-1}M_3^t/M_2^t \\ &= 4 - 1 - 1 = 2\end{aligned}$$

and this proves ii). If $(M_2 \subset M_3)$ represents a point of $\mathcal{F}_2$ that is not rational over $\mathbb{F}_{p^2}$ then $FM_2 \neq VM_2$ and L is unique. If $(M_2 \subset M_3)$ represents a $\mathbb{F}_{p^2}$-rational point, then $FM_2 = VM_2$ and the fibre is a $\mathbb{P}^1$. □

Remark 11.2 We point out that the Dieudonné module $V^{-1}M_2^t/FM_2$ is self-dual.

We now describe the morphism $\pi_1' : \mathcal{F}_1 \to \tilde{\mathcal{F}}_2$. On $\tilde{\mathcal{F}}_2$ we have by Proposition 11.1 the subspace $L \subset V^{-1}M_2^t/FM_2$. It determines a W-module $\tilde{L}$ with

$$(F, V)M_2 \subset \tilde{L} \subset V^{-1}M_2^t,$$

the inverse image of L under the projection $V^{-1}M_2^t \to V^{-1}M_2^t/FM_2$. It has the property that outside $\tilde{\pi}_2^{-1}(\mathcal{F}_2(\mathbb{F}_{p^2}))$ we have $\tilde{L} = (F, V)M_2$, where we write $\tilde{\pi}_2$ for the blow-down morphism $\tilde{\mathcal{F}}_2 \to \mathcal{F}_2$. We can now consider over a point of $\tilde{\mathcal{F}}_2$ the 3-dimensional vector space $M_2/\tilde{L}$. This should define a rank 3 vector bundle B, but as the equations show we can realize B only after an inseparable base change.

Lemma 11.3 *The threefold $\mathcal{F}_1$ is a divisor in a $\mathbb{P}^2$-bundle $\mathbb{P}(B)$ with B the rank 3 vector bundle defined by $M_2/\tilde{L}$ over a surface $\tilde{\mathcal{F}}_2'$ obtained by an inseparable base change $\tilde{\mathcal{F}}_2' \to \tilde{\mathcal{F}}_2$ of degree p.*

Proof Recall that in order to define $M_1 \subset M_2$, we chose a basis

$$v = a_5 v_0 + a_6 F x_2 + a_7 F x_3 + a_8 F x_4, \quad w = a_9 F x_2 + a_{10} F x_3 + a_{11} F x_4$$

as in (6) with $\langle v, Fv\rangle$, $\langle v, Fw\rangle$, $\langle Fv, w\rangle$ all in W. The equations (g_2) and (g_3) correspond to the inseparable base change $\tilde{\mathcal{F}}_2' \to \tilde{\mathcal{F}}_2$ given on the locus with $a_1 \neq 0$ by (11)

$$(a_9/a_{10})^p = (a_2 - a_2^{p^2})/(a_3 - a_3^{p^2})\,.$$

Then on $\tilde{\mathcal{F}}_2'$ we have the bundle $\mathbb{P}(B)$. If $a_5 \neq 0$ the morphism $\mathcal{F}_1 \to \tilde{\mathcal{F}}_2$ is defined by sending $(M_1 \subset M_2 \subset M_3)$ to the point defined by $L := M_1 \cap V^{-1} M_2^t \bmod FM_2$. Indeed, by $(F, V)M_2 \subset M_1$, the subspace L contains $(F, V)M_2 \bmod FM_2$, and L is the one-dimensional space generated by w of (6), since one can check $\langle Vw, M_2\rangle \subset W$ and if $a_5 \neq 0$, then $\langle Vv, M_2\rangle \not\subset W$.

For $a_5 \neq 0$ we find from $\langle v, Fv\rangle \in W$ an equation

$$a_1 a_8^p + a_2 a_7^p - a_3 a_6^p - a_5^{p-1}(a_1^p a_8 + a_2^p a_7 - a_3^p a_6) = 0\,.$$

This defines a rational curve with a cusp in $\mathbb{P}^2 = \mathbb{P}(M_2/\tilde{L})$. As $\mathcal{F}_1$ is defined as the closure of the space of rigid flags, we obtain that this equation defines $\mathcal{F}_1$ in $\mathbb{P}(B)$. Observe that in order to analyze this we may assume as we did in the preceding section that $a_1 = 1$ and $a_2 \neq 0$ and then $a_6 = 0$ and the curve can be written in coordinates $(a_5 : a_7 : a_8)$ as

$$a_1 a_8^p + a_2 a_7^p - a_5^{p-1}(a_1^p a_8 + a_2^p a_7) = 0\,.$$

The cusp is determined by $a_5 = 0$ and $a_8 + a_2^{1/p} a_7 = 0$. □

In particular we see that after an inseparable base change the bundle B admits a nowhere vanishing section.

12 The Hodge bundle on the supersingular locus

The description of principally polarized supersingular abelian varieties of dimension 4 via a flag gives us for each irreducible component S of S_4 a morphism $\mathcal{F}_0 \to S$ and a fibration of $\mathcal{F}_0$

$$\mathcal{F}_0 \xrightarrow{\pi_0} \mathcal{F}_1 \xrightarrow{\pi_1} \mathcal{F}_2 \to \mathcal{F}_3\,,$$

where $\mathcal{F}_i$ for $i = 0, \ldots, 3$ is the closure of the moduli space of rigid polarized flag type quotients $Y_3 \to \cdots \to Y_i$. Note that $\mathcal{F}_3$ is a point.

We have seen above that these spaces $\mathcal{F}_i$ are non-singular. In the following we view these as moduli stacks. This corresponds to dividing by the appropriate automorphism groups, here by $\mathrm{Aut}(Y_3, \eta)$.

Lemma 12.1 *For each irreducible component S of S_4 in $\mathcal{A}_4 \otimes \mathbb{F}_p$ the natural morphism $\mathcal{F}_0 \to S$ is a morphism of degree p.*

Proof Let x be a geometric point of $\mathcal{F}_2$. Let $\mathcal{F}_{1,x}$ be the fiber $\pi_1^{-1}(x)$. We claim that $\pi_0^{-1}(\mathcal{F}_{1,x}) \to \mathcal{A}_g$ is a p-to-1 map onto the image. Indeed if x is represented by $(a_1, a_2, a_3, a_4) \in k^4$ for an algebraically closed field k, then the fiber $\pi_1^{-1}(x)$ is described in $a_5, \ldots, a_{11}$ by

$$
\begin{aligned}
g_1 &:= a_1 a_8^p - a_1^p a_5^{p-1} a_8 + a_2 a_7^p - a_2^p a_5^{p-1} a_7 + a_3^p a_5^{p-1} a_6 - a_3 a_6^p = 0,\\
g_2 &:= a_1 a_{11}^p + a_2 a_{10}^p - a_3 a_9^p = 0\,,\\
g_3 &:= a_1^p a_{11} + a_2^p a_{10} - a_3^p a_9 = 0\,.
\end{aligned}
$$

