Cycle Classes of the E-O Stratification on the Moduli of Abelian Varieties

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To Yuri Ivanovich Manin on the occasion of his 70th birthday

Summary. We introduce a stratification on the space of symplectic flags on the de Rham bundle of the universal principally polarized abelian variety in positive characteristic. We study its geometric properties, such as irreducibility of the strata, and we calculate the cycle classes. When the characteristic p is treated as a formal variable these classes can be seen as a deformation of the classes of the Schubert varieties for the corresponding classical flag variety (the classical case is recovered by putting p equal to 0). We relate our stratification with the E-O stratification on the moduli space of principally polarized abelian varieties of a fixed dimension and derive properties of the latter. Our results are strongly linked with the combinatorics of the Weyl group of the symplectic group.

Key words: moduli space, abelian variety, E-O-stratification, cycle classes, Weyl group.

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1 Introduction

The moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g is defined over the integers. For the characteristic-zero fiber $\mathcal{A}_g \otimes \mathbb{C}$ we have an explicit description as an orbifold $\mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g$ with $\mathcal{H}_g$ the Siegel upper half-space of degree g. It is a recent insight, though, that perhaps the positive characteristic fibres $\mathcal{A}_g \otimes \mathbb{F}_p$ are more accessible than the characteristic-zero one. A good illustration of this is provided by the E-O stratification of $\mathcal{A}_g \otimes \mathbb{F}_p$, a stratification consisting of 2^g quasi-affine strata. It was originally defined by Ekedahl and Oort (see [Oo01]) by analyzing the structure of the kernel of

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multiplication by p of an abelian variety. It turns out that this group scheme can assume 2^g forms only, and this led to the strata. For $g = 1$ the two strata are the loci of ordinary and of supersingular elliptic curves. Some strata possess intriguing properties. For example, the stratum of abelian varieties of p-rank 0 is a complete subvariety of $\mathcal{A}_g \otimes \mathbb{F}_p$ of codimension g, the smallest codimension possible. No analogue in characteristic 0 of either this stratum or the stratification is known. In fact, Keel and Sadun [KS03] proved that complete subvarieties of $\mathcal{A}_g \otimes \mathbb{C}$ of codimension g do not exist for $g \geq 3$.

While trying to find cycle classes for the E-O strata we realized that the strata could be described as degeneration loci for maps between vector bundles, and since such loci are indexed by Young diagrams, our attention was turned towards the combinatorics of the Weyl group. When considered in this light it is clear that much of the combinatorics of [Oo01] is closely related to the Weyl group W_g of Sp_{2g}, which is the semisimple group relevant for the analytic description of $\mathcal{A}_g \otimes \mathbb{C}$. The main idea of this paper is to try to make this connection more explicit. More precisely, the combinatorics of the E-O strata is most closely related to the combinatorics associated to W_g and the Weyl subgroup corresponding to the maximal parabolic subgroup P of elements of Sp_{2g} stabilizing a maximal isotropic subspace in the $2g$-dimensional symplectic vector space. Indeed, this sub-Weyl group is S_g, the group of permutations on g letters (embedded as a sub-Weyl group in W_g), and the E-O strata are in bijection with the cosets in W_g/S_g; we shall use the notation $\mathcal{V}_\nu$ for the (open) stratum of $\mathcal{A}_g \otimes \mathbb{F}_p$ corresponding to $\nu \in W_g/S_g$ (and $\overline{\mathcal{V}}_\nu$ for its closure). The coset space W_g/S_g is also in bijection with the set of Bruhat cells in the space of maximal totally isotropic flags Sp_{2g}/P and we believe this to be no accident. (The formal relation between $\mathcal{A}_g$ and Sp_{2g}/P is that Sp_{2g}/P is the *compact dual* of $\mathcal{H}_g$.)

In order to push the analogy further we introduce a "flag space" $\mathcal{F}_g \to \mathcal{A}_g$ whose fibers are isomorphic to the fibers of the quotient morphism $\mathrm{Sp}_{2g}/B \to \mathrm{Sp}_{2g}/P$, where B is a Borel subgroup of P. In positive characteristic we define (and this definition makes sense only in positive characteristic) a stratification of $\mathcal{F}_g$, whose open strata $\mathcal{U}_w$ and their corresponding closures $\overline{\mathcal{U}}_w$ are parametrized by the elements of W_g. This stratification is very similar to the stratification by Bruhat cells of Sp_{2g}/B and their closures, the Schubert strata, which are also parametrized by the elements of W_g. Our first main result is that this is more than a similarity when one works locally; we show (cf., Therorem 8.2) that for each point of $\mathcal{F}_g$ there is a stratum-preserving local isomorphism (in the étale topology) taking the point to some point of Sp_{2g}/B. Since much is known about the local structure of the Schubert varieties we immediately get a great deal of information about the local structure of our strata. The first consequence is that $\overline{\mathcal{U}}_w$ is equidimensional of dimension equal to the length of w. A very important consequence is that the $\overline{\mathcal{U}}_w$ are all normal; this situation differs markedly from the case of the closed E-O strata, which in general are not normal. Another consequence is that the inclusion relation between the strata is given exactly by the Bruhat–Chevalley order on W_g.

(A much more sophisticated consequence is that the local structure of the ℓ-adic intersection complex for a closed stratum is the same as for the Schubert varieties and in particular that the dimensions of its fibers over the open strata of the closed stratum are given by the Kazhdan–Lusztig polynomials. We shall not, however, pursue that in this article.)

We give several applications of our results to the structure of the strata $\overline{\mathcal{U}}_w$. The first, and most important, is that by construction the strata $\mathcal{U}_w$ are defined as the loci where two symplectic flags on the same vector bundle are in relative position given by w. After having shown that they have the expected codimension and are reduced, we can use formulas of Fulton, as well as those of Pragacz and Ratajski, as crystallized in the formulas of Kresch and Tamvakis, to get formulas for the cycle classes of the strata. A result of Fulton gives such formulas (cf., Theorem 12.1) for all strata but in terms of a recursion formula that we have not been able to turn into a closed formula; however, these formulas should have independent interest and we use them to get formulas for the E-O strata as follows. If $w \in W_g$ is minimal for the Bruhat–Chevalley order in its coset wS_g, then $\mathcal{U}_w$ maps by a finite étale map to the open E-O stratum $\mathcal{V}_\nu$ corresponding to the coset $\nu := wS_g$. We can compute the degree of this map in terms of the combinatorics of the element w and we then can push down our formulas for $\overline{\mathcal{U}}_w$ to obtain formulas for the cycle classes of the E-O strata. Also the formulas of Kresch and Tamvakis can be used to give the classes of E-O strata. One interesting general consequence (cf., Theorem 13.1) is that each class is a polynomial in the Chern classes λ_i of the Hodge bundle, the cotangent bundle of the zero-section of the universal abelian variety, and the coefficients are polynomials in p. This is a phenomenon already visible in the special cases of our formula that were known previously; the oldest such example being Deuring's mass formula for the number of supersingular elliptic curves (weighted by one over the cardinalities of their groups of automorphisms) that says that this mass is $(p-1)/12$. This appears in our context as the combination of the formula $(p-1)\lambda_1$ for the class of the supersingular locus and the formula $\deg \lambda_1 = 1/12$. We interpret these results as giving rise to elements in the *p-tautological ring*; this is the ring obtained from the usual *tautological ring*, the ring generated by the Chern classes of the Hodge bundle, by extending the scalars to $\mathbb{Z}\{p\}$, the localization of the polynomial ring $\mathbb{Z}[p]$ at the polynomials with constant coefficient 1. Hence we get elements parametrized by W_g/S_g in the p-tautological ring and we show that they form a $\mathbb{Z}\{p\}$-basis for the p-tautological ring. Putting p equal to 0 maps these elements to elements of the ordinary tautological ring that can be identified with the Chow ring of Sp_{2g}/P, and these elements are the usual classes of the Schubert varieties. It seems that these results call for a p-Schubert calculus in the sense of a better understanding of these elements of the p-tautological ring and their behavior under multiplication.

However, there seems to be a more intriguing problem. We have for each $w \in W_g$ a stratum in our flag space and these strata push down to elements of the p-tautological ring under the projection map to $\tilde{\mathcal{A}}_g$ (a suitable toroidal compactification of $\mathcal{A}_g$). When setting p to 0 these elements specialize to the classes of the images of the Schubert varieties of Sp_{2g}/B_g in Sp_{2g}/P, and for them the situation is very simple: either w is minimal in its S_g coset and then the Schubert variety maps birationally to the corresponding Schubert variety of Sp_{2g}/P or it is not and then it maps to 0. When it comes to the elements of the p-tautological ring this allows us to conclude only – in the nonminimal case – that the coefficients are divisible by p, and indeed in general, they are not zero. We show that unless they map to 0 they will always map to a multiple of a class of an E-O stratum. When the element w is minimal in its S_g-coset, this stratum is indexed by the coset spanned by w. Our considerations give an extension of this map from elements minimal in their cosets to a larger class of elements. We give some examples of this extension, but in general it seems a very mysterious construction.

Another application is to the irreducibility of our strata (and hence also to the strata of the E-O stratification, since they are images of some of our strata). Since the strata are normal, this is equivalent to the connectedness of a stratum, and this connectedness can sometimes (cf. Theorem 11.5) be proved via an arithmetic argument. It is natural to ask whether this method produces all the irreducible strata, and for the characteristic large enough (the size depending on g) we can show that indeed it does. This is done using a Pieri-type formula for our strata obtained by applying a result of Pittie and Ram. A Pieri-type formula for multiplying the class of a connected cycle by an ample line bundle has as a consequence that a part of the boundary is supported by an ample line bundle and hence that this part of the boundary is connected. Applying this to λ_1, which is an ample line bundle on $\mathcal{A}_g$, allows us to show that our results are optimal; cf. [Ha07]. We are forced to assume that the characteristic is large (and are unable to specify how large), since we do not know by which power of λ_1 one needs to twist the exterior powers of the dual of the Hodge bundle to make these generated by global sections.

There is a particular element of $w_\emptyset \in W_g$ that is the largest of the elements that are minimal in their right S_g-cosets and that has the property that $\overline{\mathcal{U}}_{w_\emptyset}$ maps generically of finite degree onto $\mathcal{A}_g$. It is really the strata that are contained in this stratum that seem geometrically related to $\mathcal{A}_g$, and indeed the elements $w \in W_g$ lying below $w_\emptyset$ are the ones of most interest. (The rest of $\mathcal{F}_g$ appears mostly as a technical device for relating our strata to the Schubert varieties.) It should be of particular interest to understand the composite map $\overline{\mathcal{U}}_{w_\emptyset} \subset \mathcal{F}_g \to \mathcal{A}_g$. It follows from a result of Oort on Dieudonné modules that the inverse image of an open E-O stratum under this map is a locally constant fibration. This focuses interest on its fibers, a fiber depending only on the element of $\nu \in W_g/S_g$ that specifies the E-O stratum. We call these fibers *punctual flag spaces* (see Section 9 for details). We determine their connected components, showing in particular that two points in the same connected component can be connected by a sequence of quite simple rational curves. We also show that knowing which strata $\mathcal{U}_w$ have nonempty intersections with a given punctual flag space would determine the inclusion relations between the E-O strata.

A geometric point x of the stratum $\overline{\mathcal{U}}_{w_\emptyset}$ corresponds to a symplectic flag of subgroup schemes of the kernel of multiplication by p on the principally polarized abelian variety that is the image of x in $\mathcal{A}_g$ under the map $\mathcal{F}_g \to \mathcal{A}_g$. This is reminiscent of de Jong's moduli stack $\mathcal{S}(g,p)$ of $\Gamma_0(p)$-structures; cf. [Jo93]. The major difference (apart from the fact that $\overline{\mathcal{U}}_{w_\emptyset}$ makes sense only in positive characteristic) is that the g-dimensional element of the flag is determined by the abelian variety in our case. We shall indeed identify $\overline{\mathcal{U}}_{w_\emptyset}$ with the component of the fiber at p of $\mathcal{S}(g,p)$ that is the closure of the ordinary abelian varieties provided with a flag on the local part of the kernel of multiplication by p. As a consequence we get that that component of $\mathcal{S}(g,p)$ is normal and Cohen–Macaulay.

This paper is clearly heavily inspired by [Oo01]. The attentive reader will notice that we re-prove some of the results of that paper, sometimes with proofs that are very close to the proofs used by Oort. We justify such duplications by our desire to emphasize the relations with the combinatorics of W_g and the flag spaces. Hence, we start with (a rather long) combinatorial section in which the combinatorial aspects have been separated from the geometric ones. We hope that this way of presenting the material will be as clarifying to the reader as it has been to us. We intend to continue to exploit the approach using the flag spaces in a future paper that will deal with K3 surfaces. Since its announcement in [Ge99], our idea of connecting the E-O stratification on $\mathcal{A}_g$ with the Weyl group and filtrations on the de Rham cohomology has been taken up in other work. In this connection we want to draw attention to papers by Moonen and Wedhorn; cf. [Mo01, MW04].

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Conventions. We shall exclusively work in positive characteristic $p > 0$ (note, however, that in Section 13 the symbol p will also be a polynomial variable). After having identified final types and final elements in Section 2 we shall often use the same notation for the final type (which is a function on $\{1, \dots, 2g\}$) and the corresponding final element (which is an element of the Weyl group W_g). In Sections 10 and 11 our strata will be considered in flag spaces over not just $\mathcal{A}_g$ and $\tilde{\mathcal{A}}_g$ but also over the corresponding moduli stacks with a level structure. We shall define several natural objects, such as the Hodge bundle, over several different spaces (such as the moduli space of abelian varieties as well as toroidal compactifications of it). In order not to make the notation overly heavy, the same notation (such as $\mathbb{E}$) will normally be used for the objects on different spaces. Since the objects in question will be compatible with pullback, this should not cause confusion. Sometimes, when there might still be such a risk we shall use subscripts (such as $\mathbb{E}_{\mathcal{A}_g}$ and $\mathbb{E}_{\tilde{\mathcal{A}}_g}$) to distinguish between them. Also, when there is no risk of confusion we shall use the same name (like $\mathcal{U}_w$ and $\overline{\mathcal{U}}_w$) for a stratum and its extension to the compactified moduli space.

Our moduli objects such as $\mathcal{A}_g$ are really Deligne–Mumford stacks. However, in order to avoid what we have found to be a sometimes awkward terminology (such as "flag stacks"), we shall usually speak of them as spaces rather than stacks. In a similar vein, by for instance a "locally closed subset" of an algebraic stack we shall mean a reduced locally closed substack.

2 Combinatorics

This section is of a preparatory nature and deals with the combinatorial aspects of the E-O stratification. The combinatorics is determined by the Weyl group of the symplectic group of degree g. A general reference for the combinatorics of Weyl groups is [BL00]. We start by recalling some general notation and facts about W_g and its Bruhat–Chevalley order. We then go on to give various descriptions of the minimal elements in the S_g cosets (which we presume are well known). The short subsection on shuffles will be used to understand the rôle that the multiplicative and étale parts of the Barsotti–Tate group play in our stratification in the case of positive p-rank.

2.1 Final Elements in the Weyl Group

The Weyl group W_g of type C_g in Cartan's terminology is isomorphic to the semidirect product $S_g \ltimes (\mathbb{Z}/2\mathbb{Z})^g$, where the symmetric group S_g on g letters acts on $(\mathbb{Z}/2\mathbb{Z})^g$ by permuting the g factors. Another description of this group, and the one we shall use here, is as the subgroup of the symmetric group S_{2g} formed by elements that map any symmetric 2-element subset of $\{1,\dots,2g\}$ of the form $\{i, 2g+1-i\}$ to a subset of the same type:

$$W_g = \{\sigma \in S_{2g} \colon \sigma(i) + \sigma(2g+1-i) = 2g+1 \text{ for } i = 1,\dots,g\}.$$

The function $i \mapsto 2g+1-i$ on the set $\{1,\dots,2g\}$ will occur frequently. We shall sometimes use the notation $\bar{\imath}$ for $2g+1-i$. Using it we can say that $\sigma \in S_{2g}$ is an element of W_g precisely when $\sigma(\bar{\imath}) = \overline{\sigma(i)}$ for all i. This makes the connection with another standard description of W_g, namely as a group of *signed permutations*. An element w in this Weyl group has a *length* and a *codimension* defined by

$$\ell(w) = \#\{i < j \leq g : w(i) > w(j)\} + \#\{i \leq j \leq g : w(i) + w(j) > 2g+1\},$$

and

$$\operatorname{codim}(w) = \#\{i < j \leq g \colon w(i) < w(j)\} + \#\{i \leq j \leq g \colon w(i) + w(j) < 2g+1\}$$

and these satisfy the equality

$$\ell(w) + \operatorname{codim}(w) = g^2.$$

We shall use the following notation for elements in W_g. By $[a_1, a_2, \dots, a_{2g}]$ we mean the permutation of $\{1, 2, \dots, 2g\}$ with $\sigma(i) = a_i$. Since $\sigma(i)$ determines

$\sigma(2g+1-i)$ for $1 \le i \le g$, sometimes we use the notation $[a_1, \ldots, a_g]$ instead (when the a_i are single digits we shall often dispense with the commas and write $[a_1 \ldots a_g]$, which should cause no confusion). We shall also use cycle notation for permutations. In particular, for $1 \le i < g$ we let $s_i \in S_{2g}$ be the permutation $(i, i+1)(2g-i, 2g+1-i)$ in W_g, which interchanges i and $i+1$ (and then also $2g-i$ and $2g+1-i$) and we let $s_g = (g, g+1) \in S_{2g}$. Then the pair $(W = W_g, S = \{s_1, \ldots, s_g\})$ is a Coxeter system.

Let (W, S) be a Coxeter system and $a \in W$. If X is a subset of S we denote by W_X the subgroup of W generated by X. It is well known that for any subset X of S there exists precisely one element w of minimal length in aW_X and it has the property that every element $w' \in aW_X$ can be written in the form $w' = wx$ with $x \in W_X$ and $\ell(w') = \ell(w) + \ell(x)$. Such an element w is called an *X-reduced element*; cf. [GrLie4-6, Chapter IV, Exercises §1].

Let $W = W_g$ be the Weyl group and S the set of simple reflections. If we take $X = S \backslash \{s_g\}$, then we obtain

$$W_X = \{\sigma \in W_g \colon \sigma\{1, 2, \ldots, g\} = \{1, 2, \ldots, g\}\} \cong S_g.$$

There is a natural partial order on W_g with respect W_X, the Bruhat–Chevalley order. It is defined in terms of Schubert cells $X(w_i)$ by

$$w_1 \le w_2 \iff X(w_1) \subseteq X(w_2).$$

Equivalently, if for $w \in W_g$ we define

$$r_w(i,j) := \#\{a \le i \colon\ w(a) \le j\}, \tag{1}$$

then we have the combinatorial characterization

$$w_1 \le w_2 \iff r_{w_1}(i,j) \ge r_{w_2}(i,j) \quad \text{for all } 1 \le i, j \le 2g.$$

(Indeed, it is easy to see that it is enough to check this for all $1 \le i \le g$ and $1 \le j \le 2g$.) Chevalley has shown that $w_1 \ge w_2$ if and only if any (hence every) X-reduced expression for w_1 contains a subexpression (obtained by just deleting elements) that is a reduced expression for w_2; here reduced means that w_2 is written as a product of $\ell(w_2)$ elements of S. Again, a reference for these facts is [BL00].

We now restrict to the following case. Let V be a symplectic vector space over $\mathbb{Q}$ and consider the associated algebraic group $G = \mathrm{Sp}(V)$. If $E \subset V$ is a maximal isotropic subspace, then the stabilizer of the flag $(0) \subset E \subset V$ is a parabolic subgroup conjugate to the standard parabolic subgroup corresponding to X. Since X is a heavily used letter we shall use $H := S \backslash \{s_g\} \subset S$ instead. Hence W_H will denote the subgroup of W_g generated by the elements of H and we will also use the notation P_H for the parabolic subgroup corresponding to W_H, i.e., the subgroup of the symplectic group stabilizing a maximal totally isotropic subspace. Note that W_H consists of the permutations of W_g that stabilize the subsets $\{1, \ldots, g\}$ and $\{g+1, \ldots, 2g\}$ and that the restriction of the action of an element of W_H to $\{1,\ldots,g\}$ determines the full permutation. Therefore, we may identify W_H with S_g, the group of permutations of $\{1,\ldots,g\}$, and we shall do so without further notice. (This is of course compatible with the fact that H spans an A_{g-1}-subdiagram of the Dynkin diagram of G.) There are $2^g = |W_g|/|W_H|$ elements in W_g that are H-reduced elements. These 2^g elements will be called *final* elements of W_g. The Bruhat–Chevalley order between elements in W_g as well as the condition for being H-reduced can be conveniently expressed in terms of the concrete representation of elements of W_g as permutations in the following way.

Let A, B be to two finite subsets of $\{1,2,\ldots,g\}$ of the same cardinality. We shall write $A \prec B$ if for all $1 \le i \le |A|$ the ith-largest element of A is at most equal to the ith-largest element of B.

Lemma 2.1. *(i) If $w = [a_1a_2\ldots a_g]$ and $w' = [b_1b_2\ldots b_g]$ are two elements of W_g, then $w \le w'$ in the Bruhat–Chevalley order precisely when for all $1 \le d \le g$ we have $\{a_1,a_2,\ldots,a_d\} \prec \{b_1,b_2,\ldots,b_d\}$.*

(ii) Let $w = [a_1a_2\ldots a_g] \in W_g$. Denote the final element of wW_H by w^f. For $w' = [b_1b_2\ldots b_g] \in W_g$ we have $w^f \le w'$ in the Bruhat–Chevalley order precisely when $\{a_1,a_2,\ldots,a_g\} \prec \{b_1,b_2,\ldots,b_g\}$.

*(iii) An element $\sigma \in W_g$ is H-*reduced *(or* final*) if and only if $\sigma(i) < \sigma(j)$ for all $1 \le i < j \le g$. Also, σ is H-reduced if and only if σ sends the first $g-1$ simple roots into positive roots.*

Proof. See for instance [BL00, p. 30]. □

2.2 Final Types and Young Diagrams

There are other descriptions of final elements that are sometimes equally useful. They involve maps of $\{1,2,\ldots,2g\}$ to $\{1,2,\ldots,g\}$ and certain Young diagrams. We begin with the maps.

Definition 2.2. *A* final type *(of degree g) is a nondecreasing surjective map*

$$\nu\colon \{0,1,2,\ldots,2g\} \to \{0,1,2,\ldots,g\}$$

satisfying

$$\nu(2g-i) = \nu(i) - i + g \qquad \text{for} \quad 0 \le i \le g.$$

We always have $\nu(0) = 0$ and $\nu(2g) = g$. Note that we have either $\nu(i+1) = \nu(i)$ and then $\nu(2g-i) = \nu(2g-i-1)+1$ or $\nu(i+1) = \nu(i)+1$ and then $\nu(2g-i) = \nu(2g-i-1)$. A final type is determined by its values on $\{0,1,\ldots,g\}$. There are 2^g final types of degree g corresponding to the vectors $(\nu(i+1)-\nu(i))_{i=0}^{g-1} \in \{0,1\}^g$. The notion of a final type was introduced by Oort [Oo01].

To an element $w \in W_g$ we can associate the final type ν_w defined by

$$\nu_w(i) := i - r_w(g,i).$$

This is a final type because of the rule $r_w(g, 2g-i) - r_w(g,i) = g-i$, which follows by induction on $i \in \{1,\ldots,g\}$ from the fact that $w(2g+1-a) = 2g+1-w(a)$. It depends only on the coset wW_H of w, since a permutation of $\{a : 1 \leq a \leq g\}$ does not change the definition of

$$r_w(g,i) = \#\{a \leq g\colon w(a) \leq i\}.$$

Conversely, to a final type ν we now associate an element w_ν of the Weyl group, a permutation of $\{1,2,\ldots,2g\}$, as follows. Let

$$\beta = \{i_1, i_2, \ldots, i_k\} = \{1 \leq i \leq g\colon \nu(i) = \nu(i-1)\}$$

with $i_1 < i_2 < \cdots$ given in *increasing* order and let

$$\beta^c = \{j_1, j_2, \ldots, j_{g-k}\}$$

be the elements of $\{1,2,\ldots,g\}$ not in ξ, in *decreasing* order. We then define a permutation w_ν by mapping $1 \leq s \leq k$ to i_s and $k+1 \leq s \leq g$ to $2g+1-j_{s-k}$. The requirement that w_ν belong to W_g now completely specifies w_ν and by construction $w_\nu(i) < w_\nu(j)$ if $1 \leq i < j \leq g$. Thus w_ν is a final element of W_g. It is clear from Lemma 2.1 that we get in this way all final elements of W_g. The Bruhat–Chevalley order for final elements can also be read off from the final type ν. We have $w \geq w'$ if and only if $\nu_w \geq \nu_{w'}$. This follows from Lemma 2.1, (ii).

We summarize:

Lemma 2.3. *By associating to a final type ν the element w_ν and to a final element $w \in W_g$ the final type ν_w we get an order-preserving bijection between the set of 2^g final types and the set of final elements of W_g.*

The final types are in bijection with certain Young diagrams. Our Young diagrams will be put in a position that is opposite to the usual positioning, i.e., larger rows will be below smaller ones and the rows will be lined up to the right (see next example). Furthermore, we shall make Young diagrams correspond to partitions by associating to a diagram the parts that are the lengths of its rows. Given g we shall say that a Young diagram is *final of degree g* if its parts are $\leq g$ and no two parts are equal. They therefore correspond to subsets ξ of $\{1,2,\ldots,g\}$. We write such a ξ as $\{\xi_1,\ldots,\xi_r\}$ with $g \geq \xi_1 > \cdots > \xi_r$.

To a final type ν we now associate the Young diagram Y_ν whose associated subset ξ is defined by

$$\xi_j = \#\{i\colon 1 \leq i \leq g,\, \nu(i) \leq i-j\}.$$

A pictorial way of describing the Young diagram is by putting a stack of $i - \nu(i)$ squares in vertical position i for $1 \leq i \leq g$.

Example 2.4. This example corresponds to

$$\{\nu(i)\colon i = 1,\ldots,g\} = \{1,2,\ldots,g-5,g-5,g-4,g-4,g-3,g-3\}$$

and hence $\xi = \{5,3,1\}$ (see Figure 1).

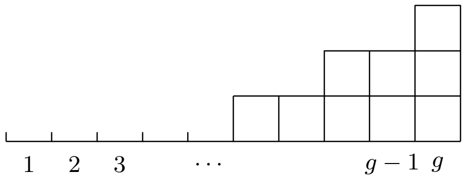


Fig. 1. Young diagram with $\xi = \{5, 3, 1\}$.

The final elements w in W_g are in one-to-one correspondence with the elements of W_g/W_H. The group W_g acts on W_g/W_H by multiplication on the left, i.e., by the permutation representation. Therefore W_g also acts on the set of final types and the set of final Young diagrams. To describe these actions we need the notion of a break point.

By a *break point* of a final type ν we mean an integer i with $1 \leq i \leq g$ such that either

1. $\nu(i-1) = \nu(i) \neq \nu(i+1)$, or
2. $\nu(i-1) \neq \nu(i) = \nu(i+1)$.

If i is not a break point of $\nu = \nu_w$ then either $\nu(i+1) = \nu(i-1)$ or $\nu(i+1) = \nu(i-1) + 2$ and therefore $\nu_{s_i w} = \nu_w$. In particular, g is always a break point. The set of break points of ν is

$$\{1 \leq i \leq g \colon \nu_{s_i w} \neq \nu_w\}.$$

Since $\nu = \nu_w$ determines a coset wW_H, we have that i is not a break point of ν if and only if $w^{-1} s_i w \in W_H$, i.e., if and only if wW_H is a fixed point of s_i acting on W_g/W_H. The action of s_i on a final type ν is as follows: if i is not a break point then ν is fixed; otherwise, replace the value of ν at i by $\nu(i) + 1$ if $\nu(i-1) = \nu(i)$ and $\nu(i) - 1$ otherwise.

If w is a final element given by the permutation $[a_1, a_2, \dots, a_g]$, then it defines a second final element, called the *complementary permutation*, defined by the permutation $[b_1, b_2, \dots, b_g]$, where $b_1 < b_2 < \cdots < b_g$ are the elements of the complement $\{1, 2, \dots, 2g\} \backslash \{a_1, \dots, a_g\}$. If ξ is the partition defining the Young diagram of w then ξ^c defines the Young diagram of the complementary permutation. The set of break points of w (that is, of the corresponding ν) and its complementary element are the same.

Lemma 2.5. *Let $w \in W_g$ be a final element with associated final type ν and complementary element v.*

(i) We have that $v = \sigma_1 w \sigma_0 = w \sigma_1 \sigma_0$, where σ_0 (respectively σ_1) is the element of S_g (respectively W_g) that maps i with $1 \leq i \leq g$ to $g + 1 - i$ (resp. to $2g + 1 - i$).

(ii) Let $i \in \{1, \dots, g\}$. If $\nu(i-1) \neq \nu(i)$ then $v^{-1}(i) = \nu(i)$ and if $\nu(i-1) = \nu(i)$ then $v^{-1}(i) = 2g + 1 - \nu(2g + 1 - i)$.

Proof. Since w maps i to a_i we have that $2g+1-a_i$ is not among the a_j and hence $b_i = 2g+1-a_{g+1-i}$ (using that both the a_i and b_i are increasing sequences). This gives $\sigma_1 w \sigma_0(i) = \sigma_1(a_{g+1-i}) = 2g+1-a_{g+1-i} = b_i$, but we note that since $w \in W_g$, it commutes with σ_1. Thus (i) holds.

If $\nu(i-1) \neq \nu(i)$ and, say, $\nu(i) = i-k$ then we have k natural numbers $1 \leq i_1 < i_2 < \cdots < i_k < i$ such that $\nu(i_j - 1) = \nu(i_j)$. By the definition of v we then have $v(i-k) = (i-k)+k$, since the k values i_j $(j = 1, \ldots, k)$ are values for v, hence not of w. The second part is checked similarly. □

Remark 2.6. The elements σ_1 and σ_0 of course have clear root-theoretic relevance: they are respectively the longest elements of W_g and S_g. Multiplication by $\sigma_1\sigma_0$ reverses the Bruhat–Chevalley order. Similarly it is clear that going from a final element to its complementary element also reverses the Bruhat–Chevalley order among the final elements and the first part of our statement says that that operation is obtained by multiplying by σ_1 and σ_0. Somewhat curiously, our use of the complementary permutation seems unrelated to these facts.

In terms of Young diagrams the description is analogous and gives us a way to write the element w_ν as a reduced product of simple reflections. To each s_i we can associate an operator on final Young diagrams. If Y is a final diagram, s_i is defined on Y by adding or deleting a box in the ith column if this gives a final diagram (only one of the two can give a final diagram) and then $s_i Y$ will be that new diagram; if neither adding nor deleting such a box gives a final Young diagram we do nothing. In terms of the description as subsets ξ, adding a box corresponds to $g+1-i \in \xi$ and $g+2-i \notin \xi$ and then $s_i\xi = (\xi \setminus \{g+1-i\}) \cup \{g+2-i\}$. It is then clear that for any final Young diagram Y there is a word $s_{i_1}s_{i_2}s_{i_3}\cdots s_{i_k}$ such that $Y = s_{i_1}s_{i_2}s_{i_3}\cdots s_{i_k}\emptyset$, where $\emptyset$ denotes the empty Young diagram. Comparison with the action of s_i on final types and the correspondence between final types and Young diagrams shows that the action of s_i on diagrams is indeed obtained from that on final types. If we now have a word $t = s_{i_1}\cdots s_{i_k}$ in the s_i we can make it act on Young diagrams by letting each individual s_i act as specified. Note that this action depends only on the image of t in W_g, but for the moment we want to consider the action by words. We define the *area* of a Young diagram Y to be the number of boxes it contains. We shall say that the word t is *building* if the area of $t\emptyset$ is equal to k, the length of the *word* (not the resulting element). This is equivalent to the action of s_{i_r} adding a box to $s_{i_{r+1}}\cdots s_{i_k}\emptyset$ for all r.

