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On a stratification of the moduli of K3 surfaces

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Abstract. In this paper we give a characterization of the height of K3 surfaces in characteristic $p > 0$. This enables us to calculate the cycle classes in families of K3 surfaces of the loci where the height is at least h. The formulas for such loci can be seen as generalizations of the famous formula of Deuring for the number of supersingular elliptic curves in characteristic p. In order to describe the tangent spaces to these loci we study the first cohomology of higher closed forms.

0. Introduction

Elliptic curves in characteristic p come in two sorts: ordinary and supersingular. The distinction can be expressed in terms of the formal group of an elliptic curve. Multiplication by p on the formal group takes the form

$$[p](t) = at^{p^h} + \text{ higher order terms}, \tag{1}$$

where $a \neq 0$ and t is a local parameter. The number h satisfies $1 \leq h \leq 2$ and is called the *height*. By definition, the elliptic curve is ordinary if $h = 1$ and supersingular if $h = 2$. There is a classical formula of Deuring for the number of supersingular elliptic curves over an algebraically closed field k of characteristic p:

$$\sum_{E \text{ supers.}/\cong} \frac{1}{\#\mathrm{Aut}(E)} = \frac{p-1}{24},$$

where the sum is over supersingular elliptic curves over k up to isomorphism.

If one views K3 surfaces as a generalization of elliptic curves, one can make a similar distinction of K3 surfaces in characteristic p by using

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the formal Brauer group as Artin showed. The formal Brauer group is a 1-dimensional formal group associated to the second étale cohomology with coefficients in the multiplicative group. Multiplication by p in this formal group has the form (1), but now we have $1 \leq h \leq 10$ or $h = \infty$, the latter if multiplication by p is zero. The height can be used to define a stratification of the moduli spaces of K3 surfaces. A generic K3 surface will have $h = 1$; those with $h = \infty$ are most special in this respect and called supersingular.

In this paper we first express the height of a K3 surface in terms of the action of the Frobenius morphism on the second cohomology group with coefficients in the sheaf $W(O_X)$ of Witt vectors of the structure sheaf O_X. The natural co-filtration $W_n(O_X)$ of $W(O_X)$ induces co-filtrations on the cohomology which correspond to approximations of the formal group. Using this characterization we can calculate the cycle classes of the strata in the moduli space where the height $\geq h$. This is done by interpreting the loci as degeneracy loci of maps between bundles. The resulting formulas can be viewed as a generalization of Deuring's formula. Generalizations of Deuring's formula to principally polarized abelian varieties were worked out in joint work of Ekedahl and one of us and can be found in [G]. The supersingular locus comes with a multiplicity.

In order to describe the tangent spaces to our strata we use differential forms rather than crystalline cohomology. We calculate the dimensions of cohomology groups $H^1(Z_i)$ and $H^1(B_i)$, where the sheaves Z_i and B_i are the sheaves of certain closed forms introduced by Illusie. We study the dimensions of the cohomology groups $H^1(Z_i)$ and $H^1(B_i)$ and of their images in $H^1(X, \Omega^1)$. We think that these spaces are quite helpful to understand the geometry of surfaces in characteristic p.

1. Witt vector cohomology

Let X be a non-singular complete variety defined over an algebraically closed field k of characteristic $p > 0$. We denote by $W_n = W_n(O_X)$ the sheaf of Witt rings of length n as defined by J.-P. Serre, cf. [S]. The sheaf $W_n(O_X)$ is a coherent sheaf of rings which comes with three operators:

i) Frobenius $F : W_n(O_X) \to W_n(O_X)$,
ii) Verschiebung $V : W_n(O_X) \to W_{n+1}(O_X)$,
iii) Restriction $R : W_{n+1}(O_X) \to W_n(O_X)$,

defined by the formulas

$$\begin{aligned} F(a_0, a_1, \ldots, a_{n-1}) &= \left(a_0^p, a_1^p, \ldots, a_{n-1}^p\right), \\ V(a_0, a_1, \ldots, a_{n-1}) &= (0, a_0, a_1, \ldots, a_{n-1}), \\ R(a_0, a_1, \ldots, a_n) &= (a_0, a_1, \ldots, a_{n-1}). \end{aligned}$$

They satisfy the relations

$$RVF = FRV = RFV = p.$$

The cohomology groups $H^i(X, W_n(O_X))$ are finitely generated $W_n(k)$-modules. The projective system $\{W_n(O_X), R\}_{n=1,2,\ldots}$ induces a sequence

$$\ldots \longleftarrow H^i(X, W_n(O_X)) \xleftarrow{R} H^i(W_{n+1}(O_X)) \longleftarrow \ldots$$

so that we can define

$$H^i(X, W(O_X)) = \text{proj. lim } H^i(X, W_n(O_X)).$$

This is a $W(k)$-module, but not necessarily a finitely generated $W(k)$-module, cf. Sect. 3. The semi-linear operators F and V act on it and they satisfy the relations $FV = VF = p$.