But g_2 is the p-th power of

$$g_2' := a_1^{1/p} a_{11} + a_2^{1/p} a_{10} - a_3^{1/p} a_9\,.$$

The space defined by g_1, g_2', g_3, say $\mathcal{F}_{1,x}'$, coincides on an open part of $\mathcal{F}_1$ with the fiber of $\mathcal{V}_{11} \to \mathcal{V}_2$ studied in [22, 9.7], where $\mathcal{V}_2$ corresponds to our $\mathcal{F}_2$ and $\mathcal{V}_{11}$ is the non-garbage component considered in [22, 9.7]. Note that $\mathcal{F}_{1,x}'$ is a closed subscheme of $\mathcal{F}_{1,x}$. Thanks to the proof by Li and Oort (cf. [22, 7.11]), the map $(\pi_0)^{-1}(\mathcal{F}_1') \to \mathcal{A}_g$ is one-to-one on its image as stacks; indeed, the proof of Li and Oort was done by fiberwise arguments. The claim follows. □

The space $\mathcal{F}_i$ carries an abelian variety $\mathcal{Y}_i$. Its cotangent bundle along the zero section may be described by Dieudonné theory. Using contravariant Dieudonné theory with the Dieudonné module M_i of a fibre Y_i of $\mathcal{Y}_i$, we have

$$\mathrm{Lie}(Y_i)^\vee = VM_i/pM_i\,.$$

The flag type quotient provides an inductive construction. For $i = 2, 1, 0$ we have the exact sequence

$$0 \to pM_{i+1}/pM_i \to VM_i/pM_i \to VM_{i+1}/pM_{i+1} \to VM_{i+1}/VM_i \to 0\,.$$

In the Grothendieck group of vector bundles on $\mathcal{F}_i$ we thus get the identity

$$[\mathrm{Lie}(\mathcal{Y}_i)^\vee] = [\pi_i^*(\mathrm{Lie}(\mathcal{Y}_{i+1})^\vee)] - [Q_i] + [Q_i^{(p)}]\,,$$

where Q_i is the locally free $\mathcal{O}_{\mathcal{F}_i}$-module of rank $i+1$ corresponding to VM_{i+1}/VM_i. Moreover, the exact sequence for $i = 1$ and $i = 2$

$$0 \to VM_i/pM_{i+1} \to VM_{i+1}/pM_{i+1} \to VM_{i+1}/VM_i \to 0$$

gives us an exact sequence of $\mathcal{O}_{\mathcal{F}_i}$-modules

$$0 \to U_i \to \pi_i^*(\mathrm{Lie}(\mathcal{Y}_{i+1})^\vee) \to Q_i \to 0$$

with U_i the locally free $\mathcal{O}_{\mathcal{F}_i}$-module defined by VM_i/pM_{i+1}. For $i=0$ we have

$$0 \to VM_0/VM_1^t \to VM_1/VM_1^t \to VM_1/VM_0 \to 0$$

and this gives a short exact sequence of $\mathcal{O}_{\mathcal{F}_0}$-modules

$$0 \to U_0 \to \pi_0^*(K_1) \to Q_0 \to 0$$

with K_1 the locally free sheaf corresponding to the Dieudonné module of $\ker(\mathcal{Y}_1 \xrightarrow{\eta_1} \mathcal{Y}_1^t)$.

In the following we will abuse the notation Q_i also for the pullback of Q_i to $\mathcal{F}_{i-1}$ in order to simplify notation. Since $\mathrm{Lie}(\mathcal{Y}_3)^\vee$ is trivial of rank 4 we get from the above the class of the Hodge bundle $\mathbb{E} = \mathrm{Lie}(\mathcal{Y}_0)^\vee$ in the Grothendieck group of vector bundle on $\mathcal{F}_0$.

Proposition 12.2 *The class of Hodge bundle of $\mathcal{Y}_0$ in the Grothendieck group of vector bundles on $\mathcal{F}_0$ is given by*

$$\begin{aligned}[\mathbb{E}] &= 4 - [Q_2] + [Q_2^{(p)}] - [Q_1] + [Q_1^{(p)}] - [Q_0] + [Q_0^{(p)}] \\ &= 4 + [U_2] - [U_2^{(p)}] - [Q_1] + [Q_1^{(p)}] - [Q_0] + [Q_0^{(p)}],\end{aligned}$$

where 4 stands for the class of trivial rank 4 bundle and where U_2 and Q_0 have rank 1, while Q_1 has rank 2.

Note that here we abuse the notation Q_i for the pull back of Q_i to $\mathcal{F}_0$.

We now set

$$\ell_i = c_1(Q_i) \quad \text{for } i = 0, 1, 2.$$

We may consider ℓ_i as a class living on $\mathcal{F}_i$, but we will denote the pull back $\pi_0^*(\ell_1)$, $\pi_1^*(\ell_2)$ and $\pi_0^*(\pi_1^*(\ell_2))$ also by ℓ_1, ℓ_2 in order to simplify notation.

Proposition 12.2 implies the following.

Proposition 12.3 *The total Chern class $c(\mathbb{E})$ of the Hodge bundle on $\mathcal{F}_0$ is given by*

$$c(\mathbb{E}) = \frac{(1-\ell_2)(1+p\,\ell_1+p^2c_2(Q_1))(1+p\,\ell_0)}{(1-p\,\ell_2)(1+\ell_1+c_2(Q_1))(1+\ell_0)}.$$

Corollary 12.4 *We have $c_2(Q_1) = (\ell_0^2+\ell_1^2-\ell_2^2)/2$. Moreover, the class ℓ_0^2 is a pullback from $\mathcal{F}_1$.*

Proof We deduce $\lambda_1 = (p-1)(\ell_0+\ell_1+\ell_2)$ and

$$2\,\lambda_2 - \lambda_1^2 = (p^2-1)(2\,c_2(Q_1) - \ell_0^2 - \ell_1^2 + \ell_2^2)$$

and since $\lambda_1^2 = 2\,\lambda_2$ on $\mathcal{A}_4$ the formula for $c(\mathbb{E})$ follows. Since Q_1 lives on $\mathcal{F}_1$ it implies that the class ℓ_0^2 is a pullback from $\mathcal{F}_1$. □

13 Loci with a-number ≥ 2 for $g = 4$

The abelian variety corresponding to the generic point of an irreducible component S of S_4 has a-number equal to 1. An irreducible component of the closed stratum of S where $a \geq 2$ is of one of two types, as shown in [22, Section 9.9]. See also [12]. A component of the preimage of the first type for the natural morphism $\mathcal{F}_0 \to S$ maps under $\mathcal{F}_0 \to \mathcal{F}_2$ to a line on $\mathcal{F}_2$, while such a component of the second type maps either dominantly to $\mathcal{F}_2$, or maps to a line of $\mathcal{F}_2$, or maps to a point of $\mathcal{F}_2(\mathbb{F}_{p^2})$.