Lemma 2.7. *(i) If ν is a final type and t is a word in the s_i such that $Y_\nu^c = t\emptyset$ then $w_\nu = w$, where w is the image of t in W_g and $\ell(w_\nu) = g(g+1)/2 - \mathrm{area}(Y_\nu)$.*

(ii) t is H-reduced if and only if t is building.

Proof. To prove (i) we begin by noting that $t\emptyset$ depends only on the image of t in W_g, so that (i) is independent of the choice of t. Hence we may prove it by choosing a particular t using induction on the area of Y_ν^c. Note that $g(g+1)/2-\text{area}(Y_\nu)=\text{area}(Y_\nu^c)$, so that the last part of (i) says that $\ell(w_\nu)=\text{area}(Y_\nu^c)$. The final type ν with $\nu(i)=0$ for $i\le g$ corresponds to a final diagram Y_ν with an empty complementary diagram. We have $w_\nu=1\in W_g$, the empty product, and it has length 0. This proves the base case of the induction. Suppose we have proved the statement for diagram Y_ν with $\text{area}(Y_\nu^c)\le a$. Adding one block to Y_ν^c to obtain $Y_{\nu'}^c$ means that for some i we have $g+1-i\in\xi^c$ and $g+2-i\notin\xi^c$, where ξ^c is the subset corresponding to Y_ν^c, and the new subset is $(\xi')^c=(\xi\setminus\{g-i\})\cup\{g-i+1\}$. This means that if $i<g$ there are $b<a\le g$ such that $w_\nu(b)=i$, $w_\nu(a)=2g-i$, $w_{\nu'}(b)=i+1$, and $w_{\nu'}(a)=2g+1-i$, and the rest of the integers between 1 and g remain unchanged. (The case $i=g$ is similar and left to the reader.) This makes it clear that we have $w_{\nu'}=s_iw_\nu$, so by the induction t maps to w_ν. It remains to establish the formula for $\ell(w_\nu)$. In the definition of $\ell(w_\nu)$ only the second term contributes, since $w_\nu(i)<w_\nu(j)$ if $i<j\le g$. Now, the only difference in the collections of sums $w(i)+w(j)$ for $i\le j$ and w equal to w_ν and $w_{\nu'}$ appears for $(i,j)=(b,a)$, and we have $w_\nu(b)+w_\nu(a)=2g$ and $w_{\nu'}(b)+w_{\nu'}(a)=2g+2$, so that the length of $w_{\nu'}$ is indeed one larger than that of w_ν.

As for (ii), we have that $t\emptyset=Y_\nu^c$, where ν is the final type of w and then (ii) is equivalent to t being H-reduced if and only if $\text{area}(Y_\nu^c)$ is equal to the length of t. However, by (i) we know that $\text{area}(Y_\nu^c)$ is equal to $\ell(w_\nu)$, and t is indeed H-reduced precisely when its length is equal to $\ell(w_\nu)$. ☐

Example 2.8. Consider again the Young diagram of the previous example but now for $g=5$ (see Figure 2). We have $\xi=\{5,3,1\}$ and thus $\xi^c=\{2,4\}$, so $w_\nu=[13579]$ and w_ν can be written as $s_4s_5s_2s_3s_4s_5$ (we emphasize that permutations act from the left on diagrams).

We now characterize final types. Besides the function $\nu=\nu_w$ defined by

$$\nu(i)=i-\#\{a\le g\colon w(a)\le i\}=i-r_w(g,i)$$

and extended by $\nu(2g-i)=\nu(i)-i+g$ for $i=0,\dots,g$, we define a function $\mu=\mu_w$ on the integers $1\le i\le 2g$ by

$$\mu(i):=\big(\max\{w^{-1}(a)\colon 1\le a\le i\}-g\big)^+,$$

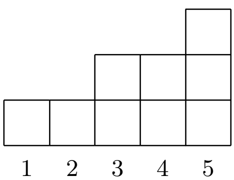


Fig. 2. Young diagram with $g=3$ and $\xi=\{5,3,1\}$.

where $(x)^+ := \max(x, 0)$. Alternatively, we have

$$\mu(i) = \min \{0 \le j \le g : r_w(g+j, i) = i\}.$$

Note that both μ and ν are nondecreasing functions taking values between 0 and g. Also for $i = 1, \ldots, 2g-1$ we have $\nu(i+1) = \nu(i)$ or $\nu(i+1) = \nu(i)+1$ and $\nu(2g) = \mu(2g) = g$. If w is a final element then ν_w is the final type associated to w. For an arbitrary w the function ν is the final type of the final element in the coset wS_g.

Lemma 2.9. *We have $\mu_w(i) \ge \nu_w(i)$ for $1 \le i \le 2g$ with equality precisely when w is a final element and then ν_w is the final type of w.*

Proof. We suppress the index w and first prove the inequality $\mu \ge \nu$. Let $1 \le i \le g$. Suppose that $\mu(i) = m$, i.e., the maximal j with $w(j)$ in $[1, i]$ is $g+m$. Then there are at most m elements from $[g+1, 2g]$ that map under w into $[1, i]$ and there are at least $i-m$ elements from $[1, g]$ with their image under w in $[1, i]$, so $i - \nu(i) \ge i - m$; in other words, $\nu(i) \le \mu(i)$. For i in the interval $[g+1, 2g]$ we consider $\nu(2g-i) = \#\{a \le g \colon w(a) > i\}$. If $\mu(2g-i) = m$ then there are at least $g-m$ elements from $[1, g]$ mapping into $[1, 2g-i]$ and thus $\nu(2g-i)$ is at most equal to m.

If w is final then w respects the order on $[1, g]$, and this implies that if $\#\{a \le g \colon w(a) \le t\} = n$ then $t-n$ elements from $[g+1, 2g]$ map under w to $[1, t]$, so the maximum element from $[g+1, 2g]$ mapping into $[1, t]$ is $g+t-n$. Hence $\mu(t) = t-n = \nu(t)$.

Conversely, if $\mu(i) = \nu(i)$ then this guarantees that $w(i) < w(j)$ for all pairs $1 \le i < j \le g$. Thus w is a final element. □

Corollary 2.10. *Let $w \in W_g$. Then w is a final element if and only if $r_w(g + \nu_w(i), i) = i$ for all $1 \le i \le g$.*

Proof. Lemma 2.9 says that if w is final then we have

$$\nu(i) = \mu(i) = \min \{0 \le j \le g : r_w(g+j, i) = i\}$$

and in particular that $\nu(i) \in \{0 \le j \le g : r_w(g+j, i) = i\}$, which gives one direction.

Conversely, if we have $r_w(g + \nu_w(i), i) = i$, then $\nu_w(i) \ge \mu_w(i)$, and then Lemma 2.9 gives that w is final. □

2.3 Canonical Types

We now deal with an iterative way of constructing the function ν starting from its values on the endpoints and applying it repeatedly.

A final type ν is given by specifying $\nu(j)$ for $j = 1, \ldots, 2g$. But it suffices to specify the values of ν for the break points of ν. Under ν an interval $[i_1, i_2]$ between two consecutive break points of ν is mapped either to an interval of length $i_2 - i_1$ or to one point. However, the image points $\nu(i_1)$ and $\nu(i_2)$ need not be break points of ν. Therefore we enlarge the set of break points to a larger set C_ν, called the *canonical domain*. We define C_ν to be the smallest subset of $\{0, 1, \ldots, 2g\}$ containing 0 and $2g$ such that if $j \in C_\nu$ then also $2g - j \in C_\nu$ and if $j \in C_\nu$ then $\nu(j) \in C_\nu$. It is obtained by starting from $R = \{0, 2g\}$ and adding the values $\nu(k)$ and $\nu(2g-k)$ for $k \in R$ and continuing till this stabilizes. The restriction of ν to C_ν is called a *canonical type*. We wish to see that the canonical domain C_ν contains the break points of ν and hence that we can retrieve ν from the canonical type of ν. To see this we need a technical lemma (its formulation is somewhat obscured by the fact that we also want to use it in another slightly different context).

Definition-Lemma 2.11. *We shall say that a subset $S \subseteq \{0, 1, \ldots, 2g\}$ is* stable *if it has the property that it contains* 0 *and is stable under $i \mapsto i^\perp := 2g - i$. For a stable subset S a map $f\colon S \to S \cap \{0, 1, \ldots, g\}$ is* adapted *to S if $f(0) = 0$ and $f(2g) = g$, if it is* contracting*, i.e., it is increasing and we have $f(j) - f(i) \leq j - i$ for $i < j$ and if it fulfills the following* complementarity condition*: For any two consecutive $i, j \in S$ (i.e., $i < j$ and there are no $k \in S$ with $i < k < j$) we have $f(j) - f(i) = j - i \Rightarrow f(j^\perp) = f(i^\perp)$.*

(i) If S is stable and f is a nonsurjective function adapted to S then there is a proper subset $T \subset S$ such that $f_{|T}$ is adapted to T.

(ii) If S is stable and f is a surjective function adapted to S then for any two consecutive $i, j \in S$ we have either $f(i) = f(j)$ or $f(j) - f(i) = j - i$.

(iii) We say that (S, f) is minimally stable *if S is stable and f is adapted to S and furthermore there is no proper stable subset $T \subset S$ for which $f_{|T}$ is adapted to it, then the function $\nu\colon \{1, 2, \ldots, 2g\} \to \{1, 2, \ldots, g\}$ obtained from f by extending it linearly between any two consecutive $i, j \in S$ is a final type, $S = C_\nu$, and ν is the unique final extension of f. Conversely, if f is the canonical type of a final type ν, then (C_ν, f) is minimally stable and in particular ν is the linear extension of its canonical type.*

Proof. For (i) consider $T = f(S) \cup (f(S))^\perp$. It is clearly stable under f and $\perp$ and contains 0. If f is not surjective, then T is a proper subset of S.

Assume now that we are in the situation of (ii). We show that if $i < j \in S$ are consecutive then either $f(j) - f(i) = j - i$ or $f(i) = f(j)$ by descending induction on $j - i$.

By induction we are going to construct a sequence $i_k < j_k \in S$, $k = 1, 2, \ldots$, of consecutive elements such that either $(i_{k-1}, j_{k-1}) = (j_k^\perp, i_k^\perp)$ or $(f(i_k), f(j_k)) = (i_{k-1}, j_{k-1})$ but not both $(i_{k-1}, j_{k-1}) = (j_k^\perp, i_k^\perp)$ and $(i_{k-2}, j_{k-2}) = (j_{k-1}^\perp, i_{k-1}^\perp)$ and in any case $j_k - i_k = j - i$. We start by putting $i_1 := i, j_1 := j$. Assume now that $i_k < j_k$ have been constructed. If we do not have $i_k, j_k \leq g$, then since $g \in S$, we must have $j_k^\perp, i_k^\perp \leq g$ and then we put $(i_{k+1}, j_{k+1}) = (j_k^\perp, i_k^\perp)$. If we do have $i_k, j_k \leq g$ then by the surjectivity of f there are $i_{k+1}, j_{k+1} \in S$ such that $f(i_{k+1}) = i_k$ and $f(j_{k+1}) = j_k$. Since f is increasing, $i_{k+1} < j_{k+1}$, and by choosing i_{k+1} to be maximal and j_{k+1} to be minimal we may assume that they are neighbors. We must have that $j_{k+1} - i_{k+1} = j_1 - i_1$. Indeed, we have $f(j_{k+1}) - f(i_{k+1}) \le j_k - i_k$, since f is contracting. If we have strict inequality we have $j - i = j_k - i_k < j_{k+1} - i_{k+1}$, and hence by the induction assumption we have either that $j_k - i_k = f(j_{k+1}) - f(i_{k+1}) = j_{k+1} - i_{k+1}$, which is a contradiction, or that $j_k = f(j_{k+1}) = f(i_{k+1}) = i_k$, which is also a contradiction. Hence we have $j_{k+1} - i_{k+1} = j_k - i_k = j - i$ and we have verified the required properties of (i_{k+1}, j_{k+1}).

There must now exist $1 \le k < \ell$ such that $(i_k, j_k) = (i_\ell, j_\ell)$, and we pick k minimal for this property. If $k = 1$ we have either $j - i = j_{\ell-1} - i_{\ell-1} = f(j_\ell) - f(i_\ell) = f(j) - f(i)$ or $j - i = j_{\ell-2} - i_{\ell-2} = f(j_{\ell-1}) - f(i_{\ell-1}) = f(i^\perp) - f(j^\perp)$, which implies that $f(i) = f(j)$ by assumptions on f. We may hence assume that $k > 1$. We cannot have both $(i_{k-1}, j_{k-1}) = (j_k^\perp, i_k^\perp)$ and $(i_{\ell-1}, j_{\ell-1}) = (j_\ell^\perp, i_\ell^\perp)$, since that would contradict the minimality of k. If $(i_{k-1}, j_{k-1}) = (j_k^\perp, i_k^\perp)$ and $(i_{\ell-1}, j_{\ell-1}) = (f(i_\ell), f(j_\ell))$ then we get $j_{k-1} - i_{k-1} = j - i = j_{\ell-1} - i_{\ell-1} = f(i_{k-1}^\perp) - f(j_{k-1}^\perp)$, which implies $f(j_{k-1}) = f(i_{k-1})$, which is either what we want in case $k = 2$ or a contradiction. Similarly, the case $(i_{\ell-1}, j_{\ell-1}) = (j_\ell^\perp, i_\ell^\perp)$ and $(i_{k-1}, j_{k-1}) = (f(i_k), f(j_k))$ leads to a contradiction, as does the case $(i_{\ell-1}, j_{\ell-1}) = (f(i_\ell), f(j_\ell))$ and $(i_{k-1}, j_{k-1}) = (f(i_k), f(j_k))$.

Finally, to prove the first part of (iii) we note that by (ii), for $i < j \in S$ consecutive we have either $f(i) = f(j)$ or $f(j) - f(i) = j - i$. This means that the linear extension ν of f has the property that for $1 \le i \le 2g$ we have either $\nu(i) = \nu(i-1)$ or $\nu(i) = \nu(i-1) + 1$, and if $\nu(i) = \nu(i-1) + 1$ we get by the conditions on f that $\nu(2g - i + 1) = \nu(2g - i)$. If for some i we have $\nu(i) = \nu(i-1)$ and $\nu(2g-i+1) = \nu(2g-i)$, we get that $g = f(2g) = \nu(2g) < g$, which is impossible by assumption, and hence ν is indeed a final type. It is clear that S fulfills the defining property of C_ν, so that $S = C_\nu$. The conditions on a final element imply that $\nu(j) - \nu(i) \le j - i$ for $j < i$ and thus that f has a unique final extension.

Conversely, if ν is a final type then C_ν clearly fulfills the required conditions and we have just noted that $\nu(j) - \nu(i) \le j - i$ for $j < i$. The complementarity condition for $f|C_\nu$ follows from the equivalence $\nu(i) = \nu(i-1) \iff \nu(2g - i + 1) = \nu(2g - i) + 1$. □

We now give an interpretation of the canonical domain in terms of the Weyl group. Let $v \in W_g$ be a final element. A *canonical fragment* of v is an interval $]i, j] := [i+1, \dots, j] \subseteq \{1, 2, \dots, 2g\}$ that is maximal with respect to the requirement that for all $k \ge 1$ the set $v^k(]i, j])$ be an interval.

Proposition 2.12. *Let $v \in W_g$ be a final element, w its complementary element, and ν the final type of w.*

(i) The set $\{1, 2, \dots, 2g\}$ is the disjoint union of the canonical fragments of v. Moreover, the canonical fragments of v are permuted by v.

(ii) If $]i, j]$ is a canonical fragment of v, and if $\nu(j) \ne \nu(j-1)$, then ν maps $]i, j]$ bijectively to $]\nu(i), \nu(j)]$.

(iii) If $]i,j]$ *is a canonical fragment of* v*, then so is* $]\bar{\jmath},\bar{\imath}]$*.*
(iv) The upper endpoints of the canonical fragments of v *together with* 0 *form exactly the canonical domain for* w*.*

Proof. If two canonical fragments I and J meet, their union K will be an interval, and since $v^k(K) = v^k(I) \cup v^k(J)$, we see that $v^k(K)$ will be an interval for all k. By the maximality we get that $I = J$. On the other hand, $]i-1,i]$ fulfills the stability condition, so that i lies in a fragment. Hence $\{1,2,\dots,2g\}$ is the disjoint union of fragments.

Now let R be the set of upper endpoints of fragments together with 0. Since $\{1,2,\dots,2g\}$ is the disjoint union of the fragments of v, it follows that if $]i,j]$ is a fragment, then i is also the upper endpoint of a fragment. Thus it follows from (iii) that R is stable under $i \mapsto \bar{\imath}$. Let now i be an upper endpoint of a fragment. We want to show that $\nu(i) \in R$, and we may certainly assume that $\nu(i) \neq 0$, and we may also, by way of contradiction, assume that i is a minimal upper endpoint for which $\nu(i)$ is not an upper endpoint. If $\nu(i) \neq \nu(i-1)$, then $v^{-1}(i) = \nu(i)$ and hence $\nu(i)$ is an upper endpoint of a fragment. Hence we may pick $j < i$ such that $\nu(i) = \nu(i-1) = \cdots = \nu(j) \neq \nu(j-1)$. Then j cannot belong to the same fragment as i, and thus there must be an upper endpoint $j \leq k < i$. Then $\nu(k) = \nu(i)$ and by minimality of i we see that $\nu(k)$ is an upper endpoint, which is a contradiction.

We therefore have shown that R contains 0 and is stable under $i \mapsto \bar{\imath}$ and ν. Hence it contains the canonical domain. Let now $j \in C_\nu \setminus \{0\}$ and let i be the largest $j \in C_\nu$ such that $i < j$. We now want to show by induction on k that $v^{-k}(I)$, $I :=]i,j]$, remains an interval for all k and that also $v^{-k}(j)$ is one of its endpoints. Now, it follows from Lemma 2.5 that $C_\nu \setminus \{0\}$ is stable under v and hence $v^{-k}(j)$ will be the only element of C_ν in $v^{-k}(I)$. Under the induction assumption, $v^{-k}(I)$ is an interval with $v^{-k}(j)$ as one of its endpoints, and hence ν is constant on $v^{-k}(I)$ by Lemma 2.11. By Lemma 2.5, an interval v^{-1} maps $v^{-k}(I)$ to the interval with $v^{-k-1}(j)$ as one of its endpoints. This means that I is contained in a fragment and $R \cap I = \{j\}$. This means that there are no elements of R between consecutive elements of C_ν and hence $R \subseteq C_\nu$. □

Corollary 2.13. *By associating to a final type* ν *its canonical type, its Young diagram, and the element* w_ν *we obtain a bijection between the following sets of cardinality* 2^g*: the set of final types, the set of canonical types, the set of final Young diagrams, and the set of final elements of* W_g*.*

2.4 Admissible Elements

The longest final element of W_g is the element

$$w_\emptyset := s_g s_{g-1} s_g s_{g-2} s_{g-1} s_g \dots s_g s_1 s_2 s_3 \dots s_g,$$

which as a permutation equals $[g+1, g+2, \dots, 2g]$. Elements of W_g that satisfy $w \leq w_\emptyset$ are called *admissible*. We now characterize these.

Lemma 2.14. *(i) An element $w \in W_g$ fulfills $w \leq w_\emptyset$ if and only if we have $w(i) \leq g + i$ for all $1 \leq i \leq g$.*

(ii) The condition that $w \leq w_\emptyset$ is equivalent to $r_w(i, g+i) = i$ for all $1 \leq i \leq g$.

Proof. The first part follows immediately from the description of the Bruhat–Chevalley order (Lemma 2.1) and the presentation of $w_\emptyset$.

For the second part one easily shows that $w(i) \leq w_\emptyset(i) = g + i$ for all $1 \leq i \leq g$ is equivalent to $r_w(i, g+i) = i$ for all $1 \leq i \leq g$, which gives the first equivalence. □

Remark 2.15. The number of elements $w \in W_g$ with $w \leq w_\emptyset$ and of given length has recently been determined by J. Sjöstrand [Sj07]. This implies in particular that the number of elements $w \in W_g$ with $w \leq w_\emptyset$ equals

$$\left(x\frac{d}{dx}\right)^g \left(\frac{1}{1-x}\right)_{\big|x=1/2},$$

a fact that we originally guessed from a computation for small g and a search in [S] leading to the sequence A000629.

We give an illustration of the various notions for the case $g = 2$.

Example 2.16. $g = 2$. The Weyl group W_2 consists of eight elements. We list (see Figure 3) the element, a reduced expression as a word (i.e., a decomposition $w = s_{i_1} \cdots s_{i_k}$ with $k = \ell(w)$), its length, the functions ν and μ, and for final elements we also give the partition defining the Young diagram.

The orbits of the complementary element will play an important role in our discussion of the canonical flag. Here we introduce some definitions pertaining to them.

w	s	ℓ	ν	μ	Y
[4321]	$s_1s_2s_1s_2$	4	$\{1,2\}$	$\{2,2\}$	
[4231]	$s_1s_2s_1$	3	$\{1,1\}$	$\{2,2\}$	
[3412]	$s_2s_1s_2$	3	$\{1,2\}$	$\{1,2\}$	$\emptyset$
[2413]	s_1s_2	2	$\{1,1\}$	$\{1,1\}$	$\{1\}$
[3142]	s_2s_1	2	$\{0,1\}$	$\{0,2\}$	
[2143]	s_1	1	$\{0,0\}$	$\{0,0\}$	
[1324]	s_2	1	$\{0,1\}$	$\{0,1\}$	$\{2\}$
[1234]	1	0	$\{0,0\}$	$\{0,0\}$	$\{1,2\}$

Fig. 3. The $g = 2$ case.

Definition 2.17. *Let $w \in W_g$ be a final element and let v be its complementary element. Assume that S is an orbit of the action of v on its fragments. Since v commutes with $i \mapsto \bar{\imath}$, we have either that S is invariant under $i \mapsto \bar{\imath}$, in which case we say that it is an* odd orbit*, or that $\overline{S}$ is another orbit, in which case we say $\{S, \overline{S}\}$ is an* even orbit pair*.*

2.5 Shuffles

Recall that a (p,q)-*shuffle* is a permutation σ of $\{1,2,\ldots,p+q\}$ for which we have $\sigma(i) < \sigma(j)$ whenever $i < j \leq p$ or $p < i < j$. It is clear that for each subset I of $\{1,2,\ldots,g\}$ there is a unique $(|I|, g-|I|)$-shuffle σ^I such that $I = \{\sigma^I(1), \sigma^I(2), \ldots, \sigma^I(|I|)\}$, and we will call it the *shuffle associated to I*. We will use the same notation for the corresponding element in W_g (i.e., fulfilling $\sigma^I(2g+1-i) = 2g+1-\sigma^I(i)$ for $1 \leq i \leq g$). By doing the shuffling from above instead of from below we get another shuffle σ_I given by $\sigma_I(i) = g+1-\sigma^I(g+1-i)$. We will use the same notation for its extension to W_g. Note that σ_I will shuffle the elements $\{g+1, g+2, \ldots, 2g\}$ in the same way that σ^I shuffles $\{1,2,\ldots,g\}$, i.e., $\sigma^I(g+i) = g+\sigma_I(i)$, which is the relation with σ^I that motivates the definition. Note that if $I = \{i_1 < \cdots < i_r\}$ and if we assume that $i_r > r$ (if this does not hold then σ^I and σ_I are the identity elements) and we let k be the smallest index such that $i_k > k$, then $\sigma^I = s_{i_k-1}\sigma^{I'}$ and $\sigma_I = s_{g+1-(i_k-1)}\sigma_{I'}$, where $I' = \{i_1, \ldots, i_k-1, \ldots, i_r\}$. We call the element $s_{i-1}ws_{g+1-(i-1)}$ for $w \in W_g$ the *ith elementary shuffle* of w, and say that I' is the *elementary reduction* of I whose *reduction index* is i_k.

We define the *height* of a shuffle associated to a subset $\{i_1, i_2, \ldots, i_k\} \subseteq \{1,2,\ldots,g\}$ to be $\sum_s (i_s - s)$. Using $w' = s_i w s_{g+1-t} \iff s_i w' s_{g+1-t} = w$, we see that starting with an element w obtained by applying a shuffle to a final element we arrive at a final element after $\mathrm{ht}(w)$ elementary shuffles.

Definition 2.18. *Let Y be a final Young diagram of degree g. The* shuffles *of Y are the elements of W_g of the form $\sigma^I w_Y \sigma_I^{-1}$ for $I \subseteq \{1,2,\ldots,g\}$.*

If $w \leq w_\emptyset$ we say that i with $1 \leq i \leq g$ is a *semi-simple index* for w if $w(i) = g+i$ (note that since $w \leq w_\emptyset$, we always have $w(i) \leq g+i$). The set of semi-simple indices will be called the *semi-simple index set* and its cardinality the *semi-simple rank*. We say that w is *semi-simply final* if the semi-simple index set has the form $[g-f+1, g]$ (where then f is the semi-simple rank). This is equivalent to w having the form $[\ldots, 2g-f+1, 2g-f+2, \ldots, 2g]$. If $w = w_Y$, Y a final Young diagram, then w is semi-simply final and the semi-simple rank is equal to g minus the length of the largest row of Y (defined to be zero if Y is empty).

Proposition 2.19. *Let $w \leq w_\emptyset$ be a semi-simply final element of semi-simple rank f and let $I \subseteq \{1,2,\ldots,2g\}$ be a subset with $\#I = f$. Put $\tilde{I} := \{g+1-i : i \in I\}$. Then $w' := \sigma^I w \sigma_I^{-1}$ is an element with $w' \leq w_\emptyset$ of semi-simple rank f and semi-simple index set $\tilde{I}$. Conversely, all $w' \leq w_\emptyset$ whose semi-simple index set is equal to I are of this form.*

Proof. Put $j := \sigma_I^{-1}(i)$. Note that $j > g-f \iff i \in \tilde{I}$. If $j > g-f$ we have $w(j) = g+j$ and hence $\sigma^I w_Y \sigma_I^{-1}(i) = \sigma^I(g+j) = g+\sigma_I(j) = g+i$. If on the other hand, $j \le g-f$, then if $w_Y(j) \le g$ holds there is nothing to prove, and if it does not hold we may write $w_Y(j) = g+k$, and since the semi-simple rank of Y is f, we have $k < j$. Then $\sigma^I w_Y \sigma_I^{-1}(i) = \sigma^I(g+j) = g+\sigma_I(k)$ and since $k < j \le g-f$, we have $\sigma_I(k) < \sigma_I(j) = i$, which gives $\sigma^I w_Y \sigma_I^{-1}(i) < g+i$.

The converse is easy and left to the reader. □

Finally, we define the *a-number* of an element $w \in W_g$ by the rule

$$a(w) := r_w(g,g).$$

If w is final with associated Young diagram Y then its a-number, also denoted by a_Y, is the largest integer a with $0 \le a \le g$ such that Y contains the diagram that corresponds to the set $\xi = \{a, a-1, a-2, \dots, 1\}$.

3 The Flag Space

3.1 The Flag space of the Hodge bundle

In this section we introduce the flag space of a principally polarized abelian scheme over a base scheme of characteristic p. We use the Frobenius morphism to produce from a chosen flag on the de Rham cohomology a second flag, whose position with respect to the first flag will be the object of study.

We let S be a scheme (or Deligne–Mumford stack) in characteristic p and let $\mathcal{X} \to S$ be an abelian scheme over S with principal polarization (everything would go through using a polarization of degree prime to p but we shall stick to the principally polarized case). We consider the de Rham cohomology sheaf $\mathcal{H}^1_{dR}(\mathcal{X}/S)$. It is defined as the hyper-direct image $\mathcal{R}^1\pi_*(O_{\mathcal{X}} \to \Omega^1_{\mathcal{X}/S})$ and is a locally free sheaf of rank $2g$ on S. The polarization (locally in the étale topology given by a relatively ample line bundle on $\mathcal{X}/S$) provides us with a symmetric homomorphism $\rho\colon \mathcal{X} \to \hat{\mathcal{X}}$, and the Poincaré bundle defines a perfect pairing between $\mathcal{H}^1_{dR}(\mathcal{X}/S)$ and $\mathcal{H}^1_{dR}(\hat{\mathcal{X}}/S)$, and thus $\mathcal{H}_{dR}(\mathcal{X}/S)$ comes equipped with a nondegenerate alternating form (cf. [Oo95])

$$\langle , \rangle\colon \mathcal{H}^1_{dR}(\mathcal{X}/S) \times \mathcal{H}^1_{dR}(\mathcal{X}/S) \to O_S.$$

Moreover, we have an exact sequence of locally free sheaves on S:

$$0 \to \pi_*(\Omega^1_{\mathcal{X}/S}) \to \mathcal{H}^1_{dR}(\mathcal{X}/S) \to R^1\pi_* O_{\mathcal{X}} \to 0.$$

We shall write $\mathbb{H}$ for the sheaf $\mathcal{H}^1_{dR}(\mathcal{X}/S)$ and $\mathbb{E}$ for the *Hodge bundle* $\pi_*(\Omega^1_{\mathcal{X}/S})$. We thus have an exact sequence

$$0 \to \mathbb{E} \to \mathbb{H} \to \mathbb{E}^\vee \to 0$$

of locally free sheaves on S. The relative Frobenius $F\colon \mathcal{X} \to \mathcal{X}^{(p)}$ and the Verschiebung $V\colon \mathcal{X}^{(p)} \to \mathcal{X}$ satisfy $F \cdot V = p \cdot \mathrm{id}_{\mathcal{X}^{(p)}}$ and $V \cdot F = p \cdot \mathrm{id}_{\mathcal{X}}$ and they induce maps, also denoted by F, respectively V, in cohomology:

$$F\colon \mathbb{H}^{(p)} \to \mathbb{H} \qquad \text{and} \qquad V\colon \mathbb{H} \to \mathbb{H}^{(p)}.$$

Of course, we have $FV = 0$ and $VF = 0$ and F and V are adjoints (with respect to the alternating form). This implies that $\mathrm{Im}(F) = \ker(V)$ and $\mathrm{Im}(V) = \ker(F)$ are maximally isotropic subbundles of $\mathbb{H}$ and $\mathbb{H}^{(p)}$ respectively. Moreover, since $dF = 0$ on $\mathrm{Lie}(\mathcal{X})$, it follows that $F = 0$ on $\mathbb{E}^{(p)}$ and thus $\mathrm{Im}(V) = \ker(F) = \mathbb{E}^{(p)}$. Verschiebung thus provides us with a bundle map (again denoted by V) $V\colon \mathbb{H} \to \mathbb{E}^{(p)}$.