2. Formal groups

Smooth formal Lie groups of dimension 1 over an algebraically closed field k of characteristic $\neq 0$ are characterized by their *height*, cf. [H], [Ma]. To a smooth formal Lie group Φ of dimension one one can associate its *covariant* Dieudonné module $M = D(\Phi)$, a free $W(k)$-module. It possesses two operators F and V with the following properties: the operator F is σ-linear, the operator V is σ^{-1}-linear and topologically nilpotent and they satisfy $FV = VF = p$. Here σ denotes the Frobenius map on k. Then M is a free $W(k)$-module with the following properties:

a) $\dim(\Phi) = \dim_k(M/VM)$,
b) $\text{height}(\Phi) = \text{rank}_W(M)$.

Note that one has the equalities

$$\text{rank}_W(M) = \dim_k(M/pM) = \dim_k(M/FM) + \dim_k(M/VM).$$

3. The formal Brauer group of Artin-Mazur

For a proper variety X/k one may consider the formal completion of the Picard group. The group of S-valued points of $\widehat{\text{Pic}}(X)$ fits into the exact sequence

$$0 \longrightarrow \widehat{\text{Pic}}(X)(S) \longrightarrow H^1(X \times S, \mathbb{G}_m) \longrightarrow H^1(X, \mathbb{G}_m)$$

for any local artinian scheme S with residue field k. Here cohomology is étale cohomology. This idea of studying infinitesimal properties of cohomology was generalized to the higher cohomology groups $H^r(X, \mathbb{G}_m)$ by Artin and

Mazur, cf. [A-M]. Their work leads to contravariant functors $\Phi^r :$ Art $\to$ Ab with

$$\Phi^r(S) = \ker\{H^r(X \times S, \mathbb{G}_m) \longrightarrow H^r(X, \mathbb{G}_m)\},$$

which under suitable circumstances are representable by formal Lie groups. For a K3 surface X this is the case and we find for $r = 2$ the formal Brauer group $\Phi = \Phi_X = \Phi^2$. Its tangent space is

$$T_\Phi = H^2(X, O_X).$$

For a K3-surface X we have two possibilities:

i) $h(\Phi) = \infty$ and $\Phi = \hat{\mathbb{G}}_a$, the formal additive group. The K3-surface is called *supersingular* (in the sense of Artin).
ii) $h(\Phi) < \infty$. Then Φ is a p-divisible formal group. Moreover, it is known that $1 \le h(\Phi) \le 10$. This follows from the following theorem of Artin, cf. [A]. We shall write simply h for $h(\Phi)$.

(3.1) Theorem. *If the formal Brauer group Φ_X of a K3 surface X is p-divisible then its height satisfies the relation $2h \le B_2 - \rho$, where B_2 is the second Betti number and ρ the rank of the Néron-Severi group.*

For the proof one combines Theorem (0.1) of [A] with Deligne's [D] result on lifting K3 surfaces, see also [I]. We give a proof in Sect. 10. This theorem implies that if $\rho = 22$ then necessarily we have $h = \infty$. If $h \ne \infty$ then it follows that $1 \le h \le 10$. One should view $h = 1$ as the generic case. It was conjectured by Artin that if $h = \infty$ then $\rho = 22$. This is known for elliptic K3 surfaces, see [A]. Note that a surface with $\rho = 22$ is called *supersingular* by Shioda, cf. [Sh].

The following result by Artin and Mazur is crucial:

(3.2) Theorem. *The Dieudonné module of the formal Brauer group Φ_X is given by*

$$D(\Phi_X) \cong H^2(X, W(O_X)).$$

For the proof we refer to [A-M]. The point to notice is that

$$D(\Phi_X) = H^2(X, D\mathbb{G}_m) = H^2(X, W(O_X)).$$

(3.3) Remark. Note that this explains why the Witt vector cohomology is sometimes not finitely generated: if $\Phi_X \cong \hat{\mathbb{G}}_a$ then $H^2(X, W(O_X))$ is not finitely generated over $W(k)$ because $D(\hat{\mathbb{G}}_a) = W(k)[[T]]$.

4. Vanishing of cohomology

We collect a number of results on the vanishing of cohomology groups for K3 surfaces that we need in the sequel.

(4.1) Lemma. *Let X be a K3 surface. We have $H^1(X, W_n(O_X)) = 0$ for all $n > 0$, hence $H^1(X, W(O_X)) = 0$.*

Proof. Since X is a K3 surface we have by definition $H^1(X, O_X) = 0$. The lemma is deduced from this by induction on n. Assume that $H^1(X, W_{n-1}(O_X)) = 0$. Then the exact sequence

$$0 \longrightarrow W_{n-1}(O_X) \xrightarrow{V} W_n(O_X) \xrightarrow{R^{n-1}} O_X \longrightarrow 0$$

induces an exact sequence

$$H^1(X, W_{n-1}(O_X)) \xrightarrow{V} H^1(W_n(O_X)) \xrightarrow{R^{n-1}} H^1(X, O_X).$$

This implies that $H^1(X, W_n(O_X)) = 0$. □

(4.2) Lemma. *For a projective surface X with $H^1(X, O_X) = 0$ the induced map $R\colon H^2(X, W_n(O_X)) \to H^2(X, W_{n-1}(O_X))$ is surjective with kernel $\cong H^2(X, O_X)$.*

Proof. This follows from the exact sequence

$$0 \to O_X \longrightarrow W_n(O_X) \xrightarrow{R} W_{n-1}(O_X) \to 0$$

and the vanishing of $H^1(X, O_X)$ and of