13.1 Loci of the first type.

The first type parametrizes flag types $M_3 \supset M_2 \supset M_1 \supset M_0$ such that there exists a totally isotropic subspace I of M_3/FM_3 such that $M_1 \subset N$ with $N \subset M_3$ the submodule generated by I and FM_3. Since the automorphism group of M_3 acts transitively on totally isotropic subspaces defined over $\mathbb{F}_{p^2}$, we may assume that $I = \langle x_1, x_2 \rangle$. In terms of abelian varieties, such a flag type can be obtained from a flag type

$$E^4 = Y_3 \xrightarrow{\rho_3} Y_2 \xrightarrow{\rho_2} Y_1 \xrightarrow{\rho_1} Y_0 \tag{12}$$

with quasi-polarization $\eta_3 : Y_3 \to Y_3^t$ with $\ker \eta_3 = E^4[F^3]$ if the composition $\rho_2\rho_3 : E^4 \to Y_1$ factors through

$$1_{E^2} \times F_{E^2} : E^4 \longrightarrow E^2 \times E^2/E^2[F] .$$

By identifying $E^2 \times E^2/E^2[F]$ with E^4 and thus factoring $\rho_2\rho_3$, we put $Z_2 = E^2 \times E^2/E^2[F] \cong E^4$ and $Z_1 = Y_1$ and then associate to it the flag

$$E^4 = Z_2 \xrightarrow{\zeta_2} Z_1 \xrightarrow{\zeta_1} Z_0 , \tag{13}$$

where $\deg(\zeta_2) = p^3$ and $\deg(\zeta_1) = p$ and $\theta_2 : Z_2 \to Z_2^t$ is a quasi-polarization with kernel equal to $E^4[p]$.

This can be described by Dieudonné modules: consider the Dieudonné module $N_2 = \langle x_1, x_2, Fx_3, Fx_4 \rangle$ with x_1, x_2, x_3, x_4 the skeleton of M_3. It satisfies $N_2^t = F^2N_2$. We choose a submodule N_1 generated by $u = ax_1 + bx_2 + cFx_3 + dFx_4$ and FN_2 with $\langle u, Fu \rangle = 0$. By viewing u as an element of N_2/FN_2 and the coefficients a, b, c, d in k we obtain an equation

$$ad^p - a^pd + bc^p - b^pc = 0 . \tag{14}$$

Then $\dim N_2/N_1 = 3$ and $\dim N_1/N_1^t = 2$. We then can choose a Dieudonné submodule N_0 with $N_1^t \subset N_0 \subset N_1$ with $\dim N_1/N_0 = 1$. The filtration $N_0 \subset N_1 \subset N_2$ corresponds to (13). The moduli of $N_2 \supset N_1$ defines a surface $\mathcal{G}_1$ in projective space $\mathbb{P}^3$ given by (14) and choosing N_0 defines a $\mathbb{P}^1$-bundle $\mathcal{G}_0 \to \mathcal{G}_1$.

Let $\mathcal{S}_1$ be the subscheme of $\mathcal{G}_1$ where $a = b = 0$ and let $\mathcal{S}_0$ be the inverse image of $\mathcal{S}_1$ under $\mathcal{G}_0 \to \mathcal{G}_1$. We now discuss how to map $\mathcal{G}_1\backslash\mathcal{S}_1$ to $\mathcal{F}_1$. Given u we choose v_0 as a multiple of u. This determines a submodule M_2 of M_3, generated by FM_3 and v_0, that contains N_1 and we set $M_1 = N_1$. Note that M_2 is generated also by $ax_1 + bx_2$ and FM_3. Then we can choose two generators v, w for M_1 modulo FM_2 and assuming that $a \neq 0$ we may choose

$$v = a_5 v_0 + cFx_3 + dFx_4 = ax_1 + bx_2 + cFx_3 + dFx_4, \quad w = Fx_2\,.$$

In terms of the coordinates in Section 10 we have

$$a = a_1a_5, \quad b = a_2a_5, \quad c = a_7, \quad d = a_8\,.$$

The fibre of $\mathcal{G}_1\backslash\mathcal{S}_1 \to \mathcal{F}_1$ over a point $v_0 = (1 : t : 0 : 0)$ of $\mathcal{F}_2$ consists of all $(a : b : c : d)$ with $d^p - a^{p-1}d + tc^p - t^p a^{p-1}c = 0$; it is defined by a Lefschetz pencil on $\mathcal{G}_1$ defined by $b = ta$. We refer to the paper [18] for such a Lefschetz fibering. The general fibre is a rational curve with one singularity given by $a = 0$.

Recall that the automorphism group of $\mathcal{F}_2$ acts transitively on the set of lines of $\mathcal{F}_2$ defined over $\mathbb{F}_{p^2}$. For each line L defined over $\mathbb{F}_{p^2}$ on the surface $\mathcal{F}_2$ we find a surface isomorphic to $\mathcal{G}_1$ that is contained in the inverse image $\pi_1^{-1}(L)$.

The fibration $\mathcal{G}_0 \to \mathcal{G}_1$ has a natural section S by taking $Z_0 = Z_2/Z_2[F]$. Note that then $N_0 = FN_2 \subset N_1$. This implies that Z_0, determined by N_0, is constant for all choices of N_1. It also implies that this section is blown down under the natural morphism $\mathcal{G}_0 \to S_4 \subset \mathcal{A}_4$ that associates to a flag type quotient (13) the isomorphism class of Z_0. We summarize:

Let $M_3 = A_{1,1}^4$ with quasi-polarization such that $M_3^t = F^3 M_3$.

Proposition 13.1 *For each totally isotropic subspace of M_3/FM_3 there is a threefold $\mathcal{G}_0$ that is a $\mathbb{P}^1$-bundle $\mathcal{G}_0 \to \mathcal{G}_1$ over a surface given by (14) with a section and a morphism $\mathcal{G}_0\backslash\mathcal{S}_0 \to \mathcal{F}_0$ whose image is a locus of supersingular abelian 4-folds with $a \geq 2$. Under $\mathcal{G}_0\backslash\mathcal{S}_0 \to \mathcal{F}_2$ it maps to a line on $\mathcal{F}_2$. Under the morphism $\mathcal{G}_0 \to \mathcal{A}_4 \otimes \mathbb{F}_p$ the section of $\mathcal{G}_0 \to \mathcal{G}_1$ is blown down.*

13.2 Loci of the second type

An irreducible component of the preimage under $\mathcal{F}_0 \to S$ of the locus of a-number ≥ 2 of the second type inside an irreducible component S of the supersingular locus S_4 is realized as follows. It is the locus of $M_\bullet$ with $M_0 \subset N \subset M_3$ for a fixed superspecial quasi-polarized Dieudonné module N with $N^t = FN$. Such N come in three sorts: the first sort with $N = FM_3$, the second with $N = \langle x_1, Fx_2, Fx_3, px_4\rangle$ and the third sort where $N = \langle x_1, x_2, px_3, px_4\rangle$ after we change generators as in the proof of Theorem 10.1.

In this subsection we treat the case where N is of the first sort, while the case of the second sort in treated in the next section and the last case is left to the reader. The first case is characterized by the condition $M_1 \subset FM_3$. This condition is determined on $\mathcal{F}_1$.

Let T be an irreducible component in $\mathcal{F}_0$ of the preimage of an irreducible component of the locus with $a \geq 2$ of the second type and first sort.