Consider the space $\mathcal{F} = \mathrm{Flag}(\mathbb{H})$ of symplectic flags on the bundle $\mathbb{H}$ consisting of flags of subbundles $\{\mathbb{E}_i\}_{i=1}^{2g}$ satisfying $\mathrm{rk}(\mathbb{E}_i) = i$, $\mathbb{E}_{g+i} = \mathbb{E}_{g-i}^{\perp}$, and $\mathbb{E}_g = \mathbb{E}$. This space is a scheme over S and it is fibered by the spaces $\mathcal{F}^{(i)}$ of partial flags

$$\mathbb{E}_i \subsetneq \mathbb{E}_{i+1} \subsetneq \cdots \subsetneq \mathbb{E}_g.$$

So $\mathcal{F} = \mathcal{F}^{(1)} = \mathrm{Flag}(\mathbb{H})$ and $\mathcal{F}^{(g)} = S$ and there are natural maps

$$\pi_{i,i+1}\colon \mathcal{F}^{(i)} \to \mathcal{F}^{(i+1)},$$

the fibers of which are Grassmann varieties of dimension i. So the relative dimension of $\mathcal{F}$ is $g(g-1)/2$. The space $\mathcal{F}^{(i)}$ is equipped with a universal partial flag. On $\mathcal{F}$ the Chern classes of the bundle $\mathbb{E}$ decompose into their roots:

$$\lambda_i = \sigma_i(\ell_1, \ldots, \ell_g) \qquad \text{with} \qquad \ell_i = c_1(\mathbb{E}_i/\mathbb{E}_{i-1}),$$

where σ_i is the ith elementary symmetric function.

On $\mathcal{F}^{(i)}$ we have the Chern classes $\ell_{i+1}, \ldots, \ell_g$ and

$$\lambda_j(i) := c_j(\mathbb{E}_i), \qquad j = 0, 1, \ldots, i.$$

Its Chow ring is generated over that of $\mathcal{A}_g$ by the monomials $\ell_1^{m_1} \cdots \ell_g^{m_g}$ with $0 \leq m_j \leq j-1$. For later use we record the following Gysin formula.

Formula 3.1. *We have $(\pi_{i,i+1})_* \ell_{i+1}^k = s_{k-i}(i+1)$, where $s_j(i+1)$ denotes the jth Segre class of $\mathbb{E}_{i+1}$ (jth complete symmetric function in the Chern roots $\ell_1, \ldots, \ell_{i+1}$).*

Given an arbitrary flag of subbundles

$$0 = \mathbb{E}_0 \subsetneq \mathbb{E}_1 \subsetneq \cdots \subsetneq \mathbb{E}_g = \mathbb{E}$$

with $\mathrm{rank}(\mathbb{E}_i) = i$ we can extend this uniquely to a symplectic filtration on $\mathbb{H}$ by putting

$$\mathbb{E}_{g+i} = (\mathbb{E}_{g-i})^{\perp}.$$

By base change we can transport this filtration to $\mathbb{H}^{(p)}$.

We introduce a second filtration by starting with the isotropic subbundle

$$\mathbb{D}_g := \ker(V) = V^{-1}(0) \subset \mathbb{H}$$

and continuing with

$$\mathbb{D}_{g+i} = V^{-1}(\mathbb{E}_i^{(p)}).$$

We extend it to a symplectic filtration by setting $\mathbb{D}_{g-i} = (\mathbb{D}_{g+i})^{\perp}$. We thus have two filtrations $\mathbb{E}_\bullet$ and $\mathbb{D}_\bullet$ on the pullback of $\mathbb{H}$ to $\mathcal{F}$.

We shall use the following notation:

$$\mathcal{L}_i = \mathbb{E}_i/\mathbb{E}_{i-1} \quad \text{and} \quad \mathcal{M}_i = \mathbb{D}_i/\mathbb{D}_{i-1} \qquad \text{for } 1 \leq i \leq 2g.$$

For ease of reference we formulate a lemma that follows immediately from definitions.

Lemma 3.2. *We have $\mathcal{M}_{g+i} \cong \mathcal{L}_i^p$, $\mathcal{L}_{2g+1-i} \cong \mathcal{L}_i^\vee$, and $\mathcal{M}_{2g+1-i} \cong \mathcal{M}_i^\vee$.*

More generally, for a family $X \to S$ of principally polarized abelian varieties we shall say that a *Hodge flag* for the family is a complete symplectic flag $\{\mathbb{E}_i\}$ of $\mathbb{H}$ for which $\mathbb{E}_g$ is equal to the Hodge bundle. By construction this is the same thing as a section of $\mathcal{F}_g \to S$. We shall also call the associated flag $\{\mathbb{D}_i\}$ the *conjugate flag* of the Hodge flag.

3.2 The canonical flag of an abelian variety

In this section we shall confirm that the *canonical filtration* of $X[p]$, (kernel of multiplication by p) by subgroup schemes of a principally polarized abelian variety X as defined by Ekedahl and Oort [Oo01] has an analogue for de Rham cohomology. Just as in [Oo01] we do this in a family $\mathcal{X} \to S$. It is the coarsest flag that is isotropic (i.e., if $\mathbb{D}$ is a member of the flag then so is $\mathbb{D}^{\perp}$) and stable under F (i.e., if $\mathbb{D}$ is a member of the flag then so is $F(\mathbb{D}^{(p)})$). The existence of such a minimal flag is proven by adding elements $\mathcal{F}^{\perp}$ and $F(\mathbb{D}^{(p)})$ for $\mathbb{D}$ already in the flag in a controlled fashion. We start by adding 0 to the flag. We then insist on three rules:

1. If we added $\mathbb{D} \subseteq \mathbb{D}_g$, then we immediately add $\mathbb{D}^{\perp}$ (unless it is already in the flag constructed so far).
2. If we added $\mathbb{D}_g \subseteq \mathbb{D}$, then we immediately add $F(\mathbb{D}^{(p)})$ (unless it is already in the flag constructed so far).
3. If neither rule (1) nor rule (2) applies, then we add $F(\mathbb{D}^{(p)})$ for the largest element $\mathbb{D}$ of the flag for which $F(\mathbb{D}^{(p)})$ is not already in the flag.

We should not, however, do this construction on S; we want to ensure that we get a filtration by vector bundles: at each stage when we want to add the image $F(\mathbb{D}^{(p)})$, we have maps $F\colon \mathbb{D}^{(p)} \to \mathbb{H}$ of vector bundles, and we then have a unique minimal decomposition of the base as a disjoint union of locally closed subschemes such that on each subscheme this map has constant rank; these are subschemes because they are given by degeneracy conditions. At the same time as we add $F(\mathbb{D}^{(p)})$ to the flag we replace the base by this disjoint union. Over this disjoint union, $F(\mathbb{D}^{(p)})$ then becomes a subbundle of $\mathbb{H}$ and whether it is equal to one of the previously defined subbundles is a locally constant condition. A simple induction then shows that we get a flag, i.e., for any two elements constructed, one is included in the other, on a disjoint union of subschemes of S. Since each element added is either the image under F of an element previously constructed or the orthogonal of such an element, it is clear that this flag is the coarsening of any isotropic flag stable under F, and it is equally clear that the decomposition of S is the coarsest possible decomposition. We shall call the (partial) flag obtained in this way the *canonical flag* of $\mathcal{X}/S$ and the decomposition of S the *canonical decomposition* of the base.

To each stratum S' of the canonical decomposition of S we associate a canonical type as follows: let $T \subseteq \{1, 2, \ldots, 2g\}$ be the set of ranks of the elements of the canonical flag and let $f\colon T \to T \cap \{1, \ldots, g\}$ be the function that to t associates $\mathrm{rk}(F(\mathbb{D}^{(p)}))$, where $\mathbb{D}$ is the element of the canonical flag of rank t. We now claim that T and f fulfill the conditions of Lemma 2.11. Clearly T contains 0, and by construction it is invariant under $i \mapsto 2g - i$. Again by construction f is increasing and has $f(0) = 0$ and $f(2g) = g$. Furthermore, if $i, j \in T$ with $i < j$ then F induces a surjective map $(\mathbb{D}/\mathbb{D}')^{(p)} \to F(\mathbb{D})/F(\mathbb{D}')$, where $\mathbb{D}$ respectively $\mathbb{D}'$ are the elements of the canonical flag for which the rank is j respectively i and hence $f(j) - f(i) = \mathrm{rk}(F(\mathbb{D})/F(\mathbb{D}')) \leq \mathrm{rk}(\mathbb{D}/\mathbb{D}') = j - i$. Finally, assume that $f(j) - f(i) = j - i$ and let $\mathbb{D}$ and $\mathbb{D}'$ be as before. Putting $\mathbb{D}_1 := F(\mathbb{D}^{(p)})$ and $\mathbb{D}'_1 := F(\mathbb{D}'^{(p)})$ these are also elements of the canonical filtration, and by assumption F induces an isomorphism $F\colon (\mathbb{D}/\mathbb{D}')^{(p)} \to \mathbb{D}_1/\mathbb{D}'_1$. The fact that it is injective means that $\mathbb{D} \cap \ker F = \mathbb{D}' \cap \ker F$, which by taking annihilators and using that $\ker F$ is its own annihilator, gives $\mathbb{D}^\perp + \ker F = \mathbb{D}'^\perp + \ker F$. In turn, this implies $F(\mathbb{D}^\perp) = (\mathbb{D}'^\perp)$ and hence that $f(2g-i) = f(2g-j)$. Now, if f is not surjective then by Lemma 2.11 there is a proper subset of T fulfilling the conditions of Lemma 2.11. This is not possible since T by construction is a minimal subset with these conditions. Hence T and f fulfill the conditions of Lemma 2.11, and hence by them we get that (f, T) is a canonical type. Let ν be its associated final type. If $0 = \mathbb{D}_0 \subset \mathbb{D}_1 \subset \cdots \subset \mathbb{D}_{2g} = \mathbb{H}$ is the canonical flag with $\mathrm{rk}\,\mathbb{D}_i = i$, we have also proved that F induces an isomorphism $(\mathbb{D}_j/\mathbb{D}_i)^{(p)} \to \mathbb{D}_{\nu(j)}/\mathbb{D}_{\nu(i)}$, which can be rephrased as an isomorphism

$$F\colon \mathbb{D}^{(p)}_{v(I)} \xrightarrow{\sim} \mathbb{D}_I,$$

where we have used the notation $\mathbb{D}_J := \mathbb{D}_i/\mathbb{D}_j$ for an interval $J =]j, i]$ and $v \in W_g$ is the complementary element of (the final element of) ν. We shall say that ν (or more properly f) is the *canonical type* of the principally polarized abelian variety $\mathcal{X}_{S'} \to S'$. (We could consider the canonical type as a locally constant function on the canonical decomposition to the set of canonical (final) types.)

Remark 3.3. Note that the canonical flag is a flag containing $\mathbb{D}_g$ and is not defined in terms of $\mathbb{E}_g$. This will later mean that the canonical flag will be a coarsening of a conjugate flag that is derived from a Hodge flag. On the one hand, this is to be expected. Since the canonical flag is just that, it will be constructed in a canonical fashion from the family of principally polarized abelian varieties. Hence it is to be expected, and it is clearly true, that the canonical flag is horizontal with respect to the Gauss–Manin connection. On the other hand, we do not want to consider just conjugate flags (or make constructions starting only with conjugate flags). The reason is essentially the same; since $\mathbb{D}_g$ (or more generally the elements of the canonical flag) is horizontal, it will not reflect first-order deformations, whereas $\mathbb{E}_g$ isn't and does. This will turn out to be of crucial importance to us and is the reason why the Hodge flags will be the primary objects, while the conjugate flags are secondary. On the other hand, when working pointwise, over an algebraically closed field, say, we may recover the Hodge flag from the conjugate flag and then it is usually more convenient to work with the conjugate flag.

Example 3.4. Let X be an abelian variety with p-rank f and $a(X) = 1$ (equivalently, on $\mathbb{E}_g$ the operator V has rank $g-1$ and semi-simple rank $g-f$). Then the canonical type is given by the numbers $\{\mathrm{rk}(\mathbb{D}_i)\}$, i.e.,

$$\{0, f, f+1, \ldots, 2g-f-1, 2g-f, 2g\},$$

and ν is given by $\nu(f) = f$, $\nu(f+1) = f$, $\nu(f+2) = f+1$, $\ldots$, $\nu(g) = g-1, \ldots,$ $\nu(2g-f-1) = g-1$, $\nu(2g-f) = g$, and $\nu(2g) = g$. The corresponding element $w \in W_g$ is $[f+1, g+1, \ldots, 2g-f-1, 2g-f+1, \ldots, 2g]$.

4 Strata on the Flag Space

4.1 The Stratification

The respective positions of two symplectic flags are encoded by a combinatorial datum, an element of a Weyl group. We shall now define strata on the flag space $\mathcal{F}$ over the base S of a principally polarized abelian scheme $X \to S$ that mark the respective positions of the two filtrations $\mathbb{E}_\bullet$ and $\mathbb{D}_\bullet$ that we have on the de Rham bundle over $\mathcal{F}$.

Intuitively, the stratum $\overline{\mathcal{U}}_w$ is defined as the locus of points x such that at x we have

$$\dim(\mathbb{E}_i \cap \mathbb{D}_j) \geq r_w(i,j) = \#\{a \leq i \colon w(a) \leq j\} \quad \text{for all} \quad 1 \leq i, j \leq 2g.$$

A more precise definition would be as degeneracy loci for some appropriate bundle maps. While this definition would work fine in our situation, where we are dealing with flag spaces for the symplectic group, it would not quite work when the symplectic group is replaced by the orthogonal group on an even-dimensional space (cf. [FP98]). With a view toward future extensions of the ideas of this paper to other situations we therefore adopt the definition that would work in general. Hence assume that we have a semi-simple group G, a Borel group B of it, a G/B-bundle $T \to Y$ (with G as structure group) over some scheme Y, and two sections $s, t\colon Y \to T$ of it. Then for any element w of the Weyl group of G we define a (locally) closed subscheme $\mathcal{U}_w$ respectively $\overline{\mathcal{U}}_w$ of Y in the following way. We choose locally (possibly in the étale topology) a trivialization of T for which t is a constant section. Then s corresponds to a map $Y \to G/B$ and we let $\mathcal{U}_w$ (respectively $\overline{\mathcal{U}}_w$) be the inverse image of the B-orbit BwB (respectively of its closure in G/B). Another trivialization will differ by a map $Y \to B$, and since BwB and its closure are B-invariant, these definitions are independent of the chosen trivializations and hence give global subschemes on Y. If s and t have the property that $Y = \mathcal{U}_w$, then we shall say that s and t are in *relative position* w and if $Y = \overline{\mathcal{U}}_w$, we shall say that s and t are in *relative position* $\leq w$.

Remark 4.1. The notation is somewhat misleading, since it suggests that $\overline{\mathcal{U}}_w$ is the closure of $\mathcal{U}_w$, which may not be the case in general. In the situation that we shall meet it will, however, be the case (cf. Corollary 8.4).

The situation to which we will apply this construction is that in which the base scheme is the space $\mathcal{F}$ of symplectic flags $\mathbb{E}_\bullet$ as above, s is the tautological section of the flag space of $\mathbb{H}$ over $\mathcal{F}$, and t is the section given by the conjugate flag $\mathbb{D}_\bullet$. From now on we shall, unless otherwise mentioned, let $\mathcal{U}_w$ and $\overline{\mathcal{U}}_w$ denote the subschemes of $\mathcal{F}$ coming from the given s and t and $w \in W_g$. In this case it is actually often more convenient to use the language of flags rather than sections of G/B-bundles and we shall do so without further mention. We shall also say that a Hodge flag $\mathbb{E}_\bullet$ is of *stamp* w respectively *stamp* less than or equal to w if $\mathbb{E}_\bullet$ and its conjugate flag $\mathbb{D}_\bullet$ are in relative position w respectively $\leq w$.

Lemma 4.2. *Over* $\mathcal{U}_w$ *we have an isomorphism* $\mathcal{L}_i \cong \mathcal{M}_{w(i)}$ *for all* $1 \leq i \leq 2g$.

Proof. By the definition of the strata we have that the image of $\mathbb{E}_i \cap \mathbb{D}_{w(i)}$ has rank one greater than the ranks of $\mathbb{E}_{i-1} \cap \mathbb{D}_{w(i)}$, $\mathbb{E}_i \cap \mathbb{D}_{w(i)-1}$, and $\mathbb{E}_{i-1} \cap \mathbb{D}_{w(i)-1}$. So the maps $\mathbb{E}_i/\mathbb{E}_{i-1} \leftarrow (\mathbb{E}_i \cap \mathbb{D}_{w(i)})/(\mathbb{E}_{i-1} \cap \mathbb{D}_{w(i)-1}) \to \mathbb{D}_{w(i)}/\mathbb{D}_{w(i)-1}$ give the isomorphism. □

When the base of the principally polarized abelian scheme is $\mathcal{A}_g$, we shall use the notation $\mathcal{F}_g$ for the space of Hodge flags. Note that a Hodge flag with respect to $X \to S$ is the same thing as a lifting over $\mathcal{F}_g \to \mathcal{A}_g$ of the classifying map $S \to \mathcal{A}_g$. The conjugate flag as well as the strata $\mathcal{U}_g$ and $\overline{\mathcal{U}}_g$ on S are then the pullbacks of the conjugate flag, respectively the strata on $\mathcal{F}_g$.

4.2 Some Important Strata

We now give an interpretation for some of the most important strata. To begin with, if one thinks instead in terms of filtrations of $X[p]$ by subgroup schemes it becomes clear that the condition $F(\mathbb{D}_i^{(p)}) \subseteq \mathbb{D}_i$ should be of interest. It can almost be characterized in terms of the strata $\overline{\mathcal{U}}_w$.

Proposition 4.3. *Let $X \to S$ be a family of principally polarized abelian varieties and $\mathbb{E}_\bullet$ a Hodge flag such that the flag is of stamp $\leq w$ and w is the smallest element with that property.*

(i) For $j \leq g$ and for all $i \in [1, \dots, g]$ we have $r_w(i, g+j) = i$ if and only if $V(\mathbb{E}_i) \subseteq \mathbb{E}_j^{(p)}$.

(ii) For $j \leq g$ we have that $r_w(g+j, i) = i$ implies that $F(\mathbb{D}_i^{(p)}) \subseteq \mathbb{D}_j$ and the converse is true if S is reduced.

(iii) We have that $V(\mathbb{E}_i) \subseteq \mathbb{E}_i^{(p)}$ for all i precisely when $w \leq w_\emptyset$. If S is reduced, $F(\mathbb{D}_i^{(p)}) \subseteq \mathbb{D}_i$ for all i precisely when $w \leq w_\emptyset$.

Proof. We have that $V(\mathbb{E}_i) \subseteq \mathbb{E}_j^{(p)}$ if and only if $\mathbb{E}_i \subseteq V^{-1}(\mathbb{E}_j^{(p)}) = \mathbb{D}_{g+j}$. On the other hand, by definition $\mathrm{rk}\, \mathbb{E}_i \cap \mathbb{D}_{g+j} \leq r_w(i, g+j)$ with equality for at least one point of S. Since $\mathrm{rk}\, \mathbb{E}_i \cap \mathbb{D}_{g+j} = i \iff \mathbb{E}_i \subseteq \mathbb{D}_{g+j}$, we get the first part.

For the second part we start by claiming that $\mathbb{E}_i^{(p)} \subseteq \mathbb{D}_j$ is implied by $F(\mathbb{D}_i^{(p)}) \subseteq \mathbb{D}_j$. Indeed, $F(\mathbb{D}_i^{(p)}) \subseteq \mathbb{D}_j$ is equivalent to $F(\mathbb{D}_i^{(p)})$ being orthogonal to $\mathbb{D}_{2g-j}$, i.e., to the condition that for $u \in \mathbb{D}_i^{(p)})$ and $v \in \mathbb{D}_{2g-j}$ we have $\langle Fu, v \rangle = 0$. This implies that $0 = \langle Fu, v \rangle = \langle u, Vv \rangle^p$ and hence $\langle u, Vv \rangle = 0$, since S is reduced, which means that $\mathbb{D}_i^{(p)} \subseteq (V(\mathbb{D}_{2g-j}))^\perp = (\mathbb{E}_{g-j}^{(p)})^\perp = \mathbb{E}_{g+j}^{(p)}$. Since S is reduced, this implies that $\mathbb{D}_i \subseteq \mathbb{E}_{g+j}$, and this in turn is equivalent to $r_w(g+j, i) = i$. The argument can be reversed (and then it does not require S to be reduced).

Finally, we have from the first part that $V(\mathbb{E}_i) \subseteq \mathbb{E}_i^{(p)}$ for all $i \leq g$ precisely when $r_w(i, g+i) = i$ for all $i \leq g$, but by induction on i that is easily seen to be equivalent to $w(i) \leq g+i$ for all $i \leq g$, which by definition means that $w \leq w_\emptyset$. Since $\mathbb{E}_g^{(p)} = V(\mathbb{H})$, the condition $V(\mathbb{E}_i) \subseteq \mathbb{E}_i^{(p)}$ for $i > g$ is trivially fulfilled.

The proof of the second equivalence is analogous in that using (ii), the condition that $F(\mathbb{D}_i^{(p)}) \subseteq \mathbb{D}_i$ is equivalent to $r_w(g+i, i) = i$. In general, $r_u(i,j) = r_{u^{-1}}(j,i)$, so that this condition is equivalent to $r_{w^{-1}}(i, g+i) = i$, and hence by the same argument as before, this condition for all i is equivalent to $w^{-1} \leq w_\emptyset$. Chevalley's characterization of the Bruhat–Chevalley order makes it clear that $u \leq v \iff u^{-1} \leq v^{-1}$, and hence we get $w^{-1} \leq w_\emptyset \iff w \leq w_\emptyset^{-1}$. However, $w_\emptyset$ is an involution. □

Remark 4.4. (i) As we shall see (cf. Corollary 8.4) the strata $\overline{\mathcal{U}}_w$ in the universal case of $\mathcal{F}_g$ are reduced.

(ii) Flags of stamp $w \leq w_\emptyset$ are called *admissible*.

We can also show that the relations between final and canonical types are reflected in relations for flags. We say that a Hodge flag is a *final flag* if it is of stamp w for a final element w. Also if $I =]i,j] \subseteq \{1,2,\dots,2g\}$ is an interval and $\mathbb{F}_\bullet$ is a complete flag of a vector bundle of rank $2g$ then we define $\mathbb{F}_I$ to be $\mathbb{F}_j/\mathbb{F}_i$.

Proposition 4.5. *Let $X \to S$ be a principally polarized abelian scheme over S and let $\mathbb{E}_\bullet$ be a final flag for it of stamp w.*

(i) The conjugate flag $\mathbb{D}_\bullet$ is a refinement of the canonical flag. In particular, w is determined by $X \to S$. More directly, we have that the final type ν associated to w is given by

$$\mathrm{rk}(\mathbb{E}_g \cap \mathbb{D}_i) = i - \nu(i)$$

for all i. In particular, the canonical decomposition of S with respect to $X \to S$ consists of a single stratum, and its canonical type is the canonical type associated to w.

(ii) Conversely, assume that S is reduced and that the canonical decomposition of S consists of a single stratum, and let ν be the final type associated to the canonical type of the canonical flag. Then any Hodge flag $\mathbb{E}_\bullet$ whose conjugate flag $\mathbb{D}_\bullet$ is a refinement of the canonical flag and for which we have $F(\mathbb{D}_i^{(p)}) \subseteq \mathbb{D}_{\nu(i)}$ for all i, is a final flag.

(iii) If I is a canonical fragment for v, the complementary element to w, then F induces a bijection $(\mathbb{D}_{v(I)})^{(p)} \xrightarrow{\sim} \mathbb{D}_I$.

Proof. We start by showing that $F(\mathbb{D}_i) = \mathbb{D}_{\nu(i)}$ for all i. Indeed, this is equivalent to $F(\mathbb{D}_i) \subseteq \mathbb{D}_{\nu(i)}$ and $\mathrm{rk}\big((\ker(F) = \mathbb{E}_g^{(p)}) \cap \mathbb{D}_i^{(p)}\big) = i - \nu(i)$ since the second condition says that $F(\mathbb{D}_i)$ has rank $\nu(i)$. Now, the condition $F(\mathbb{D}_i) \subseteq \mathbb{D}_{\nu(i)}$ is by Proposition 4.3 implied by $r_w(g+\nu(i), i) = \nu(i)$, which is true for a final element by Corollary 2.10. On the other hand, the condition $\mathrm{rk}(\mathbb{E}_g^{(p)} \cap \mathbb{D}_i^{(p)}) = i - \nu(i)$ is implied by $r_w(g,i) = \mathrm{rk}(\mathbb{E}_g \cap \mathbb{D}_i) = i - \nu(i)$, which is true by the very definition of ν. Now, the fact that $F(\mathbb{D}_i) = \mathbb{D}_j$ for all i and some j (depending on i) implies by induction on the steps of the construction of the canonical flag that $\mathbb{D}$ is a refinement of the canonical flag. The rest of the first part then follows from what we have proved.

As for (ii), assume first that $\mathbb{E}_\bullet$ has a fixed stamp w' and let ν' be its final type. Since $\mathbb{D}_\bullet$ is an extension of the Hodge flag, we get that when i is in the canonical domain of ν then $i - \nu'(i) = \mathrm{rk}(\mathbb{E}_g \cap \mathbb{D}_i) = i - \nu(i)$, so that ν and ν' coincide on the canonical domain of ν and hence they coincide by Lemma 2.11. The assumption that $F(\mathbb{D}_i^{(p)}) \subseteq \mathbb{D}_{\nu(i)} = \mathbb{D}_{\nu'(i)}$ for all i is by Proposition 4.3 equivalent to $r_w(g+\nu'(i), i) = i$ for all i, and hence Corollary 2.10 gives that w is final of type $\nu' = \nu$.

Finally, assume that $I =]i,j]$. The induced map $\mathbb{D}_j^{(p)}/\mathbb{D}_i^{(p)} \to \mathbb{D}_{\nu(j)}/\mathbb{D}_{\nu(i)}$ is always surjective, but it follows from Lemma 2.11 that either the right-hand side of this map has rank 0 or it has the same rank as the right-hand side.

If they have the same rank it induces an isomorphism $\mathbb{D}_I^{(p)} \xrightarrow{\sim} \mathbb{D}_{v^{-1}(I)}$. If the right-hand side has rank zero, then again from Lemma 2.11 the two sides of $\mathbb{D}_{\bar{\jmath}}^{(p)}/\mathbb{D}_{\bar{\imath}}^{(p)} \to \mathbb{D}_{\nu(\bar{\jmath})}/\mathbb{D}_{\nu(\bar{\imath})}$ have the same rank, and hence this map is an isomorphism and again gives an isomorphism $\mathbb{D}_I^{(p)} \xrightarrow{\sim} \mathbb{D}_{v^{-1}(I)}$. Since I is an arbitrary fragment, we conclude the proof of (ii). □

The number of final extensions of a canonical flag will now be expressed in the familiar terms of the number of flags (respectively self-dual flags) in a vector space over a finite field (respectively a vector space with a unitary form). Hence we let $\gamma_n^e(m)$ be the number of complete $\mathbb{F}_{p^m}$-flags in $\mathbb{F}_{p^m}^n$ and let $\gamma_n^o(m)$ be the number of complete $\mathbb{F}_{p^{2m}}$-flags self-dual under the unitary form $\langle (u_1,\dots,u_n),(v_1,\dots,v_n)\rangle := u_1v_1^{p^m} + \cdots + u_nv_n^{p^m}$. Recall now the definition of even and odd orbits of fragments given in Definition 2.17.

Lemma 4.6. *Let X be a principally polarized abelian variety over an algebraically closed field and $w \in W_g$ the element whose canonical type is the canonical type of X. Put*

$$\gamma(w) = \gamma_g(w) := \prod_{S=\overline{S}} \gamma^o_{\#I}(\#S/2) \prod_{\{S,\overline{S}\}} \gamma^e_{\#I}(\#S),$$

where the first product runs over the odd orbits and the second over the even orbit pairs and in both cases I indicates a member of S.

The number of final flags for X is then equal to $\gamma(w)$.

Proof. Since we are over a perfect field, any symplectic flag extending $\mathbb{D}_g$ is the conjugate flag of a unique Hodge flag. Hence we get from Proposition 4.5 that a final flag is the same thing as a flag $\mathbb{D}_\bullet$ extending the canonical flag and for which we have $F(\mathbb{D}_i^{(p)}) \subseteq \mathbb{D}_{\nu(i)}$ for all i. The condition that $\dim(\mathbb{E}_g \cap \mathbb{D}_i) = i - v(i)$ then gives that we actually have $F(\mathbb{D}_i^{(p)}) = \mathbb{D}_{\nu(i)}$. However, since $\mathbb{D}_\bullet$ refines the canonical flag, it is determined by the induced flags of the $\mathbb{D}_I$ for fragments I of v, and the stability condition $F(\mathbb{D}_i^{(p)}) = \mathbb{D}_{\nu(i)}$ transfers into a stability condition under the isomorphisms $F\colon \mathbb{D}_{v(I)}^{(p)} \xrightarrow{\sim} \mathbb{D}_I$ of Section 3.2.

Hence the problem splits up into separate problems for each orbit under v and $i \mapsto \bar{\imath}$ (the map $i \mapsto \bar{\imath}$ transfers into the isomorphism $\mathbb{D}_{\overline{I}} \cong \mathbb{D}_I^\vee$ induced by the symplectic form). Given any fragment $I \in S$, S a v-orbit of fragments of v, the flag of $\mathbb{D}_{v^k(I)}$ for any k is then determined by the flag corresponding to I by the condition that F^k-stability takes the kth Frobenius pullback of the flag on $\mathbb{D}_{v^k(I)}$ to the one of $\mathbb{D}_I$. Furthermore, the flag on $\mathbb{D}_I$ then has to satisfy the consistency condition of being stable under $F_S := F^{\#S}$.

For an even orbit pair $\{S,\overline{S}\}$ the self-duality requirement for the flag means that the flags for the elements of $\overline{S}$ are determined by those for the elements of S. Moreover, given $I \in S$ there is no other constraint on the flag on $\mathbb{D}_I$ than that it be stable under F_S. Hence we have the situation of a vector bundle $\mathcal{D}$

over our base field $\mathbf{k}$ and an isomorphism $F_S\colon \mathcal{D}^{(p^m)} \xrightarrow{\sim} \mathcal{D}$, where $m = \#S$, and we want to count the number of flags stable under F_S. Now, since $\mathbf{k}$ is algebraically closed, $\mathcal{D}_p := \{v \in \mathcal{D} : F_S(v) = v\}$ is an $\mathbb{F}_{p^m}$-vector space for which the inclusion map induces an isomorphism $\mathbf{k} \bigotimes_{\mathbb{F}_{p^m}} \mathcal{D}_p \xrightarrow{\sim} \mathcal{D}$. It then follows that F_S-stable flags correspond to $\mathbb{F}_{p^m}$-flags of $\mathcal{D}_p$.