The condition $M_1 \subset FM_3$ can be paraphrased by saying that the natural homomorphism

$$M_1/(F,V)M_2 \to M_2/FM_3 \tag{15}$$

induced by $M_1 \hookrightarrow M_2$ is zero. Let $\mathcal{L}$ be the sheaf corresponding to the module $M_1/(F,V)M_2$ and U_2 the one corresponding to VM_2/pM_3. The invertible sheaf U_2 lives on $\mathcal{F}_2$, and $\mathcal{L}$ lives on $\mathcal{F}_1$ and is invertible only outside $\pi_1^{-1}(\mathcal{F}_2(\mathbb{F}_{p^2}))$. Thus we work on the open set $\mathcal{F}_1^0$ that is the complement of $\pi_1^{-1}(\mathcal{F}_2(\mathbb{F}_{p^2}))$. The homomorphism (15) defines a homomorphism of sheaves

$$\psi : \mathcal{L} \to \pi_1^*(U_2^{(p)}) \,.$$

The locus $\mathcal{H}_1$ can now be defined as the Zariski closure in $\mathcal{F}_1$ of the zero locus $D(\psi)$ in $\mathcal{F}_1^0$ of the map ψ.

Lemma 13.2 *The cycle class of the Zariski closure in $\mathcal{F}_1$ of the zero locus $D(\psi)$ of ψ equals*

$$[\overline{D(\psi)}] = p\,\ell_1 - (p^2+1)\ell_2 + e \,,$$

where e is a class with support in the fibres of π_1 over $\mathcal{F}_2(\mathbb{F}_{p^2})$.

Proof We work on the open set $\mathcal{F}_1^0$ that is the complement of $\pi_1^{-1}(\mathcal{F}_2(\mathbb{F}_{p^2}))$. Consider the exact sequence

$$0 \to VM_2/FM_2 \cap VM_2 \to M_1/FM_2 \to M_1/(F,V)M_2 \to 0 \,.$$

If M_2 is generated by FM_3 and $v_0 = a_1x_1+a_2x_2+a_3x_3+a_4x_4$ with $(a_1 : a_2 : a_3 : a_4)$ determining a point not in $\mathcal{F}_2(\mathbb{F}_{p^2})$ then $FM_2 \cap VM_2 = pM_3$ and

$$VM_2/FM_2 \cap VM_2 = VM_2/pM_3.$$

This translates into a short exact sequence of $\mathcal{O}_{\mathcal{F}_1^0}$-modules

$$0 \to \pi_1^*(U_2) \to U_1^{(p)} \to \mathcal{L} \to 0$$

with U_1 the locally free $\mathcal{O}_{\mathcal{F}_1}$-module determined by VM_1/pM_2. We view ψ as a section of $\pi_1^*(U_2^{(p)}) \otimes \mathcal{L}^{-1}$ on $\mathcal{F}_1^0$ with class $(p+1)[U_2] - p[U_1]$. From Section 12 we use the identities $[U_2] = [4] - [Q_2]$, $[U_1] = [\mathrm{Lie}(Y_2)^\vee] - [Q_1] = [4] - [Q_2] + [Q_2^{(p)}] - [Q_1]$, hence $c_1((p+1)[U_2] - p[U_1]) = p\ell_1 - (p^2+1)\ell_2$. When taking the closure of the degeneracy locus of ψ we have to take into account a class with support in the fibres over $\mathcal{F}_2(\mathbb{F}_{p^2})$ and the result follows. □

Corollary 13.3 *A locus T of the second type and first sort as above maps dominantly to $\mathcal{F}_2$.*

Proof We claim that such a T is not contained in the fibres over $\mathcal{F}_2(\mathbb{F}_{p^2})$. Indeed, otherwise T lives inside a $\mathbb{P}^1$-bundle over the one-dimensional locus in $\mathcal{F}_1$ defined by $(a_1 : a_2 : a_3 : a_4) \in \mathbb{P}^3(\mathbb{F}_{p^2})$ and $a_5 = 0$. This contradicts that the stratum of S where $a \geq 2$ has dimension 3, see [22, 9.9]. The claim implies that T is contained in $\overline{D(\psi)}$ and the class is given by Lemma 13.2. The intersection number of $\overline{D(\psi)}$ with a generic fibre of π_1 equals the degree of ℓ_1 on such a fibre, that is, 1. That means that it intersects the generic fibre of π_1 and is irreducible. $\square$

The resulting abelian variety of a flag type with $M_1 \subset FM_3$ is defined by the filtration of Dieudonné modules $FM_3 \supset M_1 \supset M_0$. By forgetting M_1 between FM_3 and M_0, we have polarized flag types

$$\begin{tikzcd} E^4 = Z_1 \arrow[r, "\rho_1"] \arrow[d, "\eta_1"] & Z_0 \arrow[d, "\cong"] \\ E^4 = Z_1^t & Z_0^t \arrow[l] \end{tikzcd}$$

with $\ker(\eta_1) = E^4[F]$ and $\ker(\rho_1) \cong \alpha_p^2$. The choice of $\ker(\rho_1)$ in $E^4[F]$ defines a point in the Grassmann variety $G = \mathrm{Gr}(2, 4)$. Note that G can be identified with a quadric in $\mathbb{P}^5$ in terms of Plücker coordinates. If we choose a basis x_1, x_2, x_3, x_4 of the Dieudonné module of E^4 with $(F - V)x_i = 0$ and quasi-polarization with

$$\langle x_i, p\,x_{5-j}\rangle = \delta_{ij}, \quad \langle x_i, Fx_j\rangle = 0\,,$$

then M_0 can be generated by two vectors $a = \sum_{i=1}^4 a_i x_i$, $b = \sum_{i=1}^4 b_i x_i$ and the condition $\langle a, b\rangle \in W$ says

$$a_1b_4 - a_4b_1 + a_2b_3 - a_3b_2 \equiv 0 \pmod p$$

and this defines a hyperplane section $Q = H \cap G$ of the Grassmann variety. Indeed, with the Plücker coordinates $\lambda_{ij} = a_ib_j - a_jb_i$ the variety G is given by $\lambda_{12}\lambda_{34} - \lambda_{13}\lambda_{24} + \lambda_{14}\lambda_{23} = 0$ and H by $\lambda_{14} + \lambda_{23} = 0$.

Recall that we interpret moduli in the stacky way meaning that we divide by the automorphism groups of objects. We summarize.

Lemma 13.4 *The image in S_4 of an irreducible component of the locus of second type with $a \geq 2$ can be identified with (the quotient of) a hyperplane section Q of the Grassmann variety* $\mathrm{Gr}(2, 4)$.

We will analyze the case of loci of the second type contained in the fibres over $\mathbb{F}_{p^2}$-rational points on $\mathcal{F}_2$ in the next section.

14 The fibres over $\mathcal{F}_2(\mathbb{F}_{p^2})$

Here we study the fibre under $\mathcal{F}_0 \to \mathcal{F}_2$ of a rational point $\xi \in \mathcal{F}_2(\mathbb{F}_{p^2})$. Since $\mathcal{F}_0 \to \mathcal{F}_1$ is a $\mathbb{P}^1$-bundle it suffices to study the fibre under $\mathcal{F}_1 \to \mathcal{F}_2$.

The automorphism group $\mathrm{Aut}(\mathcal{F}_2)$ can be identified with the quotient by its center of the general unitary group $\mathrm{GU}_4(p^2)$ of 4-dimensional space over $\mathbb{F}_{p^2}$ that fixes the Hermitian form

$$\xi_1\bar{\xi}_4 - \xi_4\bar{\xi}_1 + \xi_2\bar{\xi}_3 - \xi_3\bar{\xi}_2$$

where $\bar{\xi} = \xi^{p^2}$. By a theorem of Witt this group acts transitively on isotropic subspaces of dimension 1 and 2. This implies that it acts transitively on the set of lines of $\mathcal{F}_2$ and on the set of $\mathbb{F}_{p^2}$-rational points, see [18, Appendix]. We thus may restrict to analyzing the fibre over the point $(1 : 0 : 0 : 0)$ of $\mathcal{F}_2$. This corresponds to the case with $M_2 \subset M_3$ generated by $v_0 = x_1$ and FM_3. The fibre $\pi_1^{-1}(\xi)$ corresponds to the choices of M_1. It can be given by a choice of basis

$$v = a_5v_0 + a_6Fx_2 + a_7Fx_3 + a_8Fx_4, \quad w = a_9Fx_2 + a_{10}Fx_3 + a_{11}Fx_4,$$

satisfying the equations g_1, g_2 and g_3 of Section 10

$$a_8^p - a_5^{p-1}a_8 = 0, \quad a_{11}^p = 0, \quad a_{11} = 0\,.$$

We distinguish whether $a_5 \neq 0$ or $a_5 = 0$.

Case i). $a_5 \neq 0$. We may assume $a_5 = 1$ and find $a_8 \in \mathbb{F}_p$. We can change basis of M_3 by $(x_1, x_2, x_3, x_4) \mapsto (x_1 + a_8Fx_4, x_2, x_3, x_4)$ and then may assume that $a_8 = 0$. Then M_1 is generated inside M_2 by $FM_2 = \langle Fx_1, F^2x_2, F^2x_3, F^2x_4\rangle$ and $v = x_1 + a_6Fx_2 + a_7Fx_3$ and $w = a_9Fx_2 + a_{10}Fx_3$. We now construct a flag of Dieudonné modules

$$F^2M_3' \subset M_1^t \subset M_1 \subset FM_3'$$

with $M_3' = \langle F^{-1}x_1, x_2, x_3, Fx_4\rangle$ and show that we can extend it to a flag type

$$M_1 \subset FM_3' \subset M_2' \subset M_3' \tag{16}$$

so that we can associate to it a point of a locus of the second type as treated in the Section 13.2 with respect to a changed basis $\langle F^{-1}x_1, x_2, x_3, Fx_4\rangle$ of M_3. To prove our claim we have to construct M_2' with $(F, V)M_2' \subset M_1$. We take $v_0' = \alpha_1F^{-1}x_1 + \alpha_2x_2 + \alpha_3x_3 + Fx_4$ and impose the following conditions

(1) $\langle v_0', Fv_0'\rangle \in W$, that is, $\alpha_1 - \alpha_1^{p^2} + \alpha_2\alpha_3^{p^2} - \alpha_2^{p^2}\alpha_3 = 0$,
(2) $Fv_0' \in M_1$, equivalently, there exists β with $Fv_0' = \alpha_1^pv + \beta w + F^2x_4$, that is, $\alpha_2^p = \alpha_1^pa_6 + \beta a_9$ and $\alpha_3^p = \alpha_1^pa_7 + \beta a_{10}$,
(3) $Vv_0' \in M_1$, equivalently, there exists γ with $Vv_0' = \alpha_1^{1/p}v + \gamma w + F^2x_4$, that is, $\alpha_2^{1/p} = \alpha_1^{1/p}a_6 + \gamma a_9$ and $\alpha_3^{1/p} = \alpha_1^{1/p}a_7 + \gamma a_{10}$.

For generic a_i (that is, $a_7a_9 - a_6a_{10}$ and a_9a_{10} not in $\mathbb{F}_{p^2}$) we find a solution. We then set

$$M_2' = Av_0' + FM_3'$$

and then by (2) we have $\langle v, w, FM_2' \rangle = \langle v, w, FM_2 \rangle = M_1$. Thus we have a filtration (16) and it gives a point of a locus $\mathcal{H}_1$ with respect to the module M_3'.

Case ii). Here $a_5 = 0$. Then by g_2 we have $a_{11} = 0$ and find that M_1 is generated by $v = a_6 F x_2 + a_7 F x_3$, $w = a_9 F x_2 + a_{10} F x_3$ and $F^2 M_3$, hence $M_1 = \langle Fx_1, Fx_2, Fx_3, F^2 x_4 \rangle$. So M_1 is fixed and this case thus yields one point. Moreover $M_1^t = \langle Fx_1, F^2x_2, F^2x_3, F^2x_4 \rangle$.

We thus see that the supersingular abelian variety corresponding to a generic point of an irreducible component $\mathcal{E}$ of the fibre over a rational point $\xi \in \mathcal{F}_2(\mathbb{F}_{p^2})$ can be viewed as the supersingular abelian variety defined by a generic point of a locus with $a \geq 2$ of the second kind with $M_1 \subset FM_3'$.

This means that there is an irreducible component S' of S_4 with model $\mathcal{F}_0'$ and a locus $\mathcal{H}_0'$ mapping dominantly to $\mathcal{F}_2'$ such that image of $\mathcal{E}$ and $\mathcal{H}_0'$ coincide in $S_4 \subset \mathcal{A}_4$.

We summarize.

Proposition 14.1 *Let S be a component of S_4 and $\mathcal{F}_0$ be the model constructed in Section 10. The fibre in $\mathcal{F}_0$ over a rational point $\xi \in \mathcal{F}_2(\mathbb{F}_{p^2})$ consists of p irreducible components. The image of each of these in S_4 is a hyperplane section of the Grassmann variety* $\mathrm{Gr}(2,4)$ *and can be seen as the image of a locus of $a \geq 2$ of the second type in another component S' of S_4.*

15 Superspecial points of S_4

The number of points of S_4 representing isomorphism classes of superspecial abelian varieties counted in the stacky sense was given in formula (2) in Section 2 and equals

$$\Sigma_4 = (p-1)(p^2+1)(p^3-1)(p^4+1)\, v(4)\,.$$

Each superspecial principally polarized abelian variety of dimension 4 defines an $\mathbb{F}_{p^2}$-rational point of $S_4 \subset \mathcal{A}_4$. By Proposition 3.1 we have $N_4 = (p^2-1)(p^6-1)\, v(4)$ irreducible components (again counted in the stacky sense) of S_4. Each irreducible component is the image of $\mathcal{F}_0$ under a degree p morphism in the stacky sense to its image in S_4 that induces a bijection between geometric points of the stacks on the open parts of a-number one.