If instead S is an odd orbit, we will together with I also have $\overline{I}$ in S. If we denote the cardinality of S by $2m$, the flag on $\mathbb{D}_I$ must be mapped to the dual flag on $\mathbb{D}_{\overline{I}}$ by $F_S := F^m$. The situation is similar to the even orbit pair situation but with a "unitary twist" as in Proposition 7.2, and we get instead a correspondence with self-dual flags. We leave the details to the reader. □

Example 4.7. For the canonical type associated to the final type of an element w of Example 3.4 we have

$$\gamma_g(w) = (p+1)(p^2+p+1)\cdots(p^{f-1}+p^{f-2}+\cdots+1).$$

Definition 4.8. *For $1 \leq f < g$ we let $w = u_f$ be the final element*

$$u_f = s_g s_{g-1} s_g \cdots s_{g-f+1} \cdots s_g s_{g-f-1} \cdots s_g \cdots s_1 \cdots s_g,$$

i.e., if we introduce $\tau_j = s_j s_{j+1} \cdots s_g$ then we have $u_f = \tau_g \tau_{g-1} \cdots \hat{\tau}_f \cdots \tau_1$. It corresponds to the Young diagram consisting of one row with $g - f$ blocks and equals the element w given in Example 3.4.

Recall the notion of the a-number $a(X)$ for a g-dimensional abelian variety X of an (algebraically closed) field k of characteristic p. It equals the dimension over k of the vector space $\mathrm{Hom}(\alpha_p, X)$ of maps of the group scheme α_p to X. Equivalently, $a(X)$ equals the dimension of the kernel of V on $H^0(X, \Omega^1_X)$. In our terms $H^0(X, \Omega^1_X) = \mathbb{E}_g$ and $\ker V = \mathbb{D}_g$, so that $a(X) = \dim \mathbb{E}_g \cap \mathbb{D}_g$. The p-rank or *semi-simple rank* f, on the other hand, can be characterized by the condition that $\dim_{\mathbf{k}} \cap_{i \leq g} (V^i \mathbb{H})^{(p^{g-i})}$ is equal to f.

Lemma 4.9. *(i) Let Y be a final Young diagram and $w \in W_g$ its final element and assume that $x = (X, \mathbb{E}_\bullet, \mathbb{D}_\bullet) \in \mathcal{U}_w(\mathbf{k})$, $\mathbf{k}$ a field. Then the p-rank of X equals the p-rank of Y.*

(ii) Let $w = u_f$ (so that $f < g$) and $x = (X, \mathbb{E}_\bullet, \mathbb{D}_\bullet) \in \mathcal{F}_g(\mathbf{k})$. Then we have $x \in \mathcal{U}_w$ (respectively $\overline{\mathcal{U}}_w$) if and only if the filtration $\mathbb{E}_\bullet$ is V-stable and the p-rank of X is f and the a-number of X is 1 (respectively the p-rank of X is $\leq f$). The image of $\mathcal{U}_w$ in $\mathcal{A}_g$ is the locus of abelian varieties of p-rank f and a-number 1.

Proof. By definition we have $r_w(i, g+i) = i$ for all $1 \leq i \leq g$. Moreover, $r_w(i, g+i-1) = i$ precisely when $i \leq g - f$. Hence by Proposition 4.3 we see that $V(\mathbb{E}_i) \subseteq \mathbb{E}_i^{(p)}$ for all $1 \leq i \leq g$, $V(\mathbb{E}_i) \subseteq \mathbb{E}_{i-1}^{(p)}$ for $i \leq g - f$, and $V(\mathbb{E}_i) \subsetneq \mathbb{E}_{i-1}^{(p)}$ for $i > g - f$. The first and last conditions mean that V induces an isomorphism $V\colon \mathbb{E}_{\{i\}} \xrightarrow{\sim} \mathbb{E}_{\{i\}}$. On the other hand, the second condition gives $V^{g-f}(\mathbb{E}_{g-f}) = 0$. Together this gives that the semi-simple rank of X is f.

For the second part, since we have $w \leq w_\emptyset$ we must check the condition on the p-rank and the a-number. By the definition on $\mathcal{U}_w$ we have $x \in \mathcal{U}_w$ if and only if $\mathrm{rk}(\mathbb{E}_g \cap \mathbb{D}_j) = 1$ for $f+1 \leq j \leq g$ and $\mathrm{rk}(\mathbb{E}_g \cap \mathbb{D}_j) = 0$ for $1 \leq j \leq f$. This implies that the kernel of V (which equals $\mathbb{D}_g \cap \mathbb{E}_g$) has rank 1 and the semi-simple rank of V on $\mathbb{E}_g$ is f. For $x \in \overline{\mathcal{U}}_w$ we get instead $\mathrm{rk}(\mathbb{E}_g \cap \mathbb{D}_j) \leq 1$ for $f+1 \leq j \leq g$ and $\mathrm{rk}(\mathbb{E}_g \cap \mathbb{D}_j) = 0$ for $1 \leq j \leq f$. □

Also, the strata $\mathcal{U}_w$ with $w \in S_g$ admit a relatively simple interpretation. Recall that an abelian variety is called *superspecial* if its a-number is equal to its dimension. This happens if and only if the abelian variety (without polarization) is geometrically isomorphic to a product of supersingular elliptic curves.

Lemma 4.10. *Let x be a geometric point of $\mathcal{F}_g$ lying over $[X] \in \mathcal{A}_g$. The following four statements are equivalent:*

1. $x \in \cup_{w \in S_g} \mathcal{U}_w$.
2. $\dim(\mathbb{E}_g \cap \mathbb{D}_g) \geq g$.
3. $\ker(V) = \mathbb{E}_g$.
4. *The underlying abelian variety X is superspecial.*

Proof. An abelian variety X is superspecial if and only if X is a product of supersingular elliptic curves, and this condition is equivalent to $\dim(\mathbb{E}_g \cap \mathbb{D}_g) \geq g$. This explains the equivalences of (2), (3), and (4). If $x \in \mathcal{U}_w$ with $w \in S_g$ then $r_w(g,g) = g$; hence (2) holds. Conversely, if X is superspecial then any filtration $\mathbb{E}_\bullet$ on $\mathbb{E}_g$ is V-stable and can be extended to a symplectic filtration. The lemma now follows from the observation that the degeneracy strata for $w \in S_g$, the Weyl group of GL_g, cover the flag space of flags on $\mathbb{E}_g$. □

Lemma 4.11. *Let x be a point of $\mathcal{U}_w$ with underlying abelian variety X. Then the a-number of X equals $a(w)$. Moreover, if $Y = \{1, 2, \ldots, a\}$ with corresponding final element $w_Y \in W_g$ then the image of $\mathcal{U}_{w_Y}$ in $\mathcal{A}_g$ is the locus T_a of abelian varieties with a-number a.*

Proof. The a-number of an abelian variety is by definition the dimension of the kernel of V on $H^0(X, \Omega^1_X)$. But this is equal to $r_w(g,g) = a(w)$. The condition that $a(X) = a$ implies that $r_w(g,g) = a$; hence $\nu(g) = g - a$. This implies that $\nu(g - a + i) \geq i$ for $i = 1, \ldots, a$. Therefore the "smallest" ν satisfying these conditions is ν_{w_Y}. □

4.3 Shuffling flags

Our first result on the stratification will concern the case in which the p-rank is positive. All in all, the étale and multiplicative parts of the kernel of multiplication by p on the abelian variety have very little effect on the space of flags on its de Rham cohomology. There is, however, one exception to this.

The most natural thing to do is to put the multiplicative part at the bottom (i.e., the first steps of the flag, and thus, by self-duality, the étale part at the top), which is what automatically happens for a final filtration (on the conjugate filtration, that is). We may, however, start with a final filtration and then "move" the μ_p-factors upward. Note that over a perfect field the kernel of multiplication by p is the direct sum of its multiplicative, local-local, and étale parts, so that this is always possible. In general, however, it is possible only after a purely inseparable extension. This means that we get an inseparable map from a stratum where not all the μ_p-factors are at the bottom to a stratum where they all are. We intend first to give a combinatorial description of the strata that can be obtained in this way from a final stratum and then to compute the degrees of the inseparable maps involved. However, since we have to compute an inseparable degree, we should work with Hodge filtrations instead of conjugate filtrations, since conjugate filtrations kill some infinitesimal information. This causes a slight conceptual problem, since the V-simple parts in a final filtration are to be found "in the middle" rather than at the top and bottom (recall that V maps the top part of the conjugate filtration to the bottom of the Hodge filtration). This will not be a technical problem, but the reader will probably be helped by keeping it in mind.

It turns out that the arguments used do not change if instead of considering shuffles of final elements we consider shuffles of semi-simply final elements. We shall treat the more general case, since we shall need it later.

Hence we pick a subset $\tilde{I} \subseteq \{1, 2, \ldots, g\}$ and let $\overline{\mathcal{U}}^{ss}_{\tilde{I}}$ be the closed subscheme of $\mathcal{F}_g$ defined by the conditions that V map $\mathbb{E}_i^{(p)}$ to $\mathbb{E}_i^{(p)}$ for all $1 \le i \le g$ and to $\mathbb{E}_{i-1}^{(p)}$ for $i \notin \tilde{I}$. Hence $\overline{\mathcal{U}}_w \subseteq \overline{\mathcal{U}}^{ss}_{\tilde{I}}$ precisely when $w \le w_\emptyset$ and the semi-simple index set of w is a subset of $\tilde{I}$. We also put

$$\mathcal{U}^{ss}_{\tilde{I}} := \overline{\mathcal{U}}^{ss}_{\tilde{I}} \setminus \cup_{\tilde{I}' \subset \tilde{I}} \overline{\mathcal{U}}^{ss}_{\tilde{I}'},$$

so that $\mathcal{U}_w \subseteq \mathcal{U}^{ss}_{\tilde{I}}$ precisely when $w \le w_\emptyset$ and its semi-simple index set is equal to $\tilde{I}$. If $I := \{g+1-i : i \in \tilde{I}\,\}$ we get from Proposition 2.19 that these w are precisely those of the form $\sigma^I w' \sigma_I^{-1}$ for the semi-simply final w'.

We are now going to construct, for every $I \subseteq \{1, 2, \ldots, g\}$, a morphism $S_I \colon \mathcal{U}^{ss}_{\tilde{I}} \to \mathcal{U}^{ss}_{\{g-f+1,\ldots,g\}}$, where $\tilde{I} := \{g+1-i : i \in I\,\}$ and $\#\tilde{I} = f$.

Let $\tilde{\imath}$ be the reduction index of the elementary reduction I' of I and put $i := g+1-\tilde{\imath}$. By Proposition 2.19 we have that $r_w(i+1, g+i) = i+1$ and $w(i) = g+i$. This means that if $\mathbb{E}_\bullet$ is the (tautological) Hodge flag on $\mathcal{U}^{ss}_{\tilde{I}}$, then $V(\mathbb{E}_{i+1}) \subseteq \mathbb{E}_i^{(p)}$ and $V(\mathbb{E}_i) \subsetneq \mathbb{E}_{i-1}^{(p)}$ everywhere on $\mathcal{U}^{ss}_{\tilde{I}}$. This means that V induces a bijection on $\mathbb{E}_{\{i\}}$ and is zero on $\mathbb{E}_{\{i+1\}}$. Let Id denote the map $s \mapsto 1 \otimes s$ from $\mathcal{T}$ to $\mathcal{T}^{(p)}$ for the sheaves involved. Then the map induced by the quotient map

$$\operatorname{Ker}(\operatorname{Id} - V)_{\mathbb{E}_{\{i+1,i\}}} \to \operatorname{Ker}(\operatorname{Id} - V)_{\mathbb{E}_{\{i\}}},$$

where the kernel is computed in the étale topology on $\mathcal{U}_w$, is an isomorphism. Also for a sheaf $\mathcal{T}$ equipped with a linear isomorphism $V \colon \mathcal{T} \to \mathcal{T}^{(p)}$, we have an isomorphism $\mathrm{Ker}(V - I) \bigotimes \mathcal{O} \to \mathcal{T}$ with $\mathcal{O}$ the structure sheaf. It follows that the short exact sequence

$$0 \to \mathbb{E}_{\{i\}} \longrightarrow \mathbb{E}_{\{i+1,i\}} \longrightarrow \mathbb{E}_{\{i+1\}} \to 0$$

splits uniquely in a way compatible with V. This means that we may define a new flag where $\mathbb{E}'_j = \mathbb{E}_j$ for $j \neq i$ and $\mathbb{E}'_i/\mathbb{E}'_{i-1} = \mathbb{E}_{i+1}/\mathbb{E}_i$. Then the classifying map $\mathcal{U}^{ss}_{\tilde{I}} \to \mathcal{F}_g$ for this new flag will have its image in $\mathcal{U}^{ss}_{\tilde{I}'}$. Repeating this process, we end up with a flag whose classifying map will have its image in $\mathcal{U}_{\{g-f+1,\dots,g\}}$, which by definition is our map S_I.

Proposition 4.12. *If $I \subseteq \{1,\dots,g\}$ then the map $S_I \colon \mathcal{U}^{ss}_{\tilde{I}} \to \mathcal{U}^{ss}_{\{g-f+1,\dots,g\}}$ is finite, radicial, and surjective.*

Proof. To get from a point of $\mathcal{U}^{ss}_{\{g-f+1,\dots,g\}}$ to one of $\mathcal{U}^{ss}_{\tilde{I}}$ one has to find a V-invariant complement to some $\mathbb{E}_i/\mathbb{E}_{i-1}$ in $\mathbb{E}_{i+1}/\mathbb{E}_{i-1}$. Since V will be zero on $\mathbb{E}_{i+1}/\mathbb{E}_i$ and bijective on $\mathbb{E}_i/\mathbb{E}_{i-1}$, a complement over the fraction field of a discrete valuation ring will extend to a complement over the discrete valuation ring (since the complement cannot meet $\mathbb{E}_i/\mathbb{E}_{i-1}$ over the special fiber), so that the map is proper. It then remains to show that the map is a bijection over an algebraically closed field. In that case $\mathbb{E}_g$ splits canonically as a sum of a V-nilpotent part and a V-semisimple part, and the bijectivity is clear. □

In order to determine the degree (necessarily of inseparability) we shall do the same factorization as in the definition of S_I, so that we may consider the situation of $\tilde{I}$ with $\tilde{i}$ and I' being the reduction index respectively elementary reduction of I. For the tautological flag $\mathbb{E}_\bullet$ on $\mathcal{U}^{ss}_{\tilde{I}}$ we have that V is an isomorphism on $\mathbb{E}_{\{i\}}$ and zero on $\mathbb{E}_{\{i+1\}}$, while the opposite is true on $\mathcal{U}^{ss}_{\tilde{I}'}$.

Lemma 4.13. *The map $\mathcal{U}^{ss}_{\tilde{I}} \to \mathcal{U}^{ss}_{\tilde{I}'}$ is flat of degree p.*

Proof. We consider the partial symplectic flag space $\mathcal{F}_g(i)$ consisting of the flags of $\mathcal{F}_g$ by removing the ith member $\mathbb{D}_i$ and its annihilator. This means that we have a $\mathbb{P}^1$-bundle $\mathcal{F}_g \to \mathcal{F}_g(i)$. Now, under this map $\mathcal{U}^{ss}_{\tilde{I}}$ and $\mathcal{U}^{ss}_{\tilde{I}'}$ map to the same subscheme $\mathcal{U} \subseteq \mathcal{F}_g(i)$, and the map $\mathcal{U}^{ss}_{\tilde{I}} \to \mathcal{U}^{ss}_{\tilde{I}'}$ is compatible with these projections. Over $\mathcal{U}$ put $\mathcal{E} := \mathbb{E}_{\{i,i+1\}}$, $\mathcal{M} := \ker(V \colon \mathcal{E} \to \mathcal{E}^{(p)})$, and $\mathcal{L} := \mathrm{Im}(V \colon \mathcal{E} \to \mathcal{E}^{(p)})$. Then on the $\mathbb{P}^1$-bundle $\pi \colon \mathcal{F}_g \to \mathcal{F}_g(i)$, the subscheme $\mathcal{U}^{ss}_{\tilde{I}'}$ is defined by the vanishing of the composite map $\mathcal{O}(-1) \to \pi^*\mathcal{E} \to \pi^*(\mathcal{E}/\mathcal{M})$ and in fact gives a section of $\mathcal{F}_g$ over $\mathcal{U}$ given by the line subbundle $\mathcal{M} \subset \mathcal{E}$. Hence it is enough to show that the projection map $\mathcal{U}^{ss}_{\tilde{I}} \to \mathcal{F}_g(i)$ is flat of degree p. We have that $\mathcal{U}^{ss}_{\tilde{I}'} \subseteq \mathcal{F}_g$ is defined by the vanishing of the composite $\mathcal{O}(-1)^{(p)} \to \pi^*\mathcal{E}^{(p)} \to \pi^*(\mathcal{E}^{(p)}/\mathcal{M})$. It is then enough to show that $\mathcal{U}^{ss}_{\tilde{I}'} \subseteq \mathcal{F}_g$ is a relative Cartier divisor, and for that it is enough to show that it is a proper subset in each fiber of $\mathcal{F}_g \to \mathcal{F}_g(i)$. This, however, is clear, since for a geometric point of $\mathcal{F}_g(i)$ there are just two points that lie in $\overline{\mathcal{U}}_\emptyset$, given by $\mathcal{M}$ and $\mathcal{L}^{p^{-1}}$. □

Composing these maps, we get the following proposition.

Proposition 4.14. *Let* $I \subseteq \{1, \dots, g\}$ *and* $\tilde{I} := \{g+1-i : i \in I\}$*. Then the map* $S_I : \mathcal{U}_I^{ss} \to \mathcal{U}_{\{1,\dots,g\}}^{ss}$ *is a finite purely inseparable map of degree* $p^{\mathrm{ht}(I)}$*.*

Proof. The flatness and the degree of S_I follow by factoring it by maps as in Lemma 4.13 and noting that the number of maps is $\mathrm{ht}(I)$. The rest then follows from Proposition 4.12. □

Remark 4.15. The result implies in particular that if w' is a shuffle of w by I, then $S_I : \mathcal{U}_{w'} \to \mathcal{U}_w$ is flat and purely inseparable of degree $p^{\mathrm{ht}(I)}$. We shall later (see Corollary 8.4) show that $\mathcal{U}_{w'}$ and $\mathcal{U}_w$ are reduced. This shows that over a generic point of $\mathcal{U}_w$ each simple shuffle toward w' really requires a finite inseparable extension of degree p. This is a kind of nondegeneracy statement that is the inseparable analogue of maximal monodromy (of which we shall see some examples later on). It can also be seen as saying that a certain Kodaira–Spencer map is injective.

4.4 The E-O strata on $\mathcal{A}_g \otimes \mathbb{F}_p$

Definition 4.16. *Let* $w \in W_g$ *be a final type. Then the E-O stratum* $\mathcal{V}_w$ *associated to* w *is the locally closed subset of* $\mathcal{A}_g$ *of points* x *for which the canonical type of the underlying abelian variety is equal to the canonical type of* w*. We let* $\overline{\mathcal{V}}_w$ *be the closure of* $\mathcal{V}_w$*.*

It is known that the dimension of $\mathcal{V}_w$ is equal to $\dim(w)$ [Oo01]. This and the fact that the E-O strata form a stratification will also follow from our results in Sections 8 and 9.3.

5 Extension to the boundary

The moduli space $\mathcal{A}_g$ admits several compactifications. The Satake or Baily–Borel compactification $\mathcal{A}_g^*$ is in some sense minimal, cf. [FC90, Chapter V]. It is a stratified space

$$\mathcal{A}_g^* = \bigsqcup_{i=0}^{g} \mathcal{A}_i.$$

Chai and Faltings define in [FC90] a class of smooth toroidal compactifications of $\mathcal{A}_g$. If $\tilde{\mathcal{A}}_g$ is such a toroidal compactification then there is a natural map $q : \tilde{\mathcal{A}}_g \to \mathcal{A}_g^*$ extending the identity on $\mathcal{A}_g$. This induces a stratification of $\tilde{\mathcal{A}}_g$:

$$\tilde{\mathcal{A}}_g = \bigsqcup_{i=0}^{g} q^{-1}(\mathcal{A}_{g-i}) = \bigsqcup_{i=0}^{g} \mathcal{A}_g^{\langle i \rangle}.$$

The stratum $\mathcal{A}_g^{\langle i \rangle}$ parametrizes the semi-abelian varieties of torus rank i.

The Hodge bundle $\mathbb{E}$ on $\mathcal{A}_g$ can be extended to a rank g vector bundle, again denoted by $\mathbb{E}$, on $\tilde{\mathcal{A}}_g$. On $\mathcal{A}_g^{\langle i \rangle}$ the Hodge bundle fits into a short exact sequence

$$0 \to \mathbb{E}' \to \mathbb{E} \to \mathbb{E}'' \to 0,$$

where $\mathbb{E}'$ is a rank $g-i$ bundle and $\mathbb{E}''$ can be identified with the cotangent bundle to the toric part of the universal semi-abelian variety along the identity section over $\mathcal{A}_g^{\langle i \rangle}$. The bundle $\mathbb{E}'$ is the pullback under $q\colon \mathcal{A}_g^{\langle i \rangle} \to \mathcal{A}_g^*$ of the Hodge bundle on $\mathcal{A}_{g-i}$.

The Verschiebung V acts in a natural way on the above short exact sequence and it preserves $\mathbb{E}'$. It induces an action on $\mathbb{E}''$ with trivial kernel because $\mathbb{E}''$ comes from the toric part and is generated by logarithmic forms.

The de Rham bundle $\mathbb{H}$ on $\mathcal{A}_g$ also admits an extension (denoted again by $\mathbb{H}$). This is the logarithmic de Rham sheaf $R^1\pi_*(\Omega^\bullet_{\tilde{\mathcal{X}}_g/\tilde{\mathcal{A}}_g}(\log))$, where the log refers to allowing logarithmic singularities along the divisor at infinity; cf. [FC90, Theorem VI, 1.1]. We have a short exact sequence

$$0 \to \mathbb{E} \to \mathbb{H} \to \mathbb{E}^\vee \to 0$$

extending the earlier mentioned sequence on $\mathcal{A}_g$. The symplectic form on $\mathbb{H}$ extends as well.

We now want to compare strata on $\mathcal{A}_g$ and $\tilde{\mathcal{A}}_g$, and for this we introduce some notation. For a given integer $1 \leq i \leq g$ we can consider the Weyl group W_{g-i} as a subgroup of W_g by letting it act on the set $\{i+1, i+2, \ldots, g, \ldots, 2g-i\}$ via the bijection $j \longleftrightarrow i+j$ for $1 \leq j \leq g-i$. More precisely, define $\rho_i\colon W_{g-i} \to W_g$ via

$$\rho_i(w)(l) = \begin{cases} i + w(l) & \text{for } 1 \leq l \leq g-i, \\ g+l & \text{for } g-i+1 \leq l \leq g. \end{cases}$$

This map respects the Bruhat–Chevalley order, and final elements are mapped to final elements.

Since symplectic flags on $\mathbb{H}$ are determined by their restriction to $\mathbb{E}$ and since we can extend $\mathbb{E}$ to $\tilde{\mathcal{A}}_g$, we can extend $\mathcal{F}_g$ to a flag space bundle $\tilde{\mathcal{F}}_g$ over $\tilde{\mathcal{A}}_g$. Then we can also consider the degeneracy loci $\mathcal{U}_w$ and $\overline{\mathcal{U}}_w$ for $\tilde{\mathcal{F}}_g$. We shall use the same notation for these extensions.

Similarly, we can define the notion of a canonical filtration for a semi-abelian variety. If $1 \to T \to A \to A' \to 0$ is a semi-abelian variety with abelian part A' and toric part T of rank t and if the function ν' on $\{0, c_1, \ldots, c_r, c_{r+1}, \ldots, c_{2r} = 2\dim(A')\}$ is the canonical type of A', then we define the canonical type of A to be the function ν on

$$\{0, t, t+c_1, \ldots, t+c_r, t+c_{r+1}, \ldots, t+c_{2r}, 2g-t, 2g\}$$

defined by $\nu(t+c_i) = t + \nu'(c_i)$. Using this definition we can extend the E-O stratification to $\tilde{\mathcal{A}}_g$.

The stratification of $\tilde{\mathcal{A}}_g$ by the strata $\mathcal{A}_g^{\langle i \rangle}$ induces a stratification of $\tilde{\mathcal{F}}_g$ by flag spaces $\tilde{\mathcal{F}}_g^{\langle i \rangle}$. Recall that the stratum $\mathcal{A}_g^{\langle i \rangle}$ admits a map $q\colon \mathcal{A}_g^{\langle i \rangle} \to \mathcal{A}_{g-i}$ induced by the natural map $\tilde{\mathcal{A}}_g \to \mathcal{A}_g^*$. Similarly, we have a natural map $\pi_i = \pi\colon \mathcal{F}_g^{\langle i \rangle} \to \mathcal{F}_{g-i}$ given by restricting the filtration on $\mathbb{E}$ to $\mathbb{E}'$.

We now describe the interplay between the two stratifications $(\mathcal{F}_g^{\langle i \rangle})_{i=1}^g$ and $(\mathcal{U}_w)_{w \in W_g}$.

Lemma 5.1. *Let $w \in W_g$ be an element with $w \leq w_\emptyset$.*

(i) We have $\mathcal{U}_w \cap \mathcal{F}_g^{\langle i \rangle} \neq \emptyset$ if and only if w is a shuffle of an element in $\rho_i(W_{g-i})$.

(ii) If $w = \rho_i(w')$ with associated degeneracy loci $\mathcal{U}_w \subset \mathcal{F}_g$ and $\mathcal{U}_{w'} \subset \mathcal{F}_{g-i}$ then we have $\mathcal{U}_w \cap \mathcal{F}_g^{\langle i \rangle} = \pi_i^{-1}(\mathcal{U}_{w'})$.

(iii) At a point x of $\tilde{\mathcal{A}}_g$ for which the torus part of the "universal" semi-abelian scheme has rank r, there is a formally smooth map from the formal completion of $\tilde{\mathcal{A}}_g$ at x to the formal multiplicative group $\hat{\mathbb{G}}_m^r$ with the following properties. The locus where the torus rank of the universal semi-abelian scheme is $s \leq r$ is the inverse image of the locus of points of $\hat{\mathbb{G}}_m^r$ where $r - s$ coordinates are 1*. The restriction of this map to the formal completion of any $\overline{\mathcal{U}}_w$ containing x is formally smooth.*

(iv) In particular, $\mathcal{U}_w$ is the closure of its intersection with $\mathcal{F}_g$.

Proof. A V-stable filtration on $\mathbb{E}$ restricts to a V-stable filtration on $\mathbb{E}'$. If $\mathcal{U}_w \cap \mathcal{F}_g^{\langle i \rangle}$ is not empty, then it determines a $w' \in W_{g-i}$ such that $\mathcal{U}_w \cap \mathcal{F}_g^{\langle i \rangle} \subseteq \pi_i^{-1}(\mathcal{U}_{w'})$. Since V is invertible on $\mathbb{E}''$, one sees that w is a shuffle of $\rho_i(w')$ and that $\mathcal{U}_w \cap \mathcal{F}_g^{\langle i \rangle} = \pi_i^{-1}(\mathcal{U}_{w'})$.

The third part is a direct consequence of the local construction of $\tilde{\mathcal{A}}_g$ using toroidal compactifications and of the universal semi-abelian variety using Mumford's construction, where it is defined by taking the quotient of a semi-abelian variety by a subgroup of the torus part, the subgroup being generated by the coordinate functions of $\hat{\mathbb{G}}_m^r$ (see [FC90, Section III.4] for details). Since $\mathbb{H}$ and $\mathbb{E}$ depend only on that universal semi-abelian scheme, it is clear that the restriction of the map to a $\overline{\mathcal{U}}_w$ is smooth.

The last part follows directly from the third. □

Note also that Lemma 5.1 is compatible with shuffling. It also results from this lemma that we can define the E-O stratification on the Satake compactification by considering either the closure of the stratum $\mathcal{V}_w$ on $\mathcal{A}_g$ or the images of the final strata $\mathcal{V}_w$ on $\tilde{\mathcal{A}}_g$.

6 Existence of boundary components

Our intent in this section is to show the existence in irreducible components of our strata of points in the smallest possible stratum $\overline{\mathcal{U}}_1$, the stratum associated to the identity element of W_g.

Proposition 6.1. *Let X be an irreducible component of any $\overline{\mathcal{U}}_w$ in $\mathcal{F}_g$. Then X contains a point of $\overline{\mathcal{U}}_1$.*

Proof. We prove this by induction on g and on the Bruhat–Chevalley order of w. The statement is clear for $g = 1$. We start off by choosing a Chai–Faltings compactification $\tilde{\mathcal{A}}_g$ of $\mathcal{A}_g$ with a semi-abelian family over it (and a "principal" cubical structure so that we get a principal polarization on the semi-abelian variety modulo its toroidal part).

What we now actually want to prove is the same statement as in the proposition but for $\tilde{\mathcal{F}}_g$ instead. Since $\overline{\mathcal{U}}_1$ is contained in $\mathcal{F}_g$, the result will follow. We start off by considering $Y := X \cap (\tilde{\mathcal{F}}_g \backslash \mathcal{F}_g)$. Assume that Y is nonempty and irreducible. If it is not, then we replace it by an irreducible component of Y. Then Y is contained in $\pi_1^{-1}(\overline{\mathcal{U}}_{w'})$ with $\rho_1(w') = w$ for some $w' \in W_{g-1}$. We claim that Y is an irreducible component of $\pi_1^{-1}(\overline{\mathcal{U}}_{w'})$. This follows from the fact that "we can freely move the toroidal part into an abelian variety," which is Lemma 5.1, part (iv).

By induction on g we can assume that $\overline{\mathcal{U}}_{w'}$ in $\mathcal{F}_{g-1}$ contains $\overline{\mathcal{U}}_{1'}$, where $1'$ is the identity element of W_{g-1}. Any component Z of $\overline{\mathcal{U}}_{\rho_1(1')}$ that lies in X and meets Y does not lie completely in the boundary $\tilde{\mathcal{F}}_g - \mathcal{F}_g$. By induction on the Bruhat–Chevalley order we can assume that $w = \rho_1(1')$ and $X = Z$. Note also that for any $w'' < w$ we have that $\overline{\mathcal{U}}_{w''}$ does not meet the boundary, and by induction on the Bruhat–Chevalley order we get that $X = \mathcal{U}_w \cap X$. On the other hand, if X does not meet the boundary we immediately get the same conclusion.

Hence we may and shall assume that Y has the property that it lies completely inside $\mathcal{U}_w$ and that it is proper. Lemma 6.2 now shows that it has an ample line bundle of finite order, which together with properness forces Y to be zero-dimensional. Now observe that $\dim Y \geq \ell(w)$ (the proof is analogous to [Ful, Theorem 14.3]). This gives $\ell(w) = 0$, and so $w = 1$ and we are reduced to a trivial case. □

The following is a version of the so-called Raynaud trick.

Lemma 6.2. *Suppose that X is a proper irreducible component of $\overline{\mathcal{U}}_w$ inside $\mathcal{F}_g$ such that $X \cap \mathcal{U}_w = X$. Then X is 0-dimensional.*

Proof. We have the variety X and two symplectic flags $\mathbb{E}_\bullet$ and $\mathbb{D}_\bullet$ that at all points of X are in the same relative position w. It follows from Lemma 4.2 that we have an isomorphism between $\mathcal{L}_i := \mathbb{E}_i/\mathbb{E}_{i-1}$ and $\mathcal{M}_{w(i)} := \mathbb{D}_{w(i)}/\mathbb{D}_{w(i)-1}$ over X, and then since we also have isomorphisms between $\mathcal{L}_i^p$ and $\mathcal{M}_{g+i}$ and $\mathcal{L}_i$ and $\mathcal{L}_{2g+1-i}^{-1}$, we conclude that all the $\mathcal{L}_i$ have finite order. On the other hand, we know from the theory of flag spaces that $\mathcal{L}_{2g} \bigotimes \mathcal{L}_{2g-1} \bigotimes \cdots \bigotimes \mathcal{L}_{g+1}$ is relatively ample, and since $\mathcal{L}_g \bigotimes \mathcal{L}_{g-1} \bigotimes \cdots \bigotimes \mathcal{L}_1$ is the pullback of an ample line bundle over the base $\mathcal{A}_g$, we conclude. □

7 Superspecial fibers

We shall now discuss the fiber of $\mathcal{F}_g \to \mathcal{A}_g$ over superspecial points. The superspecial abelian varieties are characterized by the condition that $\mathbb{E}_g = \mathbb{D}_g$, i.e., the strata $\mathcal{U}_w$ for which $w \in S_g$. Furthermore, V induces an isomorphism $\mathbb{E}/\mathbb{E}_g \xrightarrow{\sim} \mathbb{E}_g^{(p)}$. On the other hand, the polarization gives an isomorphism $(\mathbb{E}_g)^* \xrightarrow{\sim} \mathbb{E}/\mathbb{E}_g$. This leads to the following definition.

Definition 7.1. *(i) Let S be an $\mathbb{F}_p$-scheme. A p-*unitary* vector bundle is a vector bundle $\mathcal{E}$ over S together with an isomorphism $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}^*$, where $F\colon S \to S$ is the (absolute) Frobenius map.*

*(ii) Let $\mathcal{E}$ be a p-unitary vector bundle over S and let $P \to S$ be the bundle of complete flags on $\mathcal{E}$. The p-*unitary Schubert stratification *of P is the stratification given by letting $\mathcal{U}_w$, $w \in S_g$, consist of the points for which the universal flag $\mathcal{F}$ and the dual of the Frobenius pullback $(F^*\mathcal{F})^*$ are in position corresponding to w.*

A map $F^*\mathcal{E} \to \mathcal{E}^*$ of vector bundles is the same thing as a map $F^*\mathcal{E} \bigotimes_{\mathcal{O}_S} \mathcal{E} \to \mathcal{O}_S$, which in turn corresponds to a biadditive map $\langle -,- \rangle\colon \mathcal{E} \times \mathcal{E} \to \mathcal{O}_S$ fulfilling $\langle fa, b\rangle = f^p\langle a,b\rangle$ and $\langle a, fb\rangle = f\langle a,b\rangle$. We shall normally use this latter description.

All p-unitary vector bundles are trivial in the étale topology, as the following proposition shows.

Proposition 7.2. *If $\langle -,- \rangle$ is a p-unitary structure on the vector bundle $\mathcal{E}$ then*

$$E := \{a \in \mathcal{E} : \forall b \in \mathcal{E} \colon \langle b, a\rangle = \langle a, b\rangle^p\}$$

is a local (in the étale topology) system of $\mathbb{F}_{p^2}$-vector spaces. Furthermore, $\langle -,- \rangle$ induces a unitary (with respect to the involution $(-)^p$ on $\mathbb{F}_{p^2}$) structure on E. Conversely, if E is a local system of $\mathbb{F}_{p^2}$-vector spaces with a unitary structure, then $\mathcal{E} := E \bigotimes_{\mathbb{F}_{p^2}} \mathcal{O}_S$ is a p-unitary vector bundle.

These two constructions establish an equivalence between the categories of p-unitary vector bundles and that of local systems of unitary $\mathbb{F}_{p^2}$-vector spaces. In particular, all p-unitary vector bundles (of the same rank) are locally isomorphic in the étale topology.

Proof. The pairing $\langle -,- \rangle$ gives rise to an isomorphism $\psi'\colon \mathcal{E} \xrightarrow{\sim} (F^*\mathcal{E})^*$ by $a \mapsto b \mapsto \langle b,a\rangle$ and an isomorphism $\psi\colon (F^2)^*\colon \mathcal{E} \xrightarrow{\sim} (F^*\mathcal{E})^*$ by $a \mapsto b \mapsto \langle a,b\rangle^p$. The composite $\rho := \psi^{-1} \circ \psi'$ thus gives an isomorphism $\mathcal{E} \xrightarrow{\sim} (F^2)^*\mathcal{E}$. Then E is simply the kernel of $\rho - 1 \otimes id$, and the fact that $\mathcal{E} = E \bigotimes_{\mathbb{F}_{p^2}} \mathcal{O}_S$ follows from [HW61]. The pairing $\langle -,- \rangle$ then induces a unitary pairing on E, which is perfect, since $\langle -,- \rangle$ is. Conversely, it is clear that a unitary pairing on E translates to one on $\mathcal{E}$.

Finally, since all (perfect) unitary pairings on $\mathbb{F}_{p^2}$-vector spaces of fixed dimension are isomorphic, we get the local isomorphism. □

Proposition 7.2 has the following immediate corollary.

Corollary 7.3. *The flag variety fibrations of two p-unitary vector bundles of the same rank on the same base are locally isomorphic by an isomorphism preserving the p-unitary Schubert strata.*

Let us now consider the situation in which the base scheme is $\mathrm{Spec}(\mathbb{F}_{p^2})$ and $\mathcal{E}$ is an $\mathbb{F}_{p^2}$-vector space given a unitary perfect pairing. The smallest p-unitary Schubert stratum consists of flags that coincide with their unitary dual. Taking duals once more, we see that they are taken to themselves after pullback by the square of the Frobenius, hence are defined over $\mathbb{F}_{p^2}$. Furthermore, they are self-dual with respect to the unitary pairing. This should come as no surprise, since that stratum corresponds to final filtrations on superspecial abelian varieties. The next-to-lowest strata are somewhat more interesting.

Lemma 7.4. *Let V be a g-dimensional $\mathbb{F}_{p^2}$ unitary vector space and let $\mathbb{P}$ be projective space based on V. If $s = (i, i+1) \in S_g$ for some $1 \le i < g$ then the closed Schubert stratum $\mathcal{U}_s \subseteq \mathbb{P}$ consists of the flags $0 = \mathbb{E}_0 \subset \mathbb{E}_1 \subset \cdots \subset \mathbb{E}_g$, where the $\mathbb{E}_j$ for $j \neq i, g-i$ are $\mathbb{F}_{p^2}$-rational with $\mathbb{E}_j^{\perp} = \mathbb{E}_{g-j}$, $\mathbb{E}_{g-i} = (\mathbb{E}_i^{(p)})^{\perp}$ unless $i = 2g$, and $\mathbb{E}_i \neq (\mathbb{E}_{g-i}^{(p)})^{\perp}$.*

Proof. If $\mathbb{E}_\bullet$ and $\mathbb{D}_\bullet$ are two flags in $V \bigotimes R$ (R some $\mathbb{F}_{p^2}$-algebra) and they are in position s, then $\dim(\mathbb{E}_j \cap \mathbb{D}_j) = r_s(j,j) = j$ for $i \neq j$, i.e., $\mathbb{E}_j = \mathbb{D}_j$ and for i the conditions give us $\mathbb{E}_i \cap \mathbb{D}_i = \mathbb{E}_{i-1}$. In our case, where $\mathbb{D}_j = F^*\mathbb{E}_{g-j}^{\perp}$, this means $\mathbb{E}_j = F^*\mathbb{E}_{g-j}^{\perp}$ for $j \neq i$ and $\mathbb{E}_i \neq (\mathbb{E}_{g-i}^{(p)})^{\perp}$. If also $j \neq g-i$ we can use this twice and get that $\mathbb{E}_j = \mathbb{E}_j^{(p^2)}$, i.e., $\mathbb{E}_j$ is $\mathbb{F}_{p^2}$-rational. □

As a result we get the following connectedness result, analogous to [Oo01, Proposition 7.3].

Theorem 7.5. *Let V be a g-dimensional $\mathbb{F}_{p^2}$ unitary vector space and let $\mathbb{P}$ be projective space based on V. Let $S \subseteq \{1, \ldots, g-1\}$. Let $\overline{\mathcal{U}}$ be the union of the $\overline{\mathcal{U}}_{s_i}$ for $i \in S$. Then two flags $0 = A_0 \subset A_1 \subset \cdots \subset A_{g-1} \subset A_g$ and $0 = B_0 \subset B_1 \subset \cdots \subset B_{g-1} \subset B_g$ in $\overline{\mathcal{U}}_1$ lie in the same component of $\overline{\mathcal{U}}$ precisely when $B_i = A_i$ for all $i \notin S$. Furthermore, every connected component of $\overline{\mathcal{U}}$ contains an element of $\overline{\mathcal{U}}_1$.*

Proof. The last statement is clear, since every irreducible component of any $\overline{\mathcal{U}}_{s_i}$ contains a point of $\overline{\mathcal{U}}_1$. This follows from Proposition 6.1 but can also easily be seen directly.

We start by looking at the locus $\overline{\mathcal{U}}_F^i$ of a $\overline{\mathcal{U}}_{s_i}$ with $1 \le i < g$ of flags for which all the components of the flag except the dimension i and dimension $g-i$ parts are equal to a fixed (partial) $\mathbb{F}_{p^2}$-rational self-dual flag $F_\bullet$. The following claims are easily proved using Lemma 7.4.

(i) For any $1 \le i \le (g-2)/2$ or $(g+2)/2 \le i < g$ we get an element in $\overline{\mathcal{U}}_F^i$ by picking any $\mathbb{E}_{i-1} \subset \mathbb{E}_i \subset \mathbb{E}_{i+1}$ and then letting $\mathbb{E}_{g-i}$ be determined by Lemma 7.4. Hence the locus is isomorphic to $\mathbb{P}^1$ and the intersection with $\overline{\mathcal{U}}_1$ consists of the points for which $\mathbb{E}_i$ and $\mathbb{E}_{g-i}$ are $\mathbb{F}_{p^2}$-rational.

(ii) When g is even we get an element in $\overline{\mathcal{U}}_F^i$ by picking $\mathbb{E}_{g/2-1} \subset \mathbb{E}_{g/2} \subset \mathbb{E}_{g/2+1}$. Hence the locus is isomorphic to $\mathbb{P}^1$ and the intersection with $\overline{\mathcal{U}}_1$ consists of the points for which $\mathbb{E}_{g/2}$ is $\mathbb{F}_{p^2}$-rational.

(iii) When g is odd we get an element in $\overline{\mathcal{U}}_F^i$ by picking $\mathbb{E}_{(g-3)/2} \subset \mathbb{E}_{(g-1)/2} \subset \mathbb{E}_{(g+3)/2}$ for which $\overline{\mathbb{E}}_{(g-1)/2} \subset F^*\overline{\mathbb{E}}_{(g-1)/2}^{\perp}$, where $\overline{\mathbb{E}}_{(g-1)/2} = \mathbb{E}_{(g-1)/2}/\mathbb{E}_{(g-3)/2}$ and F^* comes from the $\mathbb{F}_{p^2}$-rational structure on $\mathbb{E}_{(g+3)/2}/\mathbb{E}_{(g-3)/2}$ and the scalar product is induced from that on $\mathbb{E}_g$, and then define $\mathbb{E}_{(g+1)/2}$ by the condition that $\mathbb{E}_{(g+1)/2}/\mathbb{E}_{(g-3)/2} = F^*\overline{\mathbb{E}}_{(g-1)/2}^{\perp}$. Since all nondegenerate unitary forms are equivalent, choosing a basis of $\mathbb{E}_{(g+3)/2}/\mathbb{E}_{(g-3)/2}$ for which the form has the standard form $\langle (x,y,z),(x,y,z)\rangle = x^{p+1}+y^{p+1}+z^{p+1}$ yields that $\overline{\mathcal{U}}_F^i$ is isomorphic to the Fermat curve of degree $p+1$ and hence is irreducible. The intersection with $\overline{\mathcal{U}}_1$ consists of the points for which $\mathbb{E}_{(g-1)/2}$ is $\mathbb{F}_{p^2}$-rational and then $\mathbb{E}_{(g+1)/2} = \mathbb{E}_{(g-1)/2}^{\perp}$.

(iv) When g is odd we get an element in $\overline{\mathcal{U}}_F^i$ by picking $\mathbb{E}_{(g+3)/2}$ fulfilling conditions dual to those of (iii). Hence again $\overline{\mathcal{U}}_F^i$ is irreducible and the intersection with $\overline{\mathcal{U}}_1$ consists of the points for which $\mathbb{E}_{(g+1)/2}$ is $\mathbb{F}_{p^2}$-rational and then $\mathbb{E}_{(g-1)/2} = \mathbb{E}_{(g+1)/2}^{\perp}$.

It follows from this description that two flags in $\overline{\mathcal{U}}_1$ lie in the same component of $\overline{\mathcal{U}}$ if and only if they are equivalent under the equivalence relation generated by the relations that for any unitary $\mathbb{F}_{p^2}$-flag $0 = A_0 \subset A_1 \subset \cdots \subset A_{g-1} \subset A_g$ we may replace it by any flag that is the same except for A_i and A_{g-i} for $i \in S$. The theorem then follows from the following lemma. □

Lemma 7.6. *(i) Let* $\mathbf{k}$ *be a field and let* $\mathcal{F}\ell_n$ *be the set of complete flags of vector spaces in a finite-dimensional vector space. The equivalence relation generated by the operations of modifying a flag* $E_\bullet$ *by, for any* i*, replacing* E_i *by any* i*-dimensional subspace of* E_{i+1} *containing* E_{i-1} *contains just one equivalence class.*

(ii) Let $\mathcal{F}\ell_n$ *the set of complete flags of vector spaces in an* n*-dimensional* $\mathbb{F}_{p^2}$*-vector space, self-dual with respect to a perfect unitary pairing. An* elementary modification *of such a flag* $E_\bullet$ *is obtained by either, for any* $1 \le i \le (n-1)/2$*, replacing* E_i *by any isotropic* i*-dimensional subspace of* E_{i+1} *containing* E_{i-1} *and* E_{n-i} *by its annihilator or, when* n *is even, replacing* $E_{n/2}$ *by any maximal totally isotropic subspace contained in* $E_{n/2+1}$ *and containing* $E_{n/2-1}$*. Then the equivalence relation generated by all elementary modifications contains just one equivalence class.*

Proof. Starting with (i), we prove it by induction on n, the dimension of the vector space V. Given two flags $E_\bullet$ and $F_\bullet$, if E_1 and F_1 are equal we may use induction applied to $E_\bullet/E_1$ and $F_\bullet/E_1$. We now use induction on the smallest j such that $E_1 \subseteq F_j$. The case $j = 1$ has already been taken care of. We now get a new flag $F'_\bullet$ by replacing F_{j-1} by $F_{j-2} \bigoplus E_1$, which works because $E_1 \subsetneq F_{j-1}$, and we then have $E_1 \subseteq F'_{j-1}$.

Continuing with (ii) we again use induction on n and start with two self-dual flags $E_\bullet$ and $F_\bullet$. Let us first assume that n is even, $n = 2k$. Then E_k and F_k are isotropic subspaces. If they have nontrivial intersection, then we may pick a 1-dimensional subspace contained in it and then use (i) to replace $E_\bullet$ and $F_\bullet$ by flags for which E_k and F_k are the same and $E_1 = F_1$. This implies that also $E_{n-1} = F_{n-1}$ and we may consider E_{n-1}/E_1 with its two flags induced from $E_\bullet$ and $F_\bullet$ and use induction to conclude. Assuming $F_k \cap E_k = \{0\}$ we may again use (i) to modify $F_\bullet$, keeping F_k fixed, so that $E_1 \subseteq F_{k+1}$. This means that $E_1 \bigoplus F_{k-1}$ is totally isotropic and we may replace F_k by it to obtain a new flag $F'_\bullet$ for which F'_k and E_k intersect nontrivially.

When n is odd, $n = 2k+1$, we may again use induction on n to finish if E_k and F_k intersect nontrivially. If not, we may again use (i) to reduce to the case $E_1 \subseteq F_{k+2}$ and then we may replace F_k by $E_1 \bigoplus F_{k-1}$ and F_{k+1} by its annihilator. □

8 Local structure of strata

8.1 Stratified Spaces

We now want to show that $\mathcal{F}_g$ looks locally like the space of complete symplectic flags (in $2g$-dimensional space). More precisely, we shall get an isomorphism between étale neighborhoods of points that preserves the degeneration strata. This is proved by establishing a result on suitable infinitesimal neighborhoods that involves not just the complete flag spaces but also partial ones. In order to have a convenient way of formulating such a result we introduce the following two notions:

By a *stratified space* we shall mean a scheme together with a collection of closed subschemes, called *strata*. A map between stratified spaces is said to be stratified if it maps strata into strata.

If P is a partially ordered set then a *diagram* $X_\bullet$ *of spaces over* P associates to each element q of P a scheme X_q and to each relation $q > q'$ a map $X_q \to X_{q'}$ fulfilling the condition that the composite $X_q \to X_{q'} \to X_{q''}$ equal the map $X_q \to X_{q''}$ for any $q > q' > q''$. We shall also similarly speak about a diagram of stratified spaces where both the schemes and the maps are assumed to be stratified. Given a field $\mathbf{k}$ and a $\mathbf{k}$-point x of a diagram $X_\bullet$ we may speak of its (strict) *Henselization* at x, which at each $q \in P$ is the Henselization at x of X_q.

For a positive integer g we now consider the partially ordered set P_g whose elements are the subsets of $\{1, 2, \dots, g-1\}$ and with ordering that of inclusion. We have two diagrams of stratified spaces over this set: The first, $\mathcal{F}\ell_g^\bullet$, associates to the subset S the flag space of a maximal totally isotropic subspace E of a symplectic $2g$-dimensional vector space and partial flags of subspaces of E whose dimensions form the set S. The map associated to an inclusion $S \subset S'$ is simply the map forgetting some of the elements of the flag. Similarly, we let $\mathcal{F}_g^\bullet$ be the diagram that to a subset S associates the space of flags over the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties that associates to a principally polarized abelian variety the space of flags on its Hodge bundle whose dimensions form the subset S.

The diagram $\mathcal{F}\ell_g^\bullet$ becomes a stratified diagram by considering the stratifications given by the (closed) Schubert cells with respect to some fixed complete flag. In positive characteristic p the diagram $\mathcal{F}_g^\bullet$ becomes a stratified diagram by considering the degeneracy loci given by the relative positions of the Hodge flag $\mathbb{E}_\bullet$ and the conjugate flag $\mathbb{D}$.

8.2 Height 1-Maps

For schemes in a fixed positive characteristic p we shall say that a closed immersion $S \hookrightarrow S'$ defined by the ideal sheaf $\mathcal{I}$ on S' is a *height* 1*-map* if $\mathcal{I}_S^{(p)} = 0$, where for an ideal I, we let $I^{(p)}$ be the ideal generated by the pth powers of elements of I. If R is a local ring in characteristic p with maximal ideal m_R, the *height* 1*-hull* of R is the quotient $R/m_R^{(p)}$. It has the property that its spectrum is the largest closed subscheme of $\operatorname{Spec} R$ for which the map from $\operatorname{Spec} R/m_R$ to $\operatorname{Spec} R/m_R^{(p)}$ is a height 1-map. If k is a field of characteristic p and $x\colon \operatorname{Spec} k \to S$ a k-map to a k-scheme S of characteristic p, then by the *height* 1*-neighborhood* of x we will mean the spectrum of the height 1-hull of the local ring of S at x. It is clear that taking height 1-neighborhoods of k-points is functorial under maps between pointed k-schemes. Finally, we shall say that two local rings are *height* 1*-isomorphic* if their respective height 1-hulls are isomorphic and that the height 1-hull of a k-point is *height* 1*-smooth* if it is isomorphic to the height 1-hull of a smooth k-point (i.e., is of the form $k[[t_1, \dots, t_n]]/m^{(p)}$ with $m = (t_1, \dots, t_n)$).

Theorem 8.1. *For each perfect field k of positive characteristic p and each k-point x of $\mathcal{F}_g^\bullet$ there is a k-point y of $\mathcal{F}\ell_g^\bullet$ such that the height* 1*-neighborhood of x is isomorphic to the height* 1*-neighborhood of y by a stratified isomorphism of diagrams.*

Proof. Denote also by x the point of $\mathcal{F}_g$, the space of complete flags of the Hodge bundle, associated to x as a point of the diagram $\mathcal{F}_g^\bullet$. Let $X^\bullet$ be the height 1-neighborhood of x in $\mathcal{F}_g^\bullet$ and X the height 1-neighborhood of x in $\mathcal{F}_g$. Now the ideal of the closed point of x in X has a divided power structure for which all the divided powers of order $\geq p$ are zero. This allows us to get a trivialization of the restriction of the de Rham cohomology $\mathbb{H}_X \xrightarrow{\sim} X \times W$ that is horizontal (i.e., compatible with the Gauss–Manin connection on the left and the trivial connection on the right). Now, since the absolute Frobenius map on X factors through the closed point, we get that the pullback $\mathbb{E}_\bullet^{(p)}$ is a horizontal flag, and then so is $\mathbb{D}_\bullet$, its elements being either inverse images of horizontal subbundles by the horizontal map V or duals of horizontal subbundles. We now get a map from X to the space $\mathcal{F}\ell_g$ of complete symplectic flags on W such that the pullback of the universal flag equals $\mathbb{E}_\bullet$. We may, furthermore, choose a symplectic isomorphism of W and the standard symplectic space such that $\mathbb{D}_\bullet$ is taken to the fixed complete flag. We can extend this map in a compatible fashion for all partial flag spaces giving a map from the diagram $X^\bullet$ to $\mathcal{F}\ell_g^\bullet$, and we will denote by y the k-point that is the composite of x and this map. This map is clearly a stratified map, and by the infinitesimal Torelli theorem (cf. [FC90, pp. 14–15]) it induces an isomorphism from $X^\bullet$ to $Y^\bullet$, the first height 1-neighborhood of y in $\mathcal{F}\ell_g^\bullet$. □

Theorem 8.2. *For each perfect field k of positive characteristic p and each k-point x of $\mathcal{F}_g$ there is a k-point y of $\mathcal{F}\ell_g$ such that the Henselization of x is isomorphic to the Henselization of y by a stratified isomorphism.*

Proof. The theorem provides such an isomorphism over the height 1-hull X of x. Now, over $\mathcal{O}_{\mathcal{F}_g,x}$ we may extend the trivialization of $\mathbb{H}_X$ to a trivialization of $\mathbb{H}_{\mathcal{F}_g,x}$ that also extends the trivialization of $\mathbb{D}$ (making, of course, no requirements of horizontality). This gives a map from the localization, $\tilde{X}$, of $\mathcal{F}_g$ at x to $\mathcal{F}\ell_g$ that extends the map from X to $\mathcal{F}\ell_g$. It thus induces a map from $\tilde{X}$ to $\tilde{Y}$, the localization $\mathcal{F}\ell_g^\bullet$ at y. Now, this map induces an isomorphism on tangent spaces and $\mathcal{F}_g$ is smooth. This implies that we get an induced isomorphism on Henselization and proves the theorem. □

Lemma 8.3. *Let A be a principally polarized abelian variety over an algebraically closed field. If a flag $\mathbb{D}_\bullet$ for it has type w' that is less than or equal to its canonical type, then w' is the final element corresponding to the canonical type of A.*

Proof. The flag $\mathbb{D}_\bullet$ has the property, since it is of a type $\leq$ to the canonical type, that F maps $\mathbb{D}_i^{(p)}$ into $\mathbb{D}_{\nu_w(i)}$. Consider now the set I of i's for which $\mathbb{D}_i$ is a member of the canonical flag. It clearly contains 0 and is closed under $i \mapsto \bar{\imath}$. Furthermore, if $i \in I$, then $F(\mathbb{D}_i^{(p)})$ has dimension $\nu_w(i)$ but is then equal to $\mathbb{D}_{\nu_w(i)}$, since it is contained in it. Hence I fulfills the conditions of Corollary 2.10 and hence contains the canonical domain, which means that $\mathbb{D}_\bullet$ is a refinement of the canonical flag and thus ν, the final type of A, and $\nu_{w'}$ coincide on the canonical domain of ν and are equal by Corollary 2.10, which means that w' is the final element of the canonical type. □

Corollary 8.4. *(i) Each stratum $\mathcal{U}_w$ of $\mathcal{F}_g$ is smooth of dimension $\ell(w)$ (over $\mathbb{F}_p$).*

(ii) The closed stratum $\overline{\mathcal{U}}_w$ *(again of* $\mathcal{F}_g$*) is Cohen–Macaulay, reduced, and normal of dimension* $\ell(w)$*, and* $\overline{\mathcal{U}}_w$ *is the closure of* $\mathcal{U}_w$ *in* $\mathcal{F}_g$ *for all* $w \in W_g$*.*

(iii) If w *is final then the restriction of the projection* $\mathcal{F}_g \to \mathcal{A}_g$ *to* $\mathcal{U}_w$ *is a finite surjective étale covering from* $\mathcal{U}_w$ *to* $\mathcal{V}_w$ *of degree* $\gamma_g(w)$*.*

Proof. We know that each open Schubert cell of $\mathcal{F}\ell_g$ is smooth, and each closed one is Cohen–Macaulay by a proof that runs completely along the lines of [Ful, Theorem 14.3.]. By a theorem of Chevalley (cf. [Ch94, Corollary of Proposition 3]) they are smooth in codimension 1, so by Serre's criterion (cf. [Gr65, Theorem 5.8.6]) they are normal and reduced. The same statement for the stratification of $\mathcal{F}_g$ then follows from the theorem. To finish (ii), the fact that $\overline{\mathcal{U}}_w = \overline{\mathcal{U}_w}$ follows more or less formally from the rest: if $x \in \overline{\mathcal{U}}_w$, then we know that the dimensions of all $\mathcal{U}_{w'}$ with $w' < w$ that pass through in x are $\ell(w') < \ell(w)$, but the dimension of $\overline{\mathcal{U}}_w$ at that point is $\ell(w)$, and hence x must lie in the closure of $\mathcal{U}_w$.

As for (iii), that $\mathcal{U}_w$ maps into $\mathcal{V}_w$ follows from the fact that the restriction of a final filtration to its canonical domain is a canonical filtration (Proposition 4.5). That the map $\mathcal{U}_w \to \mathcal{A}_g$ is unramified follows from the same statement for Schubert cells, which is [BGG73, Proposition 5.1]. We next prove that $\mathcal{U}_w \to \mathcal{V}_w$ is proper. Note that by Proposition 4.5 and by the fact that by definition, $\mathcal{V}_w$ is the image of $\mathcal{U}_w$, we get that the geometric points of $\mathcal{V}_w$ consist of the principally polarized abelian varieties with a canonical filtration whose canonical type corresponds to the final type of w. Hence for properness we may assume that we have a principally polarized abelian variety over a discrete valuation ring R such that both its generic and special points are of type w and we suppose that we are given a final flag over the generic point. Hence the canonical decomposition of $\operatorname{Spec} R$ for the abelian variety is equal to $\operatorname{Spec} R$, and we have a canonical flag over $\operatorname{Spec} R$. Since $\overline{\mathcal{U}}_w$ is proper, the map to it from the generic point of $\operatorname{Spec} R$ extends to a map from $\operatorname{Spec} R$ to $\overline{\mathcal{U}}_w$, hence giving a flag over its special point. This flag is then of a type $\leq w$, and hence by Lemma 8.3, its type is equal to w and the image of $\operatorname{Spec} R$ lies in $\mathcal{U}_w$, which proves properness.

Now, $\mathcal{V}_w$ being by definition the schematic image of $\mathcal{U}_w$, it is reduced because $\mathcal{U}_w$ is. Since $\mathcal{U}_w \to \mathcal{V}_w$ is unramified, it has reduced geometric fibers, and since it is finite and $\mathcal{V}_w$ is reduced, to show that it is flat it is enough to show that the cardinalities of the geometric fibers are the same for all geometric points of $\mathcal{V}_w$. This, however, is Lemma 4.6. Being finite, flat, and unramified, it is étale. That its degree is $\gamma(w)$ follows from Lemma 4.6. □

Remark 8.5. (i) The corollary is true also for the strata of $\tilde{\mathcal{F}}_g$.

(ii) Note that the degree of the map $\mathcal{U}_w \to \mathcal{V}_w$ is $\gamma_g(w)$. By looking at the proof of Lemma 4.6 it is not difficult to show that it is a covering with structure group a product of linear and unitary groups over finite fields of characteristic p.

9 Punctual flag spaces

Let M be the (contravariant) Dieudonné module of a truncated Barsotti–Tate group of level 1 over an algebraically closed field of characteristic p provided with an alternating perfect pairing (of Dieudonné modules). We let $\mathcal{F}_M$, the *punctual flag space* for M, be the scheme of self-dual admissible complete flags in M for which the middle element equals $\mathrm{Im}(V)$. It is well known that every such M occurs as the Dieudonné module of the kernel of multiplication by p on a principally polarized abelian variety. Then $\mathcal{F}_M$ is the intersection of $\overline{\mathcal{U}}_\emptyset$ and the fiber over a point of $\mathcal{A}_g$ giving rise to M. Also, by a result of Oort [Oo01], the canonical type of M determines it (over an algebraically closed field), and hence we shall also use the notation $\mathcal{F}_\nu$ where ν is a final type. For $\Gamma = (I, \mathcal{S})$ where $I \subseteq \{1, \ldots, g\}$ with $\#I$ equal to the semi-simple rank of M and $\mathcal{S}$ a complete V-stable flag of the V-semi-simple part of M, we define $\mathcal{F}_M^\Gamma$ as follows: we let $\mathcal{F}_M^\Gamma$ be the part of $\mathcal{F}_M \cap \mathcal{U}_I^{ss}$ ($\mathcal{U}_I^{ss}$ may clearly be defined directly in terms of M) for which the flag induces $\mathcal{S}$ on the V-semi-simple part. The p-rank of M is denoted by f, and we easily see that $\mathcal{F}_M$ is the disjoint union of the $\mathcal{F}_M^\Gamma$, and putting $\mathcal{F}_M^{\mathcal{S}} := \mathcal{F}_M^{(\{1,\ldots,g-f+1\},\Gamma)}$, we have maps $S_I \colon \mathcal{F}_M^I \to \mathcal{F}_M^{\mathcal{S}}$. These maps are homeomorphisms by Proposition 4.14. This can be seen directly by decomposing M as $M^{mul} \bigoplus M^{\ell\ell} \bigoplus M^{et}$, where V is bijective on M^{mul}, F on M^{et}, and F and V nilpotent on $M^{\ell\ell}$. Any element of an admissible flag over a perfect field will decompose in the same way (since that element is stable under F by definition and under V by duality) and is hence determined by its intersection with M^{mul}, $M^{\ell\ell}$, and M^{et}. By self-duality the intersection of all the elements of the flag with M^{et} is determined by that with M^{mul}, and that part is given by an arbitrary full flag of submodules of M^{mul}, which is our $\mathcal{S}$. That means that we may indeed reconstitute the whole flag from Γ and the induced flag on $M^{\ell\ell}$ and that any choice of flag on $M^{\ell\ell}$ gives rise to a flag in $\mathcal{F}_M^\Gamma$. This means that the map $\mathcal{F}_M^\Gamma \to \mathcal{F}_{M^{\ell\ell}}$ is a homeomorphism, and we may for all practical purposes focus our attention on the case that F and V are nilpotent on M (i.e., M is *local–local*). Hence in this section, *unless otherwise mentioned, the Dieudonné modules considered will be local–local.* Note that the principal interest in this section will be focused on the question of which $\mathcal{U}_w$ have nonempty intersection with $\mathcal{F}_M$ and that this problem is indeed by the above considerations immediately reduced to the local–local case.