Lemma 15.1 *We have $\#\mathcal{F}_0(\mathbb{F}_{p^2}) = (p^2+1)^3(p^3+1)(p^4+1)$.*

Proof We have $\#\mathcal{F}_2(\mathbb{F}_{p^2}) = (p^2+1)(p^4+1)$, see for example [18], hence $\#\tilde{\mathcal{F}}_2(\mathbb{F}_{p^2}) = (p^2+1)^2(p^4+1)$ and these points are the $\mathbb{F}_{p^2}$-rational points on the exceptional curves of $\tilde{\mathcal{F}}_2$. The fibre in $\mathcal{F}_1$ over a $\mathbb{F}_{p^2}$-rational point of $\tilde{\mathcal{F}}_2$ consists of a union of p lines through one point. So we find $\#\mathcal{F}_1(\mathbb{F}_{p^2}) = (p^2+1)^2(p^4+1)(p^3+1)$. Since $\mathcal{F}_0$ is a $\mathbb{P}^1$-bundle over $\mathcal{F}_1$ the formula follows. □

Let J be the set of irreducible components of S_4 and for $j \in J$ we let $\mathcal{F}_0^j$ be the corresponding smooth model. The disjoint union of these smooth models has

$$\#(\bigsqcup_{j \in J} \mathcal{F}_0^j)(\mathbb{F}_{p^2}) = N_4\, (p^2+1)^3(p^3+1)(p^4+1)$$

$\mathbb{F}_{p^2}$-rational points mapping to Σ_4 superspecial points of S_4. The variety $\mathcal{F}_0$ contains $(p^2+1)(p^4+1)$ loci $\mathcal{G}_0^n$ of the first kind, each isomorphic to $\mathcal{G}_0$. We have $\#\mathcal{G}_1(\mathbb{F}_{p^2}) = (p^2+1)(p^3+1)$ (see [18]) and $\#\mathcal{G}_0(\mathbb{F}_{p^2}) = (p^2+1)^2(p^3+1)$ since $\mathcal{G}_0$ is a $\mathbb{P}^1$-bundle over $\mathcal{G}_1$. On $\mathcal{F}_1$ these loci $\mathcal{G}_1$ of the first kind are disjoint and we see

$$\#\mathcal{F}_0(\mathbb{F}_{p^2}) = (p^2+1)(p^4+1)\,\#\mathcal{G}_0(\mathbb{F}_{p^2})\,.$$

On each component $\mathcal{G}_0^n$ a section of $\mathcal{G}_0 \to \mathcal{G}_1$ is blown down. This section has $(p^2+1)(p^3+1)$ points rational over $\mathbb{F}_{p^2}$.

Lemma 15.2 *Each superspecial point of S_4 lies on $(p+1)(p^3+1)$ irreducible components of S_4.*

Proof The number of totally isotropic subspaces of dimension 2 in a 4-dimensional unitary space over $\mathbb{F}_{p^2}$ with conjugation given by Frobenius is equal to $(p+1)(p^3+1)$. A choice of an irreducible component corresponds exactly to the choice of a totally isotropic subspace. □

We thus see that under the natural map

$$\bigsqcup_{j\in J} \mathcal{F}_0^j \longrightarrow S_4$$

the inverse image of each of the Σ_4 superspecial points of S_4 has

$$(p+1)(p^3+1) \times (p^2+1)(p^3+1) \times (p^2+1)$$

points, where the second factor corresponds to blowing down the section of $\mathcal{G}_0 \to \mathcal{G}_1$, and the third one comes from the fact that each exceptional curve on $\tilde{\mathcal{F}}_2$ intersects p^2+1 proper images of the lines defined over $\mathbb{F}_{p^2}$, in agreement with the formula

$$N_4\,(p^2+1)^3(p^3+1)(p^4+1) = \Sigma_4\,(p+1)(p^2+1)^2(p^3+1)^2\,.$$

16 The cycle class of S_4 and intersection numbers

In this section we express the cycle class of the supersingular locus S_4 for dimension $g=4$ in terms of intersection numbers.

We know that the cycle class of S_4 lies in the tautological ring and is a multiple of $\lambda_4\lambda_2$. This multiple can be determined by intersection numbers. We identify the degree of a top-dimensional Chern class with an intersection number.

Proposition 16.1 *We have $[S_4] = a\,\lambda_4\lambda_2$ with*

$$a = \frac{\lambda_3\lambda_1\,[S_4]}{v(4)} = \frac{\lambda_1^4[S_4]}{8\,v(4)}$$

with $v(4)$ the proportionality constant defined in Section 2.

Proof We have $\lambda_3\lambda_1[S_4] = a\,\lambda_4\lambda_3\lambda_2\lambda_1 = a\,v(4)$. In the tautological ring R_4 we have $\lambda_3\lambda_1 = \lambda_1^4/8$. □

We shall calculate the intersection number $[S]\cdot\lambda_3\lambda_1$ for each irreducible component S of S_4. We will do this by pulling back the Hodge bundle of $\mathcal{A}_4$ to $\mathcal{F}_0$ and calculating the degrees of the top Chern classes of the Hodge bundle on $\mathcal{F}_0$.

17 Determination of intersection numbers

Our goal is to calculate the intersection number $\lambda_1\lambda_3[S]$ for each irreducible component S of the supersingular locus. For this we calculate $\deg(\lambda_3\lambda_1)$ on the 4-dimensional variety $\mathcal{F}_0$.

By Proposition 12.3, which describes the total Chern class of the Hodge bundle, and by Corollary 12.4 this intersection number can be expressed in the intersection numbers given by the monomials of degree 4 in ℓ_0, ℓ_1, ℓ_2 evaluated at the fundamental class of $\mathcal{F}_0$. Note that we write ℓ_1 and ℓ_2 for their pullbacks to $\mathcal{F}_0$ and sometimes identify such a monomial $\ell_0^a\ell_1^b\ell_2^c$ with $\deg(\ell_0^a\ell_1^b\ell_2^c)$.

Lemma 17.1 *The following intersection numbers vanish on $\mathcal{F}_0$:*

$$\ell_0^4,\ \ell_0^2\ell_1^2,\ \ell_0^2\ell_1\ell_2,\ \ell_0^2\ell_2^2,\ \ell_0\ell_2^3,\ \ell_1^4,\ \ell_1^3\ell_2,\ \ell_1^2\ell_2^2,\ \ell_1\ell_2^3,\ \ell_2^4\,.$$

Proof Since $\dim \mathcal{F}_1 = 3$ and ℓ_0^2 is a pullback from $\mathcal{F}_1$ by Corollary 12, and ℓ_1 and ℓ_2 are also pullbacks from $\mathcal{F}_1$ we find that $\ell_0^4, \ell_0^2\ell_1^2, \ell_0^2\ell_1\ell_2, \ell_0^2\ell_2^2$ vanish.

Since the class ℓ_2 is a pullback from $\mathcal{F}_2$, which is of dimension 2, we have $\ell_2^3 = 0$, implying that $\ell_2^4 = \ell_0\ell_2^3 = \ell_1\ell_2^3 = 0$. Similarly, ℓ_2 and ℓ_1 are induced from $\mathcal{F}_1$, which is of dimension 3, hence the monomials of degree 4 in ℓ_1 and ℓ_2 vanish. □

Proposition 12.3 together with Lemma 17.1 implies the following relation.

Corollary 17.2 *We have on $\mathcal{F}_0$*

$$\deg(\lambda_3\lambda_1) = \frac{1}{2}(p-1)^4\left(\ell_0^3\ell_1 + \ell_0^3\ell_2 + \ell_0\ell_1^3 + 3\,\ell_0\ell_1^2\ell_2 + 3\,\ell_0\ell_1\ell_2^2\right).$$

Thus we need the intersection numbers defined by the five monomials in ℓ_0, ℓ_1, ℓ_2 appearing in Corollary 17.2.

The intersection numbers $(\ell_0\ell_1^3, \ell_0\ell_1^2\ell_2, \ell_0\ell_1\ell_2^2)$ on $\mathcal{F}_0$ are equal to the intersection numbers $(\ell_1^3, \ell_1^2\ell_2, \ell_1\ell_2^2)$ on $\mathcal{F}_1$ as the degree of ℓ_0 on a generic fibre of π_0 is 1.

Lemma 17.3 *We have* $\deg \ell_1\ell_2^2 = p^2(p^2+1)$ *on $\mathcal{F}_1$.*

Proof The space $\mathcal{F}_2$ can be identified with the surface in $\mathbb{P}^3$ over $\mathbb{F}_p$ given by the equation

$$x_1x_4^{p^2} - x_1^{p^2}x_4 + x_2x_3^{p^2} - x_2^{p^2}x_3 = 0$$

and ℓ_2 is represented by the pullback under π_1 of the hyperplane class h on $\mathcal{F}_2$. Therefore h^2 can be represented by an effective zero cycle of degree p^2+1. The surface $\mathcal{F}_2$ is unirational (see [18]), hence h^2 can be represented by p^2+1 times a point. The morphism π_1 is inseparable of degree p, hence the pullback of a point $\mathcal{F}_2$ is p times a fibre of $\mathcal{F}_1$. Since the degree of ℓ_1 on a fibre of π_1 is p we get $\deg(\ell_1\ell_2^2) = p\cdot p\cdot(p^2+1)$. □

Lemma 17.4 *We have on* $\mathcal{F}_0$ *the relation*

$$p\,\ell_0^3\ell_1 - (p^2+1)\,\ell_0^3\ell_2 + p\,\ell_0\ell_1^3 - (p-1)^2\,\ell_0\ell_1^2\ell_2 - (2p^2-p+2)\ell_0\ell_1\ell_2^2 = 0\,.$$

Proof This follows from the fact that λ_4 vanishes in the Chow ring of $\mathcal{A}_g$ as explained in Section 2 and the expression for λ_4 as a polynomial in the ℓ_i by Proposition 12.3 and Corollary 12.4. □

Corollary 17.5 *On* $\mathcal{F}_1$ *we have the relation*

$$p\,\ell_0^2\ell_1 - (p^2+1)\,\ell_0^2\ell_2 + p\,\ell_1^3 - (p-1)^2\,\ell_1^2\ell_2 - (2p^2-p+2)\ell_1\ell_2^2 = 0\,.$$

Proof We know that $\mathcal{F}_0$ is a $\mathbb{P}^1$-bundle over $\mathcal{F}_1$. Therefore each cycle class $\xi\in A_k(\mathcal{F}_0)$, the dimension k Chow group of $\mathcal{F}_0$, can be written uniquely as $\xi = \pi_0^*(\xi_0)+\pi_0^*(\xi_1)\ell_0$ with $\xi_0\in A_{k-1}(\mathcal{F}_1)$ and $\xi_1\in A_k(\mathcal{F}_1)$. In particular, the map $\xi_1\mapsto \pi_0^*(\xi_1)\ell_0$ is injective. The result thus follows from Lemma 17.4. □

Lemma 17.6 *We have on* $\mathcal{F}_1$ *the relation*

$$2\,\ell_0^2\ell_1 - (p-1)\ell_0^2\ell_2 + (p-1)\ell_1^2\ell_2 - 2(p^2-p+1)\ell_1\ell_2^2 = 0\,.$$

Proof We have the exact sequence of Dieudonné modules

$$0\to A\to VM_2/pM_2\to VM_2/VM_1\to 0$$

with $\mathrm{Lie}(Y_2)^\vee = VM_2/pM_2$ and $Q_1 = VM_2/VM_1$. The total Chern class of the sheaf corresponding to A has the form

$$c(A) = (1-\ell_2)(1-p\,\ell_2)^{-1}(1+\ell_1+c_2(Q_1))^{-1}\,.$$

Since $\mathrm{rank}(A)=2$ the third Chern class should vanish; this gives a relation on $\mathcal{F}_1$

$$2\ell_0^2\ell_1 - (p-1)\ell_0^2\ell_2 + (p-1)\ell_1^2\ell_2 - 2(p^2-p+1)\ell_1\ell_2^2 = 0\,.$$

□

Lemma 17.7 *On* $\mathcal{F}_1$ *we have the relation*

$$p\,\ell_0^2\ell_2 - p\,\ell_1^2\ell_2 + 2(p^2-p+1)\ell_1\ell_2^2 = 0\,.$$

Proof Let H be a hyperplane section of $\mathcal{F}_2$ with $H \cap \mathcal{F}(\mathbb{F}_{p^2}) = \emptyset$. We work on $\pi_1^{-1}(H)$. Here we have that $\dim M_2/(F, V)M_2 = 3$ and we thus have a rank 3 locally free sheaf B on H determined by $M_2/(F, V)M_2$. Because of the exact sequence

$$0 \to VM_2/VM_2 \cap FM_2 \to M_2/FM_2 \to M_2/(F, V)M_2 \to 0$$

we have the exact sequence

$$0 \to U_2 \to \mathrm{Lie}(Y_2)^{(p)\vee} \to B \to 0\,,$$

since $VM_2/VM_2 \cap FM_2 = VM_2/pM_3$. We thus find

$$[B] = [4] - [Q_2^{(p)}] + [Q_2^{(p^2)}] - [U_2] = [4] + [U_2^{(p)}] - [U_2^{(p^2)}] - [U_2]\,.$$

We also have the inclusions $(F, V)M_2 \subset M_1 \subset M_2$ on $\pi_1^{-1}(H)$ and we thus have a locally free sheaf L corresponding to $M_1/(F, V)M_2$. In the Grothendieck group we have the corresponding relation $[B] = [L] + [Q_1^{(p)}]$. Thus we find $[L] = [4] + [U_2^{(p)}] - [U_2^{(p^2)}] - [U_2] - [Q_1^{(p)}]$ and we see that the total Chern class of L is given by

$$c(L) = \frac{(1 - p\ell_2)}{(1 - p^2\ell_2)(1 - \ell_2)} \, \frac{1}{(1 + p\ell_1 + p^2 c_2(Q_1))}\,.$$

But L has rank 1, so $c_2(L) = 0$. With $c_2(Q_1) = (\ell_0^2 + \ell_1^2 - \ell_2^2)/2$ this gives

$$(p^4 - p^3 + \frac{3}{2}p^2 - p + 1)\ell_2^2 - (p^3 - p^2 + p)\ell_1\ell_2 - \frac{1}{2}p^2\ell_0^2 + \frac{1}{2}p^2\ell_1^2 = 0\,.$$

Recall now that the class of H is ℓ_2. Multiplying the preceding relation by ℓ_2 and using $\ell_2^3 = 0$ we find

$$p\ell_0^2\ell_2 - p\ell_1^2\ell_2 + 2(p^2 - p + 1)\ell_1\ell_2^2 = 0\,.$$

□

As remarked above we need five intersection numbers:

$$\ell_0^3\ell_1,\ \ell_0^3\ell_2,\ \ell_0\ell_1^3,\ \ell_0\ell_1^2\ell_2,\ \ell_0\ell_1\ell_2^2\,.$$

We know already the last one by Lemma 17.3. By multiplying the relations of Lemma 17.6 and 17.7 by ℓ_0 we find in total three relations coming from Lemmas 17.4, 17.6 and 17.7 between these five intersection numbers.