We shall make extensive use of one way to move in each $\mathcal{F}_M$:

Consider $w_\emptyset \geq w \in W_g$. Assume that we have an index $1 \leq i \leq g-1$ for which $r_w(g+i-1, i+1) \geq i+1$. This means that for a flag $\mathbb{D}_\bullet$ in $\overline{\mathcal{U}}_w$ we have that $F(\mathbb{D}_{i+1}) \subseteq \mathbb{D}_{i-1}$ or equivalently that F is zero on $\mathbb{D}_{i+1}/\mathbb{D}_{i-1}$. Hence if we replace $\mathbb{D}_i$ by any $\mathbb{D}_{i-1} \subset \mathbb{D} \subset \mathbb{D}_{i+1}$ (replacing also $\mathbb{D}_{2g-i}$ to make the flag self-dual), we shall still have an admissible flag, since $V(\mathbb{D}) \subseteq \mathbb{D}_{i-1}$. In order to construct the $\mathbb{E}$-flag, we apply V to the $\mathbb{D}$-flag, which gives us half of the $\mathbb{E}$-flag, and we complement by taking orthogonal spaces. In the $\mathbb{E}$-flag now $\mathbb{E}_{g-i}$ and $\mathbb{E}_{g+i}$ move. This construction gives a mapping from the projective line $\mathbb{P}(\mathbb{E}_{g-i+1}/\mathbb{E}_{g-i-1})$ to $\mathcal{F}_M$, and we shall therefore call this family the *simple family* of index i (with, of course, respect to M), and we shall write $P_{w,i}$ for this simple family. The condition $r_w(g+i-1,i+1) \geq i+1$ is equivalent to $r_w(g-i+1,i-1) = g-i-1$, and when it is fulfilled we shall say that $g-i$ is *movable for* w.

Proposition 9.1. *Any two points of the local flag space* $\mathcal{F}_M^\Gamma$ *can be connected by a sequence of simple families.*

Proof. We immediately reduce to the case that M is local (in which case the statement is about $\mathcal{F}_M$). We are going to identify $\mathcal{F}_M$ with the scheme of V-stable flags in $\mathrm{Im}(V)$, and we prove the statement for any Dieudonné module N with $F = 0$ and V nilpotent. Let $E_\bullet$ and $F_\bullet$ be two V-stable flags in N. If $E_1 = F_1$ then we may consider N/E_1 and use induction on the length of N to conclude. If not, we use induction on the smallest i such that $F_1 \subseteq E_i$ which we thus may assume to be > 1. We now have $F_1 \subsetneq E_{i-1}$ and hence that F_1 is a complement to E_{i-1} in E_i, so that in particular, $E_i/E_{i-2} = (E_{i-1}/E_{i-2}) \bigoplus (F_1 + E_{i-2})/E_{i-2}$. This has a consequence that V is zero on E_i/E_{i-2}, which means that every subspace of it is stable under V, so that we get a $\mathbb{P}^1$-family of flags in E_i/E_{i-2} in which both E_{i-1}/E_{i-2} and $(F_1 + E_{i-2})/E_{i-2}$ are members, so that we may move E_{i-1} so that it contains F_1. □

Recall (cf. [Oo01, 14.3]) that one defines the partial order relation on final types $\nu_1 \subseteq \nu_2$ (respectively $\nu_1 \subset \nu_2$) by the condition that $\mathcal{V}_{\nu_1} \subseteq \overline{\mathcal{V}}_{\nu_2}$ (respectively $\mathcal{V}_{\nu_1} \subsetneq \overline{\mathcal{V}}_{\nu_2}$). We shall now see that this relation can be expressed in terms of local flag spaces. For this we let M_ν be a Dieudonné module of a principally polarized truncated Barsotti–Tate group of level 1 with final type ν (there is up to isomorphism only one such M_ν, [Oo01, Theorem 9.4]).

Theorem 9.2. *(i) We have that* $\nu' \subset \nu$ *precisely when there is a* $w \in W_g$ *such that* $w \leq \nu$ *and there is a flag of type* w *in* $\mathcal{F}_{M_{\nu'}}$.

(ii) If there is a flag of type w *in* $\mathcal{F}_{M_{\nu'}}$*, then there is a* $w' \leq w$ *such that the intersection* $\mathcal{U}_{w'} \cap \mathcal{F}_{M_{\nu'}}$ *is finite.*

Proof. Consider the image in $\mathcal{A}_g$ of $\overline{\mathcal{U}}_\nu$. It is a closed subset containing $\mathcal{V}_\nu$ and hence contains $\overline{\mathcal{V}}_\nu$, and in particular it meets each fiber over a point of $\overline{\mathcal{V}}_\nu$. Consequently there is a point s in the intersection of $\overline{\mathcal{U}}_\nu$ and the fiber over a point t of $\mathcal{V}_{\nu'}$. Now, s lies in some $\mathcal{U}_w \subseteq \overline{\mathcal{U}}_\nu$ and hence fulfills $w \leq \nu$, and since $\nu \leq w_\emptyset$, s also lies in the local flag space of t, and as has been noted, this is the "same" as $\mathcal{F}_{M_{\nu'}}$. The converse is clear.

As for the second part, the proof of Lemma 6.2 shows that a $w' \leq w$ that is minimal for the condition that $\mathcal{U}_{w'} \cap \mathcal{F}_{M_{\nu'}}$ is nonempty has $\mathcal{U}_{w'} \cap \mathcal{F}_{M_{\nu'}}$ finite. □

The theorem allows us to re-prove a result of Oort [Oo01]; the E-O strata are defined in Section 4.4.

Corollary 9.3. *The E-O stratification on $\mathcal{A}_g$ is a stratification.*

Proof. The condition in 9.2, (i) says that $\nu' \subset \nu$ if and only if the closure $\overline{\mathcal{U}}_\nu$ of $\mathcal{U}_\nu$ has a nonempty intersection with the punctual flag space $\mathcal{F}_{\nu'}$. The proof there gives more precisely that a given point s of $\mathcal{V}_{\nu'}$ lies in $\overline{\mathcal{V}}_\nu$ precisely when $\overline{\mathcal{U}}_\nu$ intersects the fiber over s of the map $\mathcal{F}_g \to \mathcal{A}_g$. This condition does not depend on the point s by a result of Oort on Dieudonné modules [Oo01]. □

From Theorem 9.2 it is clear that the condition that $\mathcal{U}_w \cap \mathcal{F}_M \neq \emptyset$ is important. We shall say that an admissible $w \in W_g$ *occurs* in ν, with ν a final type, if $\mathcal{U}_w \cap \mathcal{F}_M \neq \emptyset$, and we shall write it symbolically as $w \to \nu$.

Remark 9.4. It is important to realize that a priori this relation $w \to \nu$ depends on the characteristic, which is implicit in all of this article. Hence the notation $w \xrightarrow{p} \nu$ would be more appropriate. It is our hope that the relation will a posteriori turn out to be independent of p. If not and if one is working with several primes p, the more precise notation will have to be used.

Hence we can formulate the theorem as saying that $\nu' \subset \nu$ precisely when there exists an admissible w with $w \to \nu'$ and $w \leq \nu$. Suppose final types ν and ν' are given. For an element w of minimal length in the set of minimal elements of $\{w \in W_g : \nu > w,\, w \to \nu'\}$ in the Bruhat–Chevalley order, we then have the following property. The space $\mathcal{U}_w \cap \mathcal{F}_{M_{\nu'}}$ has dimension 0 for the generic point of $\mathcal{V}_{\nu'}$. Clearly, then $\ell(w) \geq \ell(\nu')$ for every w as in Theorem 9.2.

Example 9.5. Since E-O strata on $\mathcal{A}_g$ are defined using the projection from the flag space, the closure of an E-O stratum on $\mathcal{A}_g$ need not be given by the Bruhat–Chevalley order on the set of final elements, and indeed it isn't. Oort gave the first counterexample for $g = 7$ based on products of abelian varieties. We reproduce his example and give two others, one for $g = 5$ and one for $g = 6$ that do not come from products.

(i) Let $g = 7$ and let $w_1 = [1, 2, 4, 6, 7, 10, 12]$ and $w_2 = [1, 2, 3, 7, 9, 10, 11]$. Then w_1 and w_2 are final elements of W_7 and have lengths $\ell(w_1) = 8$ and $\ell(w_2) = 9$. In the Bruhat–Chevalley order neither $w_1 \leq w_2$ nor $w_2 \leq w_1$ holds. Despite this, we have $\overline{\mathcal{V}}_{w_1} \subset \overline{\mathcal{V}}_{w_2}$. The explanation for this lies in the fact that the simple family $P_{w_1,4}$ hits the stratum U_{w_3} with w_3 the element $[1, 2, 3, 7, 6, 10, 11] = s_3 w_1 s_4$, with $w_2 > w_3$ and $w_3 \to w_1$, so by Theorem 9.2 it follows that $\overline{\mathcal{V}}_{w_1} \subset \overline{\mathcal{V}}_{w_2}$. (That there is such a simple family can be proved directly, but for now we leave it as an unsupported claim, since a proof "by hand" would be somewhat messy. A more systematic study of these phenomena will appear in a subsequent paper.) This explains the phenomenon observed in [Oo01, p. 406], (but note the misprints there). Also the element $w_2 > w_4 = [1, 2, 3, 7, 9, 5, 11] \to w_1$ will work for w_1. The element w_1 is the final element corresponding to taking the product of a Dieudonné module with final element [135] and a Dieudonné module with final element [1246], whereas similarly, w_2 appears as the "product" of the final elements [135] and [1256]. Since [1246] < [1256], there is a degeneration of a Dieudonné module of type

[1256] to one of type [1246]. This shows that this example simply expresses the fact that $\subset$ must be stable under products, whereas the Bruhat–Chevalley order isn't. (We'd like to thank Ben Moonen for pointing this out to us.)

(ii) For $g = 5$ we consider the final elements $w_1 = [1, 3, 4, 6, 9]$ and $w_2 = [1, 2, 6, 7, 8]$ of lengths 5 and 6 and the nonfinal element $w_3 = [1, 2, 6, 4, 8]$ in W_5. Then $w_3 < w_2$ and $w_3 \to w_1$, so that $\mathcal{V}_{w_1}$ lies in the closure of $\mathcal{V}_{w_2}$. But in the Bruhat–Chevalley order neither $w_1 < w_2$ nor $w_2 < w_1$ holds.

(iii) Let $g = 6$ and consider the final elements $w_1 = [1, 3, 5, 6, 9, 11]$ and $w_2 := [1, 2, 6, 8, 9, 10]$ of lengths $\ell(w_1) = 8$ and $\ell(w_2) = 9$. In the Bruhat–Chevalley order we do not have $w_1 \leq w_2$. Nevertheless, $\mathcal{V}_{w_1}$ occurs in the closure of the E-O stratum $\mathcal{V}_{w_2}$. Indeed, the admissible element $w_3 = [1, 2, 6, 8, 4, 10]$ satisfies $w_2 \geq w_3 \to w_1$: $\mathcal{U}_{w_3}$ has a nonempty intersection with the punctual flag space $\mathcal{F}_{w_1}$. This time, neither of the elements w_1 and w_2 is a product in the sense of (i). Furthermore, since $\mathcal{V}_{w_1}$ is of codimension 1 in $\mathcal{V}_{w_2}$, this example cannot be derived by taking the transitive closure of the closure under products of the Bruhat–Chevalley relation. The claim that we have $w_3 \to w_1$ and the two preceding ones will be substantiated in a subsequent paper.

There is an approach to the study of the relation of the E-O strata and the strata on $\mathcal{F}_g$ that is in some sense "dual" to the study of punctual flag spaces: that of considering the image in $\mathcal{A}_g$ of the $\mathcal{U}_w$. The following result gives a compatibility result on these images and the E-O stratification.

Proposition 9.6. *(i) The image of any $\mathcal{U}_w$, $w \in W_g$, is a union of strata $\mathcal{V}_\nu$. In particular, the image of a $\overline{\mathcal{U}}_w$ is equal to some $\overline{\mathcal{V}}_\nu$.*

(ii) For any final ν and $w \in W_g$, the maps $\mathcal{U}_w \cap \pi^{-1}\mathcal{V}_\nu \to \mathcal{V}_\nu$ and $\overline{\mathcal{U}}_w \cap \pi^{-1}\mathcal{V}_\nu \to \mathcal{V}_\nu$, where π is the projection $\mathcal{F}_g \to \mathcal{A}_g$, have the property that there is a surjective flat map $X \to \mathcal{V}_\nu$ such that the pullback of them to X is isomorphic to the product $X \times (\mathcal{F}_\nu \cap \mathcal{U}_w)$ respectively $X \times (\mathcal{F}_\nu \cap \overline{\mathcal{U}}_w)$.

(iii) A generic point of a component of $\mathcal{U}_w$ maps to the generic point of some $\mathcal{V}_\nu$, and that ν is independent of the chosen component of $\mathcal{U}_w$.

Proof. The first part follows directly from Oort's theorem (in [Oo01]) on the uniqueness of the Dieudonné module in a stratum $\mathcal{V}_\nu$, since it implies that if one fiber of $\pi^{-1}(\mathcal{V}_\nu) \to \mathcal{V}_\nu$ meets $\mathcal{U}_w$, then they all do. As for the second part, it would follow if we could prove that there is a surjective flat map $X \to \mathcal{V}_\nu$ such that the pullback of $(\mathbb{H}, \mathbb{E}, F, V, \langle -, - \rangle)$ is isomorphic to the constant data (provided by the Dieudonné module of type ν). For this we first pass to the space X_ν of bases of $\mathbb{H}$ for which the first g elements form a basis of $\mathbb{E}$, which is flat surjective over $\mathcal{V}_\nu$. Over X_ν, the data is the pullback from a universal situation, where F, V, and $\langle -, - \rangle$ are given by matrices. In this universal situation we have an action of the group G of base changes, and two points over an algebraically closed field give rise to isomorphic $(\mathbb{H}, \mathbb{E}, F, V, \langle -, - \rangle)$ precisely when they are in the same orbit. By assumption (and Oort's theorem) the image of X_ν lies in an orbit, so it is enough to show that the data over an orbit can be made constant by a flat surjective map. However, the map from G to the orbit obtained by letting g act on a fixed point of the orbit has this property.

The third part follows directly from the second. □

The proposition gives us a map $\tau_p \colon W_g \to W_g/S_g$ that to w associates the final type of the open stratum into which each generic point of $\mathcal{U}_w$ maps. We shall return to this map in Section 13.

Example 9.7. Note that the punctual flag space is in general rather easy to understand, since it depends only on the image of V and we are almost talking about the space of flags stable under a nilpotent endomorphism (remember that we have reduced to the local–local case). Almost, but not quite, since the endomorphism is semilinear rather than linear. What is complicated is the induced stratification. Already the case of $\nu = s_3 \in W_3$ is an illustrative example. We have then that $\ker V \cap \operatorname{Im} V$ is of dimension 2; in fact, we have one Jordan block for V on $\operatorname{Im} V$ of size 2 and one of size 1. The first element, $\mathbb{E}_1$, of the flag must lie in $\ker V \cap \operatorname{Im} V$, so we get a $\mathbb{P}^1$ of possibilities for it. If $\mathbb{E}_1 = \operatorname{Im} V^2$, then V is zero on $\mathbb{E}_3/\mathbb{E}_1$ and we can choose $\mathbb{E}_2/\mathbb{E}_1$ as an arbitrary subspace of $\mathbb{E}_3$ giving us a $\mathbb{P}^1$ of choices for $\mathbb{E}_2$. On the other hand, if $\mathbb{E}_1 \neq \operatorname{Im} V^2$, then $\mathbb{E}_3/\mathbb{E}_1$ has a Jordan block of size 2, and hence there is only one V-stable 1-dimensional subspace, and thus the flag is determined by $\mathbb{E}_1$. The conclusion is that the punctual flag space is the union of two $\mathbb{P}^1$'s meeting at a single point. The intersection point is the canonical filtration (which is a full flag), and one can show that the rest of the points on one component are flags of type [241] and the rest of the points on the other component are flags of type [315].

10 Pieri formulas

In this section we are going to apply a theorem of Pittie and Ram [PR99] to obtain a Pieri-type formula for our strata. (It seems to be historically more correct to speak of Pieri–Chevalley-type formulas, cf. [Ch94].) The main application of it will not be to obtain cycle class formulas, since Pieri formulas usually do not give formulas for individual strata but only for certain linear combinations. For us the principal use of these formulas will be that they show that a certain strictly positive linear combination of the boundary components will be a section of an ample line bundle (or close to ample, since one of the contributors to ampleness will be λ_1, which is ample only on the Satake compactification). This will have as consequence affineness for the open strata as well as a connectivity result for the boundary of the closed strata. We shall see in Section 13 that there is also a Pieri formula for the classes of the E-O strata, though we know very little about it.

In this section we are going to work with level structures. There are two reasons for this. The first one is that we are going to exploit the ampleness λ_1, and even formulating the notion of ampleness for a Deligne–Mumford stack is somewhat awkward. The second is that one of the consequences of our considerations will be an irreducibility criterion for strata. Irreducibility for a stratum on $\mathcal{A}_g$ does not imply irreducibility for the same stratum on the space $\mathcal{A}_{g,n}$ of principally polarized abelian varieties with level n-structure, where always $p \not| n$. In fact, irreducibility for the level-n case means irreducibility on $\mathcal{A}_g$ together with the fact that the monodromy group of the level-n cover is the maximum possible. Hence in this section we shall use $\mathcal{A}_{g,n}$ but also some toroidal compactification $\tilde{\mathcal{A}}_{g,n}$ (cf. [FC90]). Everything we have said so far applies to this situation giving us in particular $\mathcal{F}_{g,n}$ and $\tilde{\mathcal{F}}_{g,n}$, but we have the extra property that for $n \geq 3$, then $\tilde{\mathcal{A}}_{g,n}$ and hence $\tilde{\mathcal{F}}_{g,n}$ are smooth projective varieties.

We now introduce the classes $\ell_i := c_1(\mathbb{E}_{\{i\}})$ for $1 \leq i \leq 2g$ in the Chow ring $\mathrm{CH}^*(\mathcal{F}_g)$. By self-duality of the flag $\mathbb{E}_\bullet$ we have that $\ell_{2g+1-i} = -\ell_i$, and by construction $c_1(\mathbb{D}_{\{i\}}) = p\ell_{i-g} = -p\ell_{3g+1-i}$ for $g+1 \leq i \leq 2g$. (For the notation D_J see Section 3.2.) Furthermore, $\ell_1 + \cdots + \ell_g$ is the pullback from $\tilde{\mathcal{A}}_g$ of λ_1, the first Chern class of the Hodge bundle.

Now we let $M_i := c_1(\mathbb{D}_{]2g-i,2g]})$, $1 \leq i \leq g$, and start by noting that if $n = (n_1, \ldots, n_g)$ then $n \cdot M := \sum_i n_i M_i$ is relatively ample for $\tilde{\mathcal{F}}_g \to \tilde{\mathcal{A}}_g$ if $n_i > 0$ for $1 \leq i < g$. Indeed, by construction $L_i := \ell_{2g} + \cdots + \ell_{2g-i+1}$, $1 \leq i \leq g$, is the pullback from the partial flag space $\tilde{\mathcal{F}}_g[i]$ of flags with elements of rank i, $2g-i$, and g (and with the rank g-component equal to $\mathbb{E}_g$) and on $\tilde{\mathcal{F}}_g[i]$ we have that $\ell_{2g} + \cdots + \ell_{2g-i+1}$ is ample. It is well known that any strictly positive linear combination of these elements is relatively ample. From the formulas above we get that $M_i = p(L_{g-i} + \lambda_1)$ (where we put $L_0 = 0$). On the other hand, λ_1 is almost ample; it is the pullback from $\mathcal{A}_g^*$ of an ample line bundle.

Now we identify the L_i with the fundamental weights of the root system of C_g. Note that W_g acts on the ℓ_i considered as parts of the weight lattice by $\sigma(\ell_i) = \ell_{\sigma(i)}$ (keeping in mind that $\ell_{2g+1-i} = -\ell_i$) and then acts accordingly on the L_i. Let us also note that (by Chevalley's characterization of the Bruhat–Chevalley order) if $w' < w$ with $\ell(w') = \ell(w) - 1$ and if $w = s_{i_1} \cdots s_{i_k}$, then w' is of the form $s_{i_1} \cdots \widehat{s_{i_r}} \cdots s_{i_k}$, which can be rewritten as ws_α, where $s_\alpha = (s_{i_{r+1}} \cdots s_{i_k})^{-1} s_{i_r} (s_{i_{r+1}} \cdots s_{i_k})$, which thus is the reflection with respect to a unique positive root.

Theorem 10.1. *For each $1 \leq i \leq g$ and $w \in W_g$ we have that*

$$(p\lambda_1 + pL_{g-i} - wL_i)[\overline{\mathcal{U}}_w] = \sum_{w' \prec w} c^i_{w,w'} [\overline{\mathcal{U}}_{w'}] \in \mathrm{CH}^1_{\mathbb{Q}}(\overline{\mathcal{U}}_w \bigotimes \overline{\mathbb{F}}_p),$$

where $c^i_{w,w'} \geq 0$ and $w' \prec w$ means $w' \leq w$ and $\ell(w) = \ell(w') + 1$. Furthermore, $c^i_{w,w'} > 0$ precisely when $w' = ws_\alpha$ for α a positive root for which the simple root α_i appears with a strictly positive coefficient when α is written as a linear combination of the simple roots.

Proof. We shall use [PR99], which has the following setup: We fix a semi-simple algebraic group G (which in our case is the symplectic group Sp_{2g}, but using this in the notation will only confuse) with Borel group B and fix a principal B-bundle $E \to X$ over an algebraic variety X. Letting $E(G/B) \to X$ be the associated G/B-bundle, we have, because its structure group is B and not just G, Schubert varieties $\Omega_w \to X$ (which fiber by fiber are the usual Schubert varieties). For every weight $\lambda \in P$, P being the group of weights for G, we have two line bundles on $E(G/B)$; on the one hand, y^λ, obtained by regarding λ as a character of B, which gives a G-equivariant line bundle on G/B and hence a line bundle on $E(G/B)$, on the other hand, the character λ can also be used to construct, with the aid of the principal B-bundle E, a line bundle x^λ on X and then by pullback to $E(G/B)$ a line bundle also denoted by x^λ. A result of [PR99, Corollary] then says that if λ is a dominant weight then

$$y^\lambda[\mathcal{O}_{\Omega_w}] = \sum_{\eta \in \mathcal{T}_w^\lambda} x^{\eta(1)}[\mathcal{O}_{\Omega_{v(\eta,w)}}] \in K_0(E(G/B)). \tag{1}$$

Here $\mathcal{T}_w^\lambda$ is a certain set of piecewise linear paths $\eta\colon [0,1] \to P \bigotimes \mathbb{R}$ in the real vector space spanned by P; moreover, $v(\eta,w)$ is a certain element in the Weyl group of G that is always $\le w$, and $\mathcal{T}_w^\lambda$ has the property that $\eta(1) \in P$ for all its elements η. An important property of $\mathcal{T}_w^\lambda$ is that it depends only on w and λ and not on E. It follows immediately from the description of [PR99] that $v(\eta,w) = w$ in only one case, namely when η is the straight line $\eta(t) = tw\lambda$. Hence we can rewrite the formula as

$$(y^\lambda - x^{w\lambda})[\mathcal{O}_{\Omega_w}] = \sum_{\eta \in \mathcal{T}_w^\lambda}{}' x^{\eta(1)}[\mathcal{O}_{\Omega_{v(\eta,w)}}],$$

where the sum now runs over all elements of $\mathcal{T}_w^\lambda$ for which $v(\eta,w) < w$. Taking Chern characters and looking at the top term that appears in codimension $\mathrm{codim}(w)+1$, we get

$$(c_1(y^\lambda) - c_1(x^{w\lambda}))[\Omega_w] = \sum_{\eta \in \mathcal{T}_w^\lambda}{}'' [\Omega_{v(\eta,w)}], \tag{2}$$

where the sum is now over the elements of $\mathcal{T}_w^\lambda$ for which $\ell(v(\eta,w)) = \ell(w)-1$. To determine the multiplicity with which a given $[\Omega_{w'}]$ appears in the right-hand side we could no doubt use the definition of $\mathcal{T}_w^\lambda$. However, it seems easier to note that the multiplicity is independent of E, and hence we may assume that X is a point and by additivity in λ that λ is a fundamental weight λ_i. In that case one can use a result of Chevalley [Ch94, Proposition 10] to get the description of the theorem. However, we want this formula to be true not in the Chow group of $E(G/B)$ but instead in the Chow group of the relative Schubert subvariety of index w of $E(G/B)$. This, however, is no problem, since the (relative) cell decomposition shows that this Chow group injects into the Chow group of $E(G/B)$.

We now would like to claim Formula 2 in the case $X = \mathcal{F}_g$ and the principal B-bundle is the tautological bundle $\mathbb{E}_\bullet$. The deduction of (2) from (1) is purely formal; however, [PR99] claims (1) only for X a smooth variety over $\mathbb{C}$. This at least allows us to conclude (2) for $X = \mathcal{F}_{g,n} \bigotimes \mathbb{C}$, the flag space associated to a smooth toroidal compactification of $\mathcal{A}_{g,n}$, the moduli space of principally polarized abelian varieties with a principal level n-structure for $n \geq 3$ (see [FC90, Theorem 6.7, Corollary 6.9]). Using the specialization map for the Chow group, we conclude that (2) is valid for $X = \mathcal{F}_{g,n} \bigotimes \overline{\mathbb{F}}_p$. Finally, pushing down under the map $\mathcal{F}_{g,n} \to \mathcal{F}_g$ induced from the map $\overline{\mathcal{A}}_{g,n} \to \overline{\mathcal{A}}_g$ we get (2) for $X = \mathcal{F}_g$.

The final step is to pull back (2) along the section of $E(G/B)$ given by $\mathbb{D}_\bullet$. To make the pullback possible (note that the relative Schubert variety will in general not be smooth over the base), we remove the relative Schubert varieties of codimension 2 in the relative Schubert variety in question. This forces us to remove the part of $\overline{\mathcal{U}}_w$ where the section encounters the removed locus. This is, however, a codimension-2 subset by Corollary 8.4, so its removal will not affect $\mathrm{CH}^1_{\mathbb{Q}}(\overline{\mathcal{U}}_w)$. Unraveling the pullback of $(c_1(y^\lambda) - c_1(x^{w\lambda}))$ gives the theorem. $\square$

To apply the theorem we start with some preliminary results that will be used to exploit the positivity of the involved line bundles.

Lemma 10.2. *Let X be a proper (irreducible) variety of dimension > 1 and $\mathcal{L}$ a line bundle on X that is ample on some open subset $U \subseteq X$. Let $D := X \setminus U$ and let $H \subset U$ be the zero set of a section of $\mathcal{L}_{|U}$. If D is connected then so is $D \cup H$.*

Proof. By replacing the section by a power of it, we may assume that $\mathcal{L}$ is very ample, giving an embedding $U \hookrightarrow \mathbb{P}^n$. Let Z be the closure of the graph of this map in $X \times \mathbb{P}^n$; moreover, let Y be the image of Z under the projection on the second factor giving us two surjective maps $X \leftarrow Z \rightarrow Y$ and let D' be the inverse image in Z of D. Assume that $D \cup H$ is the disjoint union of the nonempty closed subsets A and B and let A' and B' be their inverse images in Z. Now, Y is irreducible of dimension > 1 and hence H'' is connected, where H'' is the hyperplane section of Y corresponding to H, so that the images of A' and B' in Y must meet. However, outside of D' the map $Z \to Y$ is a bijection, and hence the meeting point must lie below a point of D' and hence A' and B' both meet D'. This implies that A and B both meet D, which is a contradiction, since D is assumed to be connected. $\square$

Proposition 10.3. *Let $\mathcal{L}$ be the determinant $\det \mathbb{E}$ of the Hodge bundle over $\mathcal{A}_{g,n}$, $n \geq 3$ (and prime to p).*

(i) There is, for each $1 \leq i < g$, an integer m_i such that the global sections of $\Lambda^{g-i}(\mathbb{H}/\mathbb{E}) \bigotimes \mathcal{L}^{\otimes m}$ generate this bundle over $\mathcal{A}_{g,n}$ whenever $m \geq m_i$. These m_i can be chosen independently of p (but depending on g and n).

(ii) Putting $N_i := L_i + n_i \lambda_1$ for $1 \leq i < g$ and $N_g := \lambda_1$, then $\sum_i m_i N_i$ is ample on $\mathcal{F}_g$ if $m_i > 0$ for all $1 \leq i \leq g$.

(iii) Fix $w \in W_g$ and put $L := \sum_{i<g} L_i$, $N := \sum_{i<g} N_i$, and $m = \sum_i m_i$. Choose r, s, t, and u such that $rN + t\lambda_1 - wL$ respectively $sN + u\lambda - wL_g$ can be written as a positive linear combination of the N_i (using that $L_g = -\lambda_1$). Then if $p > r + sm$ and $(g-1)p > t + um$ we have that $p(L + (g-1)\lambda_1) + pm\lambda_1 - wL - mwL_g$ is ample on $\mathcal{F}_{g,n}$. The constants r, s, t, and u can be chosen independently of p.

Proof. Statement (i) follows directly from the fact that λ_1 is ample on $\mathcal{A}_{g,n}$. The independence of p follows from the existence of a model of $\mathcal{A}_{g,n}$ that exists over $\operatorname{Spec}\mathbb{Z}[1/n, \zeta_n]$.

As for (ii), we have that $\pi_*\mathcal{O}(L_i) = \Lambda^{g-i}(\mathbb{H}/\mathbb{E})$, $1 \leq i < g$, since $\pi\colon \mathcal{F}_{g,n} \to \mathcal{A}_{g,n}$ can be identified with the space of flags on $\mathbb{H}/\mathbb{E}$, and $\mathcal{O}(L_i)$ is $\det(\mathbb{H}/\mathbb{E}_{2g-i})$. By definition we then have that $\pi_*\mathcal{O}(N_i)$ is generated by global sections on $\mathcal{A}_{g,n}$. We know that on the flag space SL_g/B we have that the canonical ring $\bigoplus_\lambda H^0(\mathrm{SL}_g/B, \mathcal{L}_\lambda)$, where λ runs over the dominant weights and $\mathcal{L}_\lambda$ is the corresponding line bundle, is generated by the $H^0(\mathrm{SL}_g/B, \mathcal{L}_{\lambda_i})$, $1 \leq i < g$, where λ_i is the ith fundamental weight (see for instance [RR85]). Also $\mathcal{O}(\sum_{i<g} m_i N_i)$ is relatively very ample, and we have just shown that $\pi_*\mathcal{O}(\sum_{i<g} m_i N_i)$ is generated by global sections. Since λ_1 is ample on $\mathcal{A}_{g,n}$, we get that $\pi_*\mathcal{O}(\sum_{i\leq g} m_i N_i)$ is ample.

Continuing with (iii), we have that $N = L + n\lambda_1$, which gives $p(L + (g-1)\lambda_1) + pm\lambda_1 - wL - mwL_g = (p - (r + sm))N + (p(g-1) - (t + um)) + (rN + t\lambda_1 - wL) + m(sN + u\lambda_1 - wL_g)$. We then conclude by the definitions of r, s, t, and u and (ii). □

Remark 10.4. (i) The constants r, s, t, and u are quite small and easy to compute. We know nothing about the m_i but imagine that they would not be too large.

(ii) It would seem that the last part would not be applicable for $g = 1$, but it can be easily modified to do so. On the other hand, for $g = 1$ everything is trivial anyway.

We are now ready for the first application of the Pieri formula.

Proposition 10.5. *(i) There is a bound depending only on g and n such that if p is larger than that bound, then for an irreducible component Z of some $\mathcal{U}_w \subseteq \tilde{\mathcal{F}}_{g,n}$, $w \in W_g$, the union of the complement of Z in $\overline{Z}$, the closure of Z in $\tilde{\mathcal{F}}_{g,n}$, and the intersection of $\overline{Z}$ and $\tilde{\mathcal{F}}_{g,n} \setminus \mathcal{F}_{g,n}$ is connected if the intersection of $\overline{Z}$ with the boundary $\tilde{\mathcal{F}}_{g,n} \setminus \mathcal{F}_{g,n}$ is connected or empty.*

(ii) There is a bound depending only on g and n such that if p is larger than that bound, then for $w \in W_g$ of semi-simple rank 0 *we have that $\mathcal{U}_w$ is affine.*

Proof. By Proposition 10.3 there is a bound depending only on g and n such that if p is larger than it, then $M := p(L + (g-1)\lambda_1) + pm\lambda_1 - wL - mwL_g$ is ample on $\mathcal{F}_{g,n}$. Summing up Pieri's formula (Theorem 10.1) for $1 \leq i < g$ and

m times the formula for $i = g$ we get that $M[\overline{Z}]$ is supported on $\overline{Z}$ intersected with smaller strata. (Note that the Pieri formula is a priori—and quite likely in reality—true only modulo torsion. We may, however, simply multiply it by a highly divisible integer, and that doesn't change the support.) We then conclude by Lemma 10.2.

As for the second part, we argue as in the first part and conclude that $\overline{\mathcal{U}}_w \setminus \mathcal{U}_w$ is the support of an ample divisor in $\overline{\mathcal{U}}_w$ (as in the theorem, each component must appear, since we are summing up for $1 \leq i \leq g$, and some α_i must appear in the expansion of α) and hence $\mathcal{U}_w$ is affine. □

Remark 10.6. (i) The first part of the proposition is somewhat difficult to use because of the condition on the intersection with the boundary. In the applications of the next section it turns out that we need to apply it only when the intersection is empty.

(ii) For the second part we would like to say more generally that the image of $\mathcal{U}_w$ is affine in $\mathcal{F}^*_{g,n}$ for some appropriate definition of $\mathcal{F}^*_{g,n}$ analogous to the Satake compactification. The problem is that it doesn't seem as if some power of M would be generated by its global sections, so that we cannot define $\mathcal{F}^*_{g,n}$ as the image of $\tilde{\mathcal{F}}_{g,n}$.

11 Irreducibility properties

In this section we shall prove irreducibility of a large class of strata and also that if the characteristic is large enough and our irreducibility criterion is not fulfilled, then (with some extra conditions on the stratum) the stratum is reducible. Our proofs show two advantages of working on the flag spaces. The major one is that our strata are normal, so that irreducibility follows from connectedness. The connectedness of the closed E-O strata except $\overline{\mathcal{V}}_1$ is proven in [Oo01], but the $\overline{\mathcal{V}}_w$ are most definitely not locally connected and hence that does not say very much about the irreducibility. In the converse direction we also make use of the Pieri formula.

Definition-Lemma 11.1. *Let $\{Z_\alpha\}$ be a* stratification *of a Deligne–Mumford stack X of finite type over a field, by which we mean that the strata Z_α are locally closed reduced substacks of X such that the closure $\overline{Z_\alpha}$ of a stratum is the union of strata. By the k-*skeleton *of the stratification we mean the union of the strata of dimension $\leq k$ (which is a closed substack). The* boundary *of a stratum Z_α is the complement of Z_α in its closure. Assume furthermore that each Z_α is irreducible and that for Z_α of dimension strictly greater than some fixed N we have that its boundary is connected (and in particular nonempty). Then the intersection of a connected union Z of closed strata $\overline{Z_\alpha}$ with the N-skeleton is connected.*

Proof. It is enough by induction to prove that if we remove a stratum Z' from Z, whose dimension is maximal and $> N$, then the result remains connected.

Assume that $Z \setminus Z'$ is the disjoint union of two closed subschemes Z_1 and Z_2. By assumption, the boundary of Z' is connected and hence lies in Z_1, say. This means that $Z' \cup Z_1$ is closed and disjoint from Z_2, which by the connectedness of Z implies that Z_2 is empty. □

Proposition 11.2. *There is a bound depending only on g and n such that the following is true if p is larger than that bound:*

Let $X \subseteq \tilde{\mathcal{F}}_{g,n}$ be a connected union of irreducible components of closed strata $\overline{\mathcal{U}}_w$ (for possibly different w) that lie inside of $\mathcal{F}_{g,n}$. Then the intersection of X with the 1-skeleton of the stratification is connected.

Proof. This follows directly from Proposition 10.5 and Lemma 11.1. (Note that for level 1 or 2 we may pass to a higher level in order to apply the proposition.) □

We now want to interpret this proposition (and its converse, which will be true for any p) in arithmetical terms. Hence we define the 1-*skeleton graph* of level n as the following edge-colored graph: Its vertices are the points of $\overline{\mathcal{U}}_1 \subset \mathcal{F}_{g,n}$, i.e., isomorphism classes of principally polarized superspecial g-dimensional abelian varieties A together with a level-n structure and a complete flag $0 = \mathbb{D}_0 \subset \mathbb{D}_1 \subset \cdots \subset \mathbb{D}_g = H^0(A, \Omega^1_A)$ on $H^0(A, \Omega^1_A)$ for which $\mathbb{D}^{\perp}_{g-i} = V^{-1}\mathbb{D}^{(p)}_i$. For each $1 \leq i \leq g$ we connect two vertices by an edge of color i if there is an irreducible component of $\overline{\mathcal{U}}_{s_i}$ that contains them.

Lemma 11.3. *If $S \subseteq \{1, \ldots, g\}$ has the property that it contains g and for every $1 \leq i < g$ we have that either i or $g - i$ belongs to S, then the subgraph of the 1-skeleton graph consisting of all vertices and all edges of colors $i \in S$ is connected.*

Proof. This follows from [Oo01, Proposition 7.3] and Theorem 7.5. □

For a subset $S \subseteq \{1, \ldots, g\}$ the S-*subgraph* of the 1-skeleton graph is the subgraph with the same vertices and with only the edges whose color is in S. This definition allows us to formulate our irreducibility conditions.

Theorem 11.4. *(i) Let $w \in W_g$ and let $S := \{1 \leq i \leq g : s_i \leq w\}$. If the S-subgraph of the 1-skeleton graph is connected, then $\overline{\mathcal{U}}_w \subseteq \mathcal{A}_g$ is irreducible.*

(ii) There is a bound depending only on g such that if p is larger than that bound the following is true: if $w \in W_g$ is admissible and either final or of semi-simple rank 0 and if $S := \{1 \leq i \leq g : s_i \leq w\}$, then there is a bijection between the irreducible components of $\overline{\mathcal{U}}_w \subseteq \mathcal{A}_g$ and the connected components of the 1-skeleton graph.

Proof. The first part is clear, since Proposition 6.1 says that each connected component meets $\overline{\mathcal{U}}_1$, and then by the assumption on connectedness of the S-subgraph $\overline{\mathcal{U}}_w$ is connected; but by Corollary 8.4 it is normal and hence is irreducible.

As for the second part, assume first that w is final but of positive semi-simple rank. This means that its Young diagram does not contain a row of length g, and hence by Lemma 2.7 and the Chevalley characterization of the Bruhat–Chevalley order we have that $s_i \leq w$ for all $1 \leq i \leq g$ and hence the S-subgraph is connected by Lemma 11.3, which makes the statement trivially true. We may therefore assume that the semi-simple rank is 0 and hence that $\overline{\mathcal{U}}_w$ lies entirely in $\mathcal{F}_{g,n}$. In that case the result follows from Proposition 11.2 and the fact that two irreducible components of two $\overline{\mathcal{U}}_{s_i}$ meet only at $\overline{\mathcal{U}}_1$. □

Projecting down to $\mathcal{A}_g$ we get the following corollary, which shows irreducibility for many E-O strata.

Theorem 11.5. *Let $w \in W_g$ be a final element whose Young diagram Y has the property that there is a $\lceil (g+1)/2 \rceil \leq i \leq g$ such that Y does not contain a row of length i. Then $\overline{\mathcal{V}}_w$ is irreducible and the total space of the étale cover $\mathcal{U}_w \to \mathcal{V}_w$ is connected.*

Proof. This follows from Theorem 11.4 and Lemmas 2.7 and 11.3. □

Example 11.6. For $g = 2$ the locus of abelian surfaces of p-rank ≤ 1 is irreducible. For $g = 3$ all E-O strata except the superspecial locus ($Y = \{1, 2, 3\}$) and the Moret–Bailly locus ($Y = \{2, 3\}$) are irreducible.

In [Ha07], S. Harashita has proved that the number of irreducible components of an E-O stratum that is contained in the supersingular locus is given as a class number and as a consequence that, except possibly for small p, these strata are reducible. As has been proved by Oort (cf. [Ha07, Proposition 5.2]) these strata are exactly the ones to which Theorem 11.5 does not apply.

We shall finish this section by showing that the 1-skeleton graph can be described in purely arithmetic terms very strongly reminiscent of the results of Harashita. Note that even for final elements our results are not formally equivalent to Harashita's, since we are dealing with the set of components of the final strata in $\mathcal{F}_g$, whereas Harashita is dealing with their images in $\mathcal{A}_g$. In any case, our counting of the number of components uses Theorem 11.4 and hence is valid only for sufficiently large p, whereas Harashita's are true unconditionally.

We start by giving a well known description of the vertices of the 1-skeleton graph (see for instance [Ek87]) valid when $g > 1$. We fix a supersingular elliptic curve E and its endomorphism ring $\mathbf{D}$ that is provided with the Rosati involution $*$. To simplify life we assume, as we may, that E is defined over $\mathbb{F}_p$ and hence $\mathbf{D}$ contains the Frobenius map F. It has the property that $\mathbf{D}F = F\mathbf{D} = \mathbf{D}F\mathbf{D}$, the unique maximal ideal containing p. Furthermore, we have that $\mathbf{D}/\mathbf{D}F \cong \mathbb{F}_{p^2}$. There is then a bijection between isomorphism classes of $\mathbf{D}$-*lattices* M (i.e., right modules torsion-free and finitely generated as abelian groups) of rank g (i.e., of rank $4g$ as abelian groups) and isomorphism classes of g-dimensional abelian varieties A. The correspondence associates to the abelian variety A the $\mathbf{D}$-module $\mathrm{Hom}(E, A)$. Polarizations on A then correspond to positive definite *unitary forms*, i.e., a biadditive map $\langle -,- \rangle \colon M \times M \to \mathbf{D}$ such that $\langle md, n \rangle = \langle m, n \rangle dn$, $\langle n, m \rangle = (\langle m, n \rangle)^*$, and $m \neq 0 \Rightarrow \langle m, m \rangle > 0$. The polarization is principal precisely when the form is *perfect*, i.e., the induced map of right $\mathbf{D}$-modules $M \to \mathrm{Hom}_{\mathbf{D}}(M, \mathbf{D})$ given $n \mapsto (m \mapsto \langle m, n \rangle)$ is an isomorphism. In general we put $M^* := \mathrm{Hom}_{\mathbf{D}}(M, \mathbf{D})$, and then the form induces an embedding $M \to M^*$, which makes the image of finite index. More precisely, on $M \bigotimes \mathbb{Q}$ we get a nondegenerate pairing with values in $\mathbf{D} \bigotimes \mathbb{Q}$ and then we may identify M^* with the set $\{n \in M \bigotimes \mathbb{Q} : \forall m \in M \colon \langle m, n \rangle \in \mathbf{D}\}$. Using this we get a $\mathbf{D} \bigotimes \mathbb{Q}/\mathbb{Z}$-valued unitary perfect form on M^*/M given by $\langle \overline{m}, \overline{n} \rangle := \langle m, n \rangle \bmod \mathbf{D}$. As usual, superlattices $M \subseteq N$ over $\mathbf{D}$ correspond to totally isotropic submodules of M^*/M.

If now $S \subseteq \{0, \ldots, g\}$ is stable under $i \mapsto g - i$ then an *arithmetic S-flag* consists of the choice of unitary forms on $\mathbf{D}$-modules M_i of rank g for $i \in S$ and compatible isometric embeddings $M_i \hookrightarrow M_j$ whenever $i < j$ fulfills the following conditions:

- For all $i \in S$ with $i \geq g/2$ we have that M_i^*/M_i is killed by F of $\mathbf{D}$ and can hence be considered as a $\mathbf{D}/m = \mathbb{F}_{p^2}$-vector space with a perfect unitary form.
- We have that $FM_i^* = M_{g-i}$ for all $i \in S$.
- The length of M_j/M_i for $i < j$ is equal to $j - i$.

Remark 11.7. (i) Note that we allow S to be empty, in which case there is exactly one S-flag.

(ii) As follows (implicitly) from the proof of the next proposition, the isomorphism class of an element of an arithmetic S-flag tensored with $\mathbb{Q}$ is independent of the flag. Hence we may consider only lattices in a fixed unitary form over $\mathbf{D} \bigotimes \mathbb{Q}$ and then think of M_i as a sublattice of M_j.

Proposition 11.8. *Let $S \subseteq \{1, \ldots, g\}$ and let $\overline{S} \subseteq \{0, \ldots, g\}$ be the set of integers of the form i or $g - i$ for $i \in S$. Then the set of isomorphism classes of $\overline{S}$-flags is in bijection with the set of connected components of the S-subgraph of the* 1*-skeleton graph.*

Proof. This follows from the discussion above and Theorem 7.5 once we have proven that an S-flag can be extended to a $\{0, \ldots, g\}$-flag. Assume first that $g \notin S$ and let $i \in S$ be the largest element in S. By assumption we have $FM_i^* = M_{g-i} \subseteq M_i \subset M_i^*$ and we have that the length of M_i^*/FM_i^* is g, whereas again by assumption, that of M_i/M_{g-i} is $2i - g$. Together this gives that the length of M_i^*/M_i is $g - (2i - g) = 2(g - i)$. Since the form on M_i^*/M_i is a nondegenerate unitary $\mathbb{F}_{p^2}$-form and since all such forms are equivalent, we get that there is a $(g - i)$-dimensional totally isotropic (and hence its own orthogonal) subspace of M_i^*/M_i; this then gives an $M_i \subset M_g \subset M_i^*$ and since M_g/M_i is its own orthogonal, we get that the pairing on M_g is perfect. We then put $M_0 := FM_g$, and the rest of the flag extension is immediate. □

12 The Cycle Classes

If one wishes to exploit our stratification on $\mathcal{F}_g$ and the E-O stratification on $\mathcal{A}_g$ fully, one needs to know the cohomology classes (or Chow classes) of the (closed) strata. In this section we show how to calculate these classes.

The original idea for the determination of the cycle classes can be illustrated well by the p-rank strata. If X is a principally polarized abelian variety of dimension g that is general in the sense that its p-rank is g, then its kernel of multiplication by p contains a direct sum of g copies of μ_p, the multiplicative group scheme of order p. The unit tangent vector of μ_p gives a tangent vector to X at the origin. By doing this in the universal family we thus see that on a suitable level cover of the moduli we have g sections of the Hodge bundle over the open part of ordinary abelian varieties. If the abelian variety loses p-rank under specialization, the g sections thus obtained become dependent and the loci where this happens have classes represented by a multiple of the Chern classes of the Hodge bundle.

To calculate the cycle classes of the E-O strata on $\mathcal{A}_g \otimes \mathbb{F}_p$ we shall use the theory of degeneration cycles of maps between vector bundles. To this end we shall apply formulas of Fulton for degeneracy loci of symplectic bundle maps to calculate the classes of the $\overline{\mathcal{U}}_w$ and formulas of Pragacz and Ratajski and of Kresch and Tamvakis for calculating those of $\overline{\mathcal{V}}_\nu$.

12.1 Fulton's formulas

Over the flag space $\mathcal{F}_g$ we have the pullback of the de Rham bundle and the two flags $\mathbb{E}_\bullet$ and $\mathbb{D}_\bullet$ on it. We denote by ℓ_i the roots of the Chern classes of $\mathbb{E}$ so that $c_1(\mathbb{E}_i) = \ell_1 + \ell_2 + \cdots + \ell_i$. We then have $c_1(\mathbb{D}_{g+i}) - c_1(\mathbb{D}_{g+i-1}) = p\ell_i$.

Recall (cf. Section 4.1 and Section 5) that for each element $w \in W_g$ we have degeneracy loci $\overline{\mathcal{U}}_w$ in $\mathcal{F}_g$ respectively $\tilde{\mathcal{F}}_g$. Their codimensions equal the length $\ell(w)$, and it thus makes sense to consider the cycle class $u_w = [\overline{\mathcal{U}}_w]$ in $\mathrm{CH}_{\mathbb{Q}}^{\mathrm{codim(w)}}(\tilde{\mathcal{F}}_g)$, where we write $\mathcal{F}_g$ instead of $\mathcal{F}_g \otimes \mathbb{F}_p$.

Fulton's setup in [Fu96] is the following (or more precisely the part that interests us): We have a symplectic vector bundle H over some scheme X and two full symplectic flags $0 \subset \cdots \subset E_2 \subset E_1 = H$ and $0 \subset \cdots \subset D_2 \subset D_1 = H$. For each $w \in W_g$ one defines the degeneracy locus $\overline{\mathcal{U}}_w$ by $\{x \in X : \forall i,j\colon \dim(\mathbb{E}_{i,x} \cap \mathbb{D}_{j,x}) \le r_w(i,j)\}$ (of course this closed subset is given a scheme structure by considering these conditions as rank conditions for maps of vector bundles). Fulton then defines a polynomial in two sets of variables x_i and y_j, $i,j = 1,\ldots,g$, such that if this polynomial is evaluated as $x_i = c_1(E_i/E_{i+1})$ and $y_j = c_1(D_j/D_{j+1})$, then it gives the class of $\overline{\mathcal{U}}_w$ provided that $\overline{\mathcal{U}}_w$ has the expected codimension codim(w) (and X is Cohen–Macaulay). The precise definition of these polynomials is as follows: For a partition $\mu = \{\mu_1 > \mu_2 > \cdots > \mu_r > 0\}$ with $r \le g$ and $\mu_1 \le g$ one defines a Schur function

$$\Delta_\mu(x) := \det(x_{\mu_i+j-i})_{1\le i,j\le r}$$

in the variables x_i and puts

$$\Delta(x,y) := \Delta_{(g,g-1,\ldots,1)}(\sigma_i(x_1,\ldots,x_g)+\sigma_i(y_1,\ldots,y_g)),$$

where σ_i is the ith elementary symmetric function. One then considers the "divided difference operators" ∂_i on the polynomial ring $\mathbb{Z}[x_1,\ldots,x_g]$ by

$$\partial_i(F(x)) = \begin{cases} \frac{F(x)-F(s_i x)}{x_i - x_{i+1}} & \text{if } i < g, \\ \frac{F(x)-F(s'_g x)}{2x_g} & \text{if } i = g, \end{cases}$$

where s_i interchanges x_i and x_{i+1} for $i = 1,\ldots,g-1$, but s'_g sends x_g to $-x_g$ and leaves the other x_i unchanged. We write an element $w \in W_g$ as a product $w = s_{i_\ell} s_{i_{\ell-1}} \cdots s_{i_1}$ with $\ell = \ell(w)$ and set

$$P_w := \partial_{i_\ell} \cdots \partial_{i_1}\Big(\prod_{i+j\le g} (x_i - y_j)\cdot \Delta\Big). \tag{4}$$

An application of Fulton's formulas gives the following.

Theorem 12.1. *Let $w = s_{i_\ell} s_{i_{\ell-1}} \cdots s_{i_1}$ with $\ell = \ell(w)$ be an element of the Weyl group W_g. Then the cycle class $u_w := [\overline{\mathcal{U}}_w]$ in $\mathrm{CH}_{\mathbb{Q}}^{\mathrm{codim}(w)}(\tilde{\mathcal{F}}_g)$ is given by*

$$u_w = \partial_{i_1} \cdots \partial_{i_\ell} \left(\prod_{i+j\le g} (x_i - y_j)\cdot \Delta(x,y) \right)_{|x_i = -\ell_i, y_j = p\ell_j} .$$

Proof. By construction $\overline{\mathcal{U}}_w$ is the degeneracy locus of the flags $\mathbb{E}_\bullet$ and $\mathbb{D}_\bullet$. By Corollary 8.4 they are Cohen–Macaulay and have the expected dimension, and hence the degeneracy cycle class is equal to the class of $\overline{\mathcal{U}}_w$. □

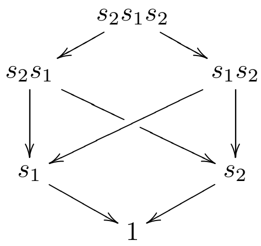
For a final element $w \in W_g$ the map $\overline{\mathcal{U}}_w \to \overline{\mathcal{V}}_w$ is generically finite of degree $\gamma_g(w)$. By applying the Gysin map to the formula of Theorem 12.1 using Formula 3.1 we can in principle calculate the cohomology classes of all the pushdowns of final strata, hence of the E-O strata.

Example 12.2. $g = 2$.

The Weyl group W_2 consists of eight elements; we give the cycle classes in $\tilde{\mathcal{F}}_2$ and the pushdowns on $\tilde{\mathcal{A}}_2$:

w	s	ℓ	$[\overline{\mathcal{U}}_w]$	$\pi_*([\overline{\mathcal{U}}_w])$
$[4,3]$	$s_1s_2s_1s_2$	4	1	0
$[4,2]$	$s_1s_2s_1$	3	$(p-1)\lambda_1$	0
$[3,4]$	$s_2s_1s_2$	3	$-\ell_1 + p\ell_2$	$1+p$
$[2,4]$	s_1s_2	2	$(1-p)\ell_1^2 + (p^2-p)\lambda_2$	$(p-1)\lambda_1$
$[3,1]$	s_2s_1	2	$(1-p^2)\ell_1^2 + (1-p)\ell_1\ell_2 + (1-p)\ell_2^2$	$p(p-1)\lambda_1$
$[2,1]$	s_1	1	$(p-1)(p^2+1)\lambda_1\lambda_2$	0
$[1,3]$	s_2	1	$(p^2-1)\ell_1^2(\ell_1 - p\ell_2)$	$(p-1)(p^2-1)\lambda_2$
$[1,2]$	1	0	$-(p^4-1)\lambda_1\lambda_2\ell_1$	$(p^4-1)\lambda_1\lambda_2$

In the flag space $\mathcal{F}_2$ the stratum corresponding to the empty diagram is $U_{s_2s_1s_2}$ and the strata contained in its closure are the four final ones $U_{s_2s_1s_2}$, $U_{s_1s_2}$, U_{s_2}, and U_1 and the two nonfinal ones U_{s_1} and $U_{s_2s_1}$. The Bruhat–Chevalley order on these is given by the following diagram:



The four final strata $U_{s_2s_1s_2}$, $U_{s_1s_2}$, U_{s_2}, and U_1 lie étale of degree 1 over the p-rank 2 locus, the p-rank 1 locus, the locus of abelian surfaces with p-rank 0 and a-number 1, and the locus of superspecial abelian surfaces ($a = 2$). The locus U_{s_1} is an open part of the fibers over the superspecial points. The locus $U_{s_2s_1}$ is of dimension 2 and lies finite but inseparably of degree p over the p-rank 1 locus. Then E_1 corresponds to an α_p and E_2/E_1 to a μ_p. In the final type locus $U_{s_1s_2}$ the filtration is $\mu_p \subset \mu_p \oplus \alpha_p$. Note that this description is compatible with the calculated classes of the loci.

We have implemented the calculation of the Gysin map in Macaulay2 (cf. [M2]) and calculated all cycle classes for $g \leq 5$. For $g = 3, 4$ the reader will find the classes in the appendix. (The Macaulay2 code for performing the calculations can be found at `http://www.math.su.se/ teke/strata.m2`.) We shall return to the qualitative consequences one can draw from Theorem 12.1 in the next section.

12.2 The *p*-rank strata

It is very useful to have closed formulas for the cycle classes of important strata. We give the formulas for the strata defined by the p-rank and by the a-number. The formulas for the p-rank strata can be derived immediately from the definition of the strata.

Let V_f be the closed E-O stratum of $\tilde{A}_g$ of semi-abelian varieties of p-rank $\leq f$. It has codimension $g - f$. To calculate its class we consider the element $w_\emptyset$, the longest final element. The corresponding locus $\overline{\mathcal{U}}_\emptyset$ is a generically finite cover of $\mathcal{A}_g$ of degree $\gamma_g(w_\emptyset) = \prod_{i=1}^{g-1}(p^i + p^{i-1} + \cdots + 1)$. The map of $\mathcal{U}_\emptyset$ to the p-rank g locus is finite. The space $\overline{\mathcal{U}}_\emptyset$ contains the degeneracy loci $\mathcal{U}_w$ for all final elements $w \in W_g$. The condition that a point x of $\mathcal{F}_g$ lie in $\overline{\mathcal{U}}_\emptyset$ is that the filtration $\mathbb{E}_i$ for $i = 1, \ldots, g$ be stable under V. By forgetting part of the flag and considering flags $\mathbb{E}_j$ with $j = i, \ldots, g$ we find that $\overline{\mathcal{U}}_\emptyset \to \mathcal{A}_g$ is fibered by generically finite morphisms

$$\overline{\mathcal{U}}_\emptyset = \overline{\mathcal{U}}^{(1)} \xrightarrow{\pi_1} \overline{\mathcal{U}}^{(2)} \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_{g-1}} \overline{\mathcal{U}}^{(g)} = \mathcal{A}_g.$$

We shall write $\pi_{i,j}$ for the composition $\pi_j \pi_{j-1} \cdots \pi_i \colon \overline{\mathcal{U}}^{(i)} \to \overline{\mathcal{U}}^{(j)}$ and $\pi_\emptyset = \pi_{1,g}$.

Since V_{g-1} is given by the vanishing of the map $\det(V)\colon \det(\mathbb{E}_g) \to \det(\mathbb{E}^{(p)})$, the class of V_{g-1} is $(p-1)\lambda_1$. The pullback of V_{g-1} to $\overline{\mathcal{U}}_\emptyset$ decomposes in g irreducible components

$$\pi_\emptyset^{-1}(V_{g-1}) = \cup_{i=1}^{g} Z_i,$$

where Z_i is the degeneracy locus of the induced map $\phi_i = V_{|\mathcal{L}_i}\colon \mathcal{L}_i \to \mathcal{L}_i^{(p)}$. Note that the Z_i are the $\overline{\mathcal{U}}_w$ for the w that are shuffles of the final element u_{g-1} (see Section 4.2) defining the (open) E-O stratum of p-rank f, and Z_g is the stratum corresponding to the element $u_{g-1} \in W_g$. An abelian variety of p-rank $g-1$ (and thus with a-number 1) has a unique subgroup scheme α_p. The index i of Z_i indicates where this subgroup scheme can be found (i.e., its Dieudonné module lies in $\mathbb{E}_i$ but not in $\mathbb{E}_{i-1}$).

It follows from the definition of Z_i as degeneracy set that the class of Z_i on $\overline{\mathcal{U}}_\emptyset$ equals $(p-1)\ell_i$, since ϕ_i can be interpreted as a section of $\mathcal{L}_i^{(p)} \otimes \mathcal{L}_i^{-1}$. We also know by Section 4.3 that the map $Z_i \to Z_{i+1}$ is inseparable. Therefore $\pi_\emptyset([Z_i]) = p^{n(i)}\pi_\emptyset([Z_g])$ for some integer $n(i) \geq g-i$. Using the fact that $(\pi_\emptyset)_*([Z_g]) = \gamma_g(u_{g-1})[V_{g-1}] = \deg(\pi_{1,g-1})[V_{g-1}]$, we see that

$$(\pi_\emptyset)_*(\pi_\emptyset^*([V_{g-1}]) = \sum_{i=1}^{g} (\pi_\emptyset)_*([Z_i]) = \sum_{i=1}^{g} p^{n(i)} \deg(\pi_{1,g-1})[V_{g-1}],$$

while on the other hand,

$$(\pi_\emptyset)_*(\pi_\emptyset^*([V_{g-1}]) = \deg(\pi_\emptyset)[V_{g-1}] = (1+p+\cdots+p^{g-1}) \deg(\pi_{1,g-1})[V_{g-1}].$$

Comparison yields that $n(i) = g-i$ and so we obtain

$$(\pi_\emptyset)_*(\ell_i) = p^{g-i} \deg(\pi_{1,g-1})\lambda_1$$

and

$$(\pi_\emptyset)_*([Z_i]) = (p-1)p^{g-i} \deg(\pi_{1,g-1})\lambda_1.$$

Lemma 12.3. *In the Chow groups with rational coefficients of* $\overline{\mathcal{U}}^{(i)}$ *and* $\overline{\mathcal{U}}^{(i+1)}$ *we have for the pushdown of the* j*th Chern class* $\lambda_j(i)$ *of* $\mathbb{E}_i$ *the relations*

$$\pi_*^i \lambda_j(i) = p^j (p^{i-j} + p^{i-j-1} + \cdots + p + 1)\lambda_j(i+1)$$

and

$$p^{f(g-f)}(\pi_{1,g})_*(\ell_g \ell_{g-1} \cdots \ell_{f+1}) = (\pi_{1,g})_*(\ell_1 \ell_2 \cdots \ell_{g-f}).$$

Proof. The relation $(\pi_1)_*([Z_1]) = p\,[Z_2]$ translates into the case $j=1$ and $i=1$. Using the push–pull formula and the relations $\pi_i^*(\lambda_j(i+1) = \ell_{i+1}\lambda_{j-1}(i) + \lambda_j(i)$, the formulas for the pushdowns of the $\lambda_j(i)$ follow by induction on j and i. □

We now calculate the class of all p-rank strata V_f.

Theorem 12.4. *The class of the locus V_f of semi-abelian varieties of p-rank $\leq f$ in the Chow ring $\mathrm{CH}_{\mathbb{Q}}(\tilde{\mathcal{A}}_g)$ equals*

$$[V_f] = (p-1)(p^2-1)\cdots(p^{g-f}-1)\,\lambda_{g-f},$$

where λ_i denotes the ith Chern class of the Hodge bundle.