Corollary 17.8 *We have* $\deg(\ell_0^3\ell_1) = p(p^2 + 1)(p^2 - p + 1)$.

Proof The sum of p times the relation of 17.6 and $(p - 1)$ times that of 17.7 gives the relation $2p\,\ell_0^3\ell_1 - 2(p^2 - p + 1)\ell_0\ell_1\ell_2^2 = 0$. □

Using the three relations and Lemma 17.3 our five intersection numbers depend on one unknown.

Corollary 17.9 *With* $x = \deg(\ell_0\ell_1^2\ell_2)$ *we find that*

$$\deg\begin{bmatrix} \ell_0^3\ell_1 \\ \ell_0^3\ell_2 \\ \ell_0\ell_1^3 \\ \ell_0\ell_1^2\ell_2 \\ \ell_0\ell_1\ell_2^2 \end{bmatrix} = \begin{bmatrix} p(p^2+1)(p^2-p+1) \\ x-2p(p^2+1)(p^2-p+1) \\ 2(p-1+1/p)x-(p^2+1)^2(2p^2-3p+2) \\ x \\ p^2(p^2+1) \end{bmatrix}.$$

Remark 17.10 We have on $\mathcal{F}_0$

$$\deg\lambda_1^4 = 8\,(p-1)^4(p^2+p+1)\left(\frac{\deg(\ell_0\ell_1^2\ell_2)}{p}-(p^2+1)(p-1)^2\right).$$

Since λ_1 is ample on S_4 this should be positive and this gives

$$\deg(\ell_0\ell_1^2\ell_2) > p(p^2+1)(p-1)^2\,.$$

We now determine the last intersection number. Recall that the second Chern class $c_2(Q_1)$ satisfies $c_2(Q_1) = (\ell_0^2+\ell_1^2-\ell_2^2)/2$. Furthermore, recall the cycle class $[\overline{D(\psi)}]$ of a 'horizontal' $a \geq 3$-locus on $\mathcal{F}_1$ given by

$$[\overline{D(\psi)}] = p\,\ell_1-(p^2+1)\ell_2+e$$

with e a class with support in the exceptional fibres as given in Lemma 13.2.

Proposition 17.11 *We have* $c_2(Q_1)\cdot[\overline{D(\psi)}] = 0$ *and* $c_2(Q_1)\cdot e = 0$.

Proof Since Q_1 is the tautological quotient of the $O_{\mathcal{F}_1}$-module associated to M_2/FM_2 by the universal rank 2 subbundle U_1, the second Chern class can be realized as the class of the locus where the fibre of U_1 contains a fixed vector. For this we choose an element v' of M_2/FM_2 that has the property that over each affine part of $\mathcal{F}_2$ with $a_i \neq 0$ (for $i = 1,\ldots,4$) it is of the form

$$v' = \alpha_5\,v_0+\alpha_6\,Fx_2+\alpha_7Fx_3+\alpha_8Fx_4$$

with the property that the equation $g_2 = 0$, that is,

$$a_1\,\alpha_8^p-a_1^p\alpha_5^{p-1}\alpha_8+a_2\alpha_7^p-a_2^p\alpha_5^{p-1}\alpha_7+a_3^p\alpha_5^{p-1}\alpha_6-a_3\alpha_6^p = 0$$

has no solutions with $(a_1,a_2,a_3,a_4)\in\mathbb{F}_{p^2}$ with $a_i \neq 0$. Indeed, choosing $\alpha_5 \neq 0$, α_6 and α_7 there are only finitely many α_8 satisfying this equation. Then since $\alpha_5 \neq 0$, we see that this locus has zero intersection with $\overline{D(\psi)}$. We get $c_2(Q_1)\cdot[\overline{D(\psi)}] = 0$. By the requirement that we put over $\mathcal{F}_2(\mathbb{F}_{p^2})$ we see that also $c_2(Q_1)\cdot e = 0$. □

Corollary 17.12 *We have* $(\ell_0^2 + \ell_1^2 - \ell_2^2)(p\ell_1 - (p^2+1)\ell_2) = 0$.

Proof Recall that $c_2(Q_1) = (\ell_0^2 + \ell_1^2 - \ell_2^2)/2$ and $[\overline{D(\psi)}] = p\ell_1 - (p^2+1)\ell_2 + e$ with e a class with support in the exceptional fibres. □

By combining Corollary 17.9 and Corollary 17.12 we can determine all the intersection numbers.

Corollary 17.13 *We have on* $\mathcal{F}_0$

$$\deg \begin{bmatrix} \ell_0^3\ell_1 \\ \ell_0^3\ell_2 \\ \ell_0\ell_1^3 \\ \ell_0\ell_1^2\ell_2 \\ \ell_0\ell_1\ell_2^2 \end{bmatrix} = p\,(p^2+1) \begin{bmatrix} p^2-p+1 \\ -p^2+p-1 \\ -(p-1)^2 \\ p^2-p+1 \\ p \end{bmatrix}.$$

Finally we are ready to calculate the coefficient $f_4(p)$ of Theorem 1.2.

Theorem 17.14 *The class of the supersingular locus* $S_4 \subset \mathcal{A}_4 \otimes \mathbb{F}_p$ *in the Chow ring of* $\tilde{\mathcal{A}}_4 \otimes \mathbb{F}_p$ *equals*

$$[S_4] = (p-1)^3(p^3-1)(p^4-1)(p^6-1)\lambda_4\lambda_2\,.$$

Proof For each irreducible component S of S_4 we calculate the degree of $\lambda_3\lambda_1$ on the model $\mathcal{F}_0$ of S. Indeed, we have $[S_4] = a\lambda_4\lambda_2$ with $a = \lambda_3\lambda_1[S_4]/v(4)$ by Proposition 16.1. A calculation using Corollary 17.13 and taking into account the degree p of the map $\mathcal{F}_0 \to S$ (see Lemma 12.1) yields that $\deg(\lambda_3\lambda_1)$ on S equals $1/p$ times the degree on $\mathcal{F}_0$ of

$$\begin{aligned}&(p^2-3p+1)\ell_0^3\ell_1 + (2p^2-2p+2)\ell_0^3\ell_2 + (p^2-3p+1)\ell_0\ell_1^3 \\ &\quad + 4(p-1)^2\ell_0\ell_1^2\ell_2 + (5p^2-7p+5)\ell_0\ell_1\ell_2^2\end{aligned}$$

and this equals $(p-1)^4(p^2+p+1)(p^2+1)$. Multiplying this with the number of irreducible components $(p^2-1)(p^6-1)v(4)$ we find the coefficient $a = (p-1)^3(p^3-1)(p^4-1)(p^6-1)$. □

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