Proof. The class of the final stratum $\overline{\mathcal{U}}_{u_f}$ on $\overline{\mathcal{U}}_{\emptyset}$ is given by the formula

$$(p-1)^{g-f}\ell_g\ell_{g-1}\cdots\ell_{f+1},$$

since it is the simultaneous degeneracy class of the maps ϕ_j for $j = f+1,\ldots,g$. By pushing down this class under $\pi_{\emptyset} = \pi_{1,g}$ we find using Lemma 12.3 and the notation $\lambda_j(i) = c_j(\mathbb{E}_i)$ that

$$\begin{aligned}(\pi_{1,g})_*(\ell_g\ell_{g-1}\cdots\ell_{f+1}) &= p^{-f(g-f)}(\pi_{1,g})_*(\ell_1\ell_2\cdots\ell_{g-f})\\ &= p^{-f(g-f)}\,(\pi_{1,g})_*(\pi_{1,g-f}^*(\lambda_{g-f}(g-f)))\\ &= p^{-f(g-f)}\,\deg(\pi_{1,g-f})\,(\pi_{g-f,g})_*(\lambda_{g-f}(g-f)).\end{aligned}$$

Applying Lemma 12.3 repeatedly we obtain

$$\begin{aligned}(\pi_{g-f,g})_*(\lambda_{g-f}(g-f)) &= p^{f(g-f)}(1+p)(1+p+p^2)\cdots(1+\cdots+p^{f-1})\lambda_g\\ &= p^{f(g-f)}\,\gamma_g(u_f)\,\lambda_g,\end{aligned}$$

with $\gamma_g(u_f)$ the number of final filtrations refining the canonical filtration of u_f. Hence we get $(\pi_{\emptyset})_*(\overline{\mathcal{U}}_{u_f}) = (p-1)^{g-f}\deg(\pi_{1,g-f})\gamma_g(u_f)\lambda_g$. On the other hand, we have that $(\pi_{\emptyset})_*(\overline{\mathcal{U}}_{u_f}) = \gamma_g(u_f)\,[V_f]$. All together these formulas prove the result. □

12.3 The *a*-number Strata

Another case in which we can find attractive explicit formulas is that of the E-O strata $\mathcal{V}_w$ with w the element of W_g associated to $Y = \{1,2,\ldots,a\}$. We denote these by T_a. Here we can work directly on $\mathcal{A}_g$. The locus T_a on $\mathcal{A}_g$ may be defined as the locus $\{x \in \mathcal{A}_g\colon \mathrm{rank}(V)|_{\mathbb{E}_g} \leq g-a\}$. We have $T_{a+1} \subset T_a$ and $\dim(T_g) = 0$. We apply now formulas of Pragacz and Ratajski [PR97] for the degeneracy locus for the rank of a self-adjoint bundle map of symplectic bundles globalizing the results in isotropic Schubert calculus from [Pr91]. Before we apply their result to our case we have to introduce some notation.

Define for a vector bundle A with Chern classes a_i the expression

$$Q_{ij}(A) := a_ia_j + 2\sum_{k=1}^{j}(-1)^k a_{i+k}a_{j-k} \quad \text{for} \quad i > j.$$

A subset $\beta = \{g \geq \beta_1 > \cdots > \beta_r \geq 0\}$ of $\{1, 2, \ldots, g\}$ (with r even, note that β_r may be zero) is called admissible, and for such subsets we set

$$Q_\beta = \mathrm{Pfaffian}(x_{ij}),$$

where the matrix (x_{ij}) is antisymmetric with entries $x_{ij} = Q_{\beta_i,\beta_j}$. Applying the formula of Pragacz–Ratajski to our situation gives the following result:

Theorem 12.5. *The cycle class $[T_a]$ of the reduced locus T_a of abelian varieties with a-number $\geq a$ is given by*

$$[T_a] = \sum_\beta Q_\beta(\mathbb{E}^{(p)}) \cdot Q_{\rho(a)-\beta}(\mathbb{E}^*),$$

where the sum is over the admissible subsets β contained in the subset $\rho(a) = \{a, a-1, a-2, \ldots, 1\}$.

Example 12.6.

$$\begin{aligned}
[T_1] &= (p-1)\lambda_1 \\
[T_2] &= (p-1)(p^2+1)\lambda_1\lambda_2 - (p^3-1)2\lambda_3 \\
&\cdots \\
[T_g] &= (p-1)(p^2+1)\cdots(p^g+(-1)^g)\lambda_1\lambda_2\cdots\lambda_g.
\end{aligned}$$

As a corollary we obtain a result of one of us [Ek87] on the number of principally polarized abelian varieties with $a = g$.

Corollary 12.7. *We have*

$$\sum_X \frac{1}{\#\mathrm{Aut}(X)} = (-1)^{g(g+1)/2}2^{-g}\left[\prod_{j=1}^{g}(p^j+(-1)^j)\right]\cdot\zeta(-1)\zeta(-3)\cdots\zeta(1-2g),$$

where the sum is over the isomorphism classes (over $\bar{\mathbb{F}}_p$) of principally polarized abelian varieties of dimension g with $a = g$, and $\zeta(s)$ is the Riemann zeta function.

Proof. Combine the formula for T_g with the Hirzebruch–Mumford proportionality theorem (see [Ge99]), which says that

$$\deg(\lambda_1\lambda_2\cdots\lambda_g) = (-1)^{\frac{g(g+1)}{2}}\prod_{j=1}^{g}\frac{\zeta(1-2j)}{2},$$

when interpreted for the stack $\mathcal{A}_g$. □

The formulas for the cycles classes of the p-rank strata and the a-number strata can be seen as generalizations of the classical formula of Deuring (known as Deuring's mass formula), which states that

$$\sum_E \frac{1}{\#\mathrm{Aut}(E)} = \frac{p-1}{24},$$

where the sum is over the isomorphism classes over $\bar{\mathbb{F}}_p$ of supersingular elliptic curves. It is obtained from the formula for V_{g-1} or T_1 for $g=1$, i.e., $[V_1] = (p-1)\lambda_1$, by observing that the degree of λ_1 is $1/12$ the degree of a generic point of the stack $\tilde{\mathcal{A}}_1$.

One can obtain formulas for all the E-O strata by applying the formulas of Pragacz–Ratajski or those of Kresch–Tamvakis [KT02, Corollary 4]. If Y is a Young diagram given by a subset $\{\xi_1, \ldots, \xi_r\}$ we call $|\xi| = \sum_{i=1}^r \xi_i$ the weight and r the length of ξ. Moreover, we need the *excess* $e(\xi) = |\xi| - r(r+1)/2$ and the *intertwining number* $e(\xi, \eta)$ of two strict partitions with $\xi \cap \eta = \emptyset$ by

$$e(\xi,\eta) = \sum_{i\geq 1} i\,\#\{j : \xi_i > \eta_j > \xi_{i+1}\}$$

(where we use $\xi_k = 0$ if $k > r$). We put $\rho_g = \{g, g-1, \ldots, 1\}$ and $\xi' = \rho_g - \xi$ and have then $e(\xi, \xi') = e(\xi)$. The formula obtained by applying the result of Kresch and Tamvakis interpolates between the formulas for the two special cases, the p-rank strata and a-number strata, as follows:

Theorem 12.8. *For a Young diagram given by a partition ξ we have*

$$[\overline{\mathcal{V}}_Y] = (-1)^{e(\xi)+|\xi'|} \sum_\alpha Q_\alpha(\mathbb{E}^{(p)}) \sum_\beta (-1)^{e(\alpha,\beta)} Q_{(\alpha\cup\beta)'}(\mathbb{E}^*) \det(c_{\beta_i - \xi'_j}(\mathbb{E}^*_{g-\xi'_j})),$$

where the sum is over all admissible α and all admissible β that contain ξ' with length $\ell(\beta) = \ell(\xi')$ and $\alpha \cap \beta = \emptyset$.

12.4 Positivity of tautological classes

The Hodge bundle possesses certain positivity properties. It is well known that the determinant of the Hodge bundle (represented by the class λ_1) is ample on $\mathcal{A}_g$. Over $\mathbb{C}$ this is a classical result, while in positive characteristic this was proven by Moret–Bailly [MB85]. On the other hand, the Hodge bundle itself is not positive in positive characteristic. For example, for $g = 2$ the restriction of $\mathbb{E}$ to a line from the p-rank 0 locus is $O(-1) \oplus O(p)$, [MB81]. But our Theorem 12.4 implies the following nonnegativity result.

Theorem 12.9. *The Chern classes $\lambda_i \in \mathrm{CH}_{\mathbb{Q}}(\mathcal{A}_g \otimes \mathbb{F}_p)$ $(i = 1, \ldots, g)$ of the Hodge bundle $\mathbb{E}$ are represented by effective classes with $\mathbb{Q}$–coefficients.*

13 Tautological rings

We shall now interpret the results of previous sections in terms of tautological rings. Recall that the tautological ring of $\tilde{\mathcal{A}}_g$ is the subring of $\mathrm{CH}_{\mathbb{Q}}(\tilde{\mathcal{A}}_g)$ generated by the Chern classes λ_i. To obtain maximal precision we shall use the subring and *not* the $\mathbb{Q}$-subalgebra (but note that this is still a subring of $\mathrm{CH}_{\mathbb{Q}}(\tilde{\mathcal{A}}_g)$ not of the integral Chow ring $\mathrm{CH}^*(\tilde{\mathcal{A}}_g)$). As a graded ring it is isomorphic to the Chow ring $\mathrm{CH}^*(\mathrm{Sp}_{2g}/P_H)$ and as an abstract graded ring it is generated by the λ_i with relations coming from the identity $1 = (1+\lambda_1+\cdots+\lambda_g)(1-\lambda_1+\cdots+(-1)^g\lambda_g)$. This implies that it has a $\mathbb{Z}$-basis consisting of the square-free monomials in the λ_i. (Note, however, that the degree maps from the degree $g(g+1)/2$ part are not the same; on Sp_{2g}/P_H the degree of $\lambda_1\cdots\lambda_g$ is ± 1, whereas for $\tilde{\mathcal{A}}_g$ it is given by the Hirzebruch–Mumford proportionality theorem as in the previous section.) Since $\tilde{\mathcal{F}}_g \to \tilde{\mathcal{A}}_g$ is an SL_g/B-bundle, we can express $\mathrm{CH}_{\mathbb{Q}}(\tilde{\mathcal{F}}_g)$ as an algebra over $\mathrm{CH}_{\mathbb{Q}}(\tilde{\mathcal{A}}_g)$; it is the algebra generated by the ℓ_i, and the relations are that the elementary symmetric functions in them are equal to the λ_i. This makes it natural to define the *tautological ring* of $\tilde{\mathcal{F}}_g$ to be the subring of $\mathrm{CH}_{\mathbb{Q}}(\tilde{\mathcal{F}}_g)$ generated by the ℓ_i. It will then be the algebra over the tautological ring of $\tilde{\mathcal{A}}_g$ generated by the ℓ_i and with the relations that say that the elementary symmetric functions in the ℓ_i are equal to the λ_i. Again this means that the tautological ring for $\tilde{\mathcal{F}}_g$ is isomorphic to the integral Chow ring of Sp_{2g}/B_g, the space of full symplectic flags in a $2g$-dimensional symplectic vector space. Note furthermore that the Gysin maps for $\mathrm{Sp}_{2g}/B_g \to \mathrm{Sp}_{2g}/P_H$ and $\tilde{\mathcal{F}}_g \to \tilde{\mathcal{A}}$ are both given by Formula 3.1.

Theorem 12.1 shows in particular that the classes of the $\overline{\mathcal{U}}_w$ and $\overline{\mathcal{V}}_\nu$ lie in the respective tautological rings. However, we want both to compare the formulas for these classes with the classical formulas for the Schubert varieties and to take into account the variation of the coefficients of the classes when expressed in a fixed basis for the tautological ring. Hence since the rest of this section is purely algebraic, we shall allow ourselves the luxury of letting p temporarily be also a polynomial variable. We then introduce the ring $\mathbb{Z}\{p\}$, which is the localization of the polynomial ring $\mathbb{Z}[p]$ at the multiplicative subset of polynomials with constant coefficient equal to 1. Hence evaluation at 0 extends to a ring homomorphism $\mathbb{Z}\{p\} \to \mathbb{Z}$, which we shall call the *classical specialization*. An element of $\mathbb{Z}\{p\}$ is thus invertible precisely when its classical specialization is invertible. By a modulo n consideration we see that an integer polynomial with 1 as constant coefficient can have no integer zero $n \neq \pm 1$. This means in particular that evaluation at a prime p induces a ring homomorphism $\mathbb{Z}\{p\} \to \mathbb{Q}$ taking the variable p to the integer p, which we shall call the *characteristic p specialization* (since p will be an integer only when this phrase is used, there should be no confusion because of our dual use of p). We now extend scalars of the two tautological rings from $\mathbb{Z}$ to $\mathbb{Z}\{p\}$ and we shall call them the *p-tautological rings*. We shall also need to express the condition that an element is in the subring obtained by extension to a subring of $\mathbb{Z}\{p\}$ and we shall then say that the element *has coefficients* in the subring. We may consider the Fulton polynomial P_w of (4) as a polynomial with coefficients in $\mathbb{Z}\{p\}$, and when we evaluate them on elements of the tautological ring as in Theorem 12.1 we get elements $[\overline{\mathcal{U}}_w]$ of the p-tautological ring of $\tilde{\mathcal{F}}_g$. If ν is a final element we can push down the formula for $[\overline{\mathcal{U}}_\nu]$ using Formula 3.1 and then we get an element in the p-tautological ring of $\tilde{\mathcal{A}}_g$. We then note that $\gamma(w)$ is a polynomial in p with constant coefficient equal to 1, and hence we can define $[\overline{\mathcal{V}}_\nu] := \gamma(w)^{-1}\pi_*[\overline{\mathcal{U}}_\nu]$, where $\pi\colon \tilde{\mathcal{F}}_g \to \tilde{\mathcal{A}}_g$ is the projection map. By construction these elements map to the classes of $\overline{\mathcal{U}}_w$, respectively $\overline{\mathcal{V}}_\nu$, under specialization to characteristic p. We shall need to compare them with the classes of the Schubert varieties. To be specific we shall define the Schubert varieties of Sp_{2g}/B_g by the condition $\dim E_i \cap D_j \geq r_w(i,j)$, where $D_\bullet$ is a fixed reference flag (and then the Schubert varieties of Sp_{2g}/P_H are the images of the Schubert varieties of Sp_{2g}/B_g for final elements of W_g).

Theorem 13.1. *(i) The classes $[\overline{\mathcal{U}}_w]$ and $[\overline{\mathcal{V}}_\nu]$ in the p-tautological ring of $\tilde{\mathcal{F}}_g$ map to the classes of the corresponding Schubert varieties under classical specialization.*

(ii) The classes $[\overline{\mathcal{U}}_w]$ and $[\overline{\mathcal{V}}_\nu]$ form a $\mathbb{Z}\{p\}$-basis for the respective p-tautological rings.

(iii) The coefficients of $[\overline{\mathcal{U}}_w]$ and $[\overline{\mathcal{V}}_\nu]$ when expressed in terms of the polynomials in the λ_i are in $\mathbb{Z}[p]$.

(iv) For $w \in W_g$ we have that $\ell(w) = \ell(\tau_p(w))$ (see Section 9 for the definition of τ_p) precisely when the specialization to characteristic p of $\pi_*[\overline{\mathcal{U}}_w]$ is nonzero. In particular, there is a unique map $\tau\colon W_g \to (W_g/S_g)\bigsqcup\{0\}$ such that $\tau(w) = 0$ precisely when $\pi_*[\overline{\mathcal{U}}_w] = 0$, which implies that $\ell(w) \neq \ell(\tau_p(w))$ and is implied by $\ell(w) \neq \ell(\tau_p(w))$ for all sufficiently large p. Furthermore, if $\tau(w) \neq 0$ then $\ell(w) = \ell(\tau(w))$ and $\pi_*[\overline{\mathcal{U}}_w]$ is a nonzero multiple of $[\overline{\mathcal{V}}_{\tau(w)}]$.

Proof. The first part is clear, since putting $p = 0$ in our formulas gives the Fulton formulas for $x_i = -\ell_i$ and $y_i = 0$, which are the Fulton formulas for the Schubert varieties in Sp_{2g}/B_g. One then obtains the formulas for the Schubert varieties of Sp_{2g}/P_H by pushing down by Gysin formulas. The remaining compatibility needed is that the classical specialization of $\gamma(w)$ is the degree of the map from the Schubert variety of Sp_{2g}/B_g for a final element to the corresponding Schubert variety of Sp_{2g}/P_H. However, the classical specialization of $\gamma(w)$ is 1, and the map between Schubert varieties is an isomorphism between Bruhat cells.

As for the second part, we need to prove that the determinant of the matrix expressing the classes of the strata in terms of a basis of the tautological ring (say given by monomials in the ℓ_i respectively the λ_i) is invertible. Given that an element of $\mathbb{Z}\{p\}$ is invertible precisely when its classical specialization is, we are reduced to proving the corresponding statement in the classical case. However, there it follows from the cell decomposition given by the Bruhat cells, which give that the classes of the Schubert cells form a basis for the integral Chow groups.

To prove (iii) it is enough to verify the conditions of Proposition 13.2. Hence consider $w \in W_g$ (respectively a final element ν) and consider an element m in the tautological ring of degree complementary to that of $[\overline{\mathcal{U}}_w]$ (respectively $[\overline{\mathcal{V}}_\nu]$). By the projection formula the degree of $m[\overline{\mathcal{U}}_w]$ (respectively $m[\overline{\mathcal{V}}_\nu]$) is the degree of the restriction of m to $\overline{\mathcal{U}}_w$ (respectively $\overline{\mathcal{V}}_\nu$), and it is enough to show that the denominators of these degrees are divisible only by a finite number of primes (independently of the characteristic p). However, if the characteristic is different from 3 we may pull back to the moduli space with a level 3 structure, and there the degree is an integer, since the corresponding strata are schemes. Hence the denominator divides the degree of the level 3 structure covering $\tilde{\mathcal{A}}_{g,3} \to \tilde{\mathcal{A}}_g$, which is independent of p.

Finally for (iv), it is clear that in the Chow ring of $\tilde{\mathcal{A}}_g$ the class $\pi_*[\overline{\mathcal{U}}_w]$ is nonzero precisely when $\pi\colon [\overline{\mathcal{U}}_w] \to \overline{\mathcal{V}}_{\tau_p(w)}$ is generically finite, since all fibers have the same dimension by Proposition 9.6. This latter fact also gives that it is generically finite precisely when $\overline{\mathcal{U}}_w$ and $\overline{\mathcal{V}}_{\tau_p(w)}$ have the same dimension, which is equivalent to $\ell(w) = \ell(\tau_p(w))$. When this is the case, we get that $\pi_*[\overline{\mathcal{U}}_w]$ is a nonzero multiple of $[\overline{\mathcal{V}}_{\tau_p(w)}]$, again since the degree over each component of $\overline{\mathcal{V}}_{\tau_p(w)}$ is the same by Proposition 9.6. Consider now instead $\pi_*[\overline{\mathcal{U}}_w]$ in the p-tautological ring and expand $\pi_*[\overline{\mathcal{U}}_w]$ as a linear combination of the $[\overline{\mathcal{V}}_\nu]$. Then what we have just shown is that for every specialization to characteristic p, at most one of the coefficients is nonzero. This implies that in the p-tautological ring at most one of the coefficients is nonzero. If it is zero then $\pi_*[\overline{\mathcal{U}}_w]$ is always zero in all characteristic p specializations, and we get $\ell(w) \neq \ell(\tau_p(w))$ for all p. If it is nonzero, then the coefficient is nonzero for all sufficiently large p. This proves (iv). □

To complete the proof of the theorem we need to prove the following proposition.

Proposition 13.2. *Let a be an element of the p-tautological ring for $\tilde{\mathcal{F}}_g$ or $\tilde{\mathcal{A}}_g$. Assume that there exists an $n \neq 0$ such that for all elements b of the tautological ring of complementary degree and all sufficiently large primes p we have that $\deg(\overline{a}\overline{b}) \in \mathbb{Z}[n^{-1}]$, where $\overline{a}$ and $\overline{b}$ are the specializations to characteristic p of a, respectively b. Then the coefficients of a are in $\mathbb{Z}[p]$.*

Proof. If $r(x)$ is one of the coefficients of a, then the assumptions say that $r(p) \in \mathbb{Z}[n^{-1}]$ for all sufficiently large primes p. Write r as $g(x)/f(x)$ where f and g are integer polynomials with no common factor. Thus there are integer polynomials $s(x)$ and $t(x)$ such that $s(x)f(x) + t(x)g(x) = m$, where m is a nonzero integer. If g is nonconstant there are arbitrarily large primes ℓ such that there is an integer k with $\ell | f(k)$ (by, for instance, the fact that there is a prime that splits completely in a splitting field of f). By Dirichlet's theorem on primes in arithmetic progressions there are arbitrarily large primes p such that $f(p) \neq 0$ and $f(p) \equiv f(k) \equiv 0 \bmod \ell$. By making ℓ so large that $\ell \nmid m$, we get that $\ell \nmid g(q)$ (since $s(q)f(q) + t(q)g(q) = m$) and hence ℓ appears in the denominator of $r(q)$. By making ℓ so large that $\ell \nmid n$, we conclude. □

Example 13.3. If w is a shuffle of a final element ν we have $\tau(w)=\nu$.

We can combine Theorem 13.1 with our results on the punctual flag spaces to give an algebraic criterion for inclusion between E-O strata.

Corollary 13.4. *Let ν' and ν be final elements. Then for sufficiently large p we have that $\nu'\subseteq\nu$ if there are $w,w'\in W_g$ for which $w'\le w$, $\tau(w)=\nu$, and $\tau(w')=\nu'$.*

Proof. Assume that there are $w,w'\in W_g$ for which $w'\le w$, $\tau(w)=\nu$, and $\tau(w')=\nu'$. By Proposition 9.6 we have for $\pi\colon\tilde{\mathcal{F}}_g\to\tilde{\mathcal{A}}_g$, the image relations $\pi(\overline{\mathcal{U}}_w)=\overline{\mathcal{V}}_{\nu_1}$ and $\pi(\overline{\mathcal{U}}_{w'})=\overline{\mathcal{V}}_{\nu_1'}$ for some ν_1 and ν_1', and by the theorem $\nu_1=\tau(w)$ and $\nu_1'=\tau(w')$ for sufficiently large p. Since $w'\le w$ we have that $\overline{\mathcal{U}}_{w'}\subseteq\overline{\mathcal{U}}_w$, which implies that $\pi\overline{\mathcal{U}}_{w'}\subseteq\pi(\overline{\mathcal{U}}_w)$. □

14 Comparison with $\mathcal{S}(g,p)$

We shall now make a comparison with de Jong's moduli stack $\mathcal{S}(g,p)$ of $\Gamma_0(p)$-structures (cf. [Jo93]). Recall that for a family $\mathcal{A}\to S$ of principally polarized g-dimensional abelian varieties a $\Gamma_0(p)$-*structure* consists of the choice of a flag $0\subset H_1\subset\cdots\subset H_g\subset\mathcal{A}[p]$ of flat subgroup schemes with H_i of order p^i and H_g totally isotropic with respect to the Weil pairing. We shall work exclusively in characteristic p and denote by $\overline{\mathcal{S}(g,p)}$ the mod p fiber of $\mathcal{S}(g,p)$. We now let $\mathcal{S}(g,p)^0$ be the closed subscheme of $\overline{\mathcal{S}(g,p)}$ defined by the condition that the group scheme H_g be of height 1. This means that the (relative) Frobenius map $F_{\mathcal{A}/\mathcal{S}(g,p)}$, where $\pi\colon\mathcal{A}\to\mathcal{S}(g,p)$ is the universal abelian variety, is zero on it. For degree reasons we then get that H_g equals the kernel of $F_{\mathcal{A}/\mathcal{S}(g,p)}$. Using the principal polarization we may identify the Lie algebra of π with $R^1\pi_*\mathcal{O}_{\mathcal{A}}$, and hence we get a flag $0\subset\mathrm{Lie}(H_1)\subset\mathrm{Lie}(H_2)\subset\cdots\subset\mathrm{Lie}(H_g)=R^1\pi_*\mathcal{O}_{\mathcal{A}}$. By functoriality this is stable under V. Completing this flag by taking its annihilator in $\mathcal{E}$ gives a flag in $\overline{\mathcal{U}}_{w_\emptyset}$, thus giving a map $\mathcal{S}(g,p)^0\to\overline{\mathcal{U}}_{w_\emptyset}$.

Theorem 14.1. *The canonical map $\mathcal{S}(g,p)^0\to\overline{\mathcal{U}}_{w_\emptyset}$ is an isomorphism. In particular, $\mathcal{S}(g,p)^0$ is the closure of its intersection with the locus of ordinary abelian varieties and is normal and Cohen–Macaulay.*

Proof. Starting with the tautological flag $\{\mathbb{E}_i\}$ on $\overline{\mathcal{U}}_{w_\emptyset}$ we consider the induced flag $\{\mathbb{E}_{g+i}/\mathbb{E}_g\}$ in $R^1\pi_*\mathcal{O}_{\mathcal{A}}$. This is a V-stable flag of the Lie algebra of a height 1 group scheme, so by, for instance, [Mu70, Theorem §14], any V-stable subbundle comes from a subgroup scheme of the kernel of $F_{\mathcal{A}/\mathcal{S}(g,p)}$ and thus the flag $\{\mathbb{E}_{g+i}/\mathbb{E}_g\}$ gives rise to a complete flag of subgroup schemes with H_g equal to the kernel of the Frobenius map and hence a map from $\overline{\mathcal{U}}_{w_\emptyset}$ to $\mathcal{S}(g,p)^0$ that clearly is the inverse of the canonical map.

The rest of the theorem now follows from Corollary 8.4. □

15 Appendix $g = 3, 4$

15.1 Admissible Strata for $g = 3$

In the following matrix one finds the loci lying in $\overline{\mathcal{U}}_{w_\emptyset}$. In the sixth column we give for each w an example of a final ν such that $w \to \nu$.

ℓ	Y	w	ν	word	$w \to \nu$	
0	$\{1,2,3\}$	[123]	$\{0,0,0\}$	Id	[123]	Superspecial
1		[132]	$\{0,0,0\}$	s_2	[123]	Fiber over s.s.
1		[213]	$\{0,0,0\}$	s_1	[123]	Fiber over s.s.
1	$\{2,3\}$	[124]	$\{0,0,1\}$	s_3	[124]	Moret-Bailly
2		[142]	$\{0,0,1\}$	s_3s_2	[135]	
2		[214]	$\{0,0,1\}$	s_3s_1	[135]	
2		[231]	$\{0,0,0\}$	s_1s_2	[123]	Fiber over s.s.
2		[312]	$\{0,0,0\}$	s_2s_1	[123]	Fiber over s.s.
2	$\{1,3\}$	[135]	$\{0,1,1\}$	s_2s_3	[135]	$f = 0$, $a = 2$
3		[153]	$\{0,1,1\}$	$s_2s_3s_2$	[236]	Shuffle of $\{1,2\}$
3		[241]	$\{0,0,1\}$	$s_3s_1s_2$	[124]	
3		[315]	$\{0,1,1\}$	$s_2s_3s_1$	[124]	
3		[321]	$\{0,0,0\}$	$s_1s_2s_1$	[123]	Fiber over s.s.
3		[412]	$\{0,0,1\}$	$s_3s_2s_1$	[236]	Shuffle of $\{1,2\}$
3	$\{3\}$	[145]	$\{0,1,2\}$	$s_3s_2s_3$	[145]	$f = 0$
3	$\{1,2\}$	[236]	$\{1,1,1\}$	$s_1s_2s_3$	[236]	$a = 2$
4		[154]	$\{0,1,2\}$	$s_3s_2s_3s_2$	[246]	Shuffle of $\{2\}$
4		[326]	$\{1,1,1\}$	$s_1s_2s_3s_1$	[236]	
4		[351]	$\{0,1,1\}$	$s_2s_3s_1s_2$	[236]	
4		[415]	$\{0,1,2\}$	$s_3s_2s_3s_1$	[246]	Shuffle of $\{2\}$
4		[421]	$\{0,0,1\}$	$s_3s_1s_2s_1$	[236]	
4	$\{2\}$	[246]	$\{1,1,2\}$	$s_3s_1s_2s_3$	[246]	$f = 1$
5		[426]	$\{1,1,2\}$	$s_3s_1s_2s_3s_1$	[356]	Shuffle of $\{1\}$
5		[451]	$\{0,1,2\}$	$s_3s_2s_3s_1s_2$	[356]	Shuffle of $\{1\}$
5	$\{1\}$	[356]	$\{1,2,2\}$	$s_2s_3s_1s_2s_3$	[356]	$f = 2$
6	$\{\}$	[456]	$\{1,2,3\}$	$s_3s_2s_3s_1s_2s_3$	[456]	$f = 3$

15.2 E-O Cycle Classes for $g = 3$

We give the cycle classes of the (reduced) E-O strata for $g = 3$.

Y	class
$\emptyset$	1
$\{1\}$	$(p-1)\lambda_1$
$\{2\}$	$(p-1)(p^2-1)\lambda_2$
$\{1,2\}$	$(p-1)(p^2+1)\lambda_1\lambda_2 - 2(p^3-1)\lambda_3$
$\{3\}$	$(p-1)(p^2-1)(p^3-1)\lambda_3$
$\{1,3\}$	$(p-1)^2(p^3+1)\lambda_1\lambda_3$
$\{2,3\}$	$(p-1)^2(p^6-1)\lambda_2\lambda_3$
$\{1,2,3\}$	$(p-1)(p^2+1)(p^3-1)\lambda_1\lambda_2\lambda_3$

15.3 E-O Cycle Classes for $g = 4$

We give the cycle classes of the (reduced) E-O strata for $g = 4$.

Y	class
$\emptyset$	1
$\{1\}$	$(p-1)\lambda_1$
$\{2\}$	$(p-1)(p^2-1)\lambda_2$
$\{1,2\}$	$(p-1)(p^2+1)\lambda_1\lambda_2 - 2(p^3-1)\lambda_3$
$\{3\}$	$(p-1)(p^2-1)(p^3-1)\lambda_3$
$\{1,3\}$	$(p-1)^2(p+1)((p^2-p+1)\lambda_1\lambda_3 - 2(p^2+1)\lambda_4)$
$\{2,3\}$	$(p-1)^2((p^6-1)\lambda_2\lambda_3 - (2p^6+p^5-p-2)\lambda_1\lambda_4$
$\{1,2,3\}$	$(p-1)(p^2{+}1)((p^3+1)((p^3-1)\lambda_1\lambda_2\lambda_3 - 2(3p^3{+}p^2-p+3)\lambda_2\lambda_4)$
$\{4\}$	$(p-1)(p^2-1)(p^3-1)(p^4-1)\lambda_4$
$\{1,4\}$	$(p-1)^3(p+1)(p^4+1)\lambda_1\lambda_4$
$\{2,4\}$	$(p-1)^3(p^8-1)\lambda_2\lambda_4$
$\{1,2,4\}$	$(p-1)^2(p^4-1)((p^2+1)\lambda_1\lambda_2 - 2(p^2+p+1)\lambda_3)\lambda_4$
$\{3,4\}$	$(p-1)^2(p^2+1)(p^3-1)(p^2-p+1)((p+1)^2\lambda_3 - p\lambda_1\lambda_2)\lambda_4$
$\{1,3,4\}$	$(p-1)^2(p^4-1)(p^6-1)\lambda_1\lambda_3\lambda_4$
$\{2,3,4\}$	$(p-1)(p^6-1)(p^8-1)\lambda_2\lambda_3\lambda_4$
$\{1,2,3,4\}$	$(p-1)(p^2+1)(p^3-1)(p^4+1)\lambda_1\lambda_2\lambda_3\lambda_4$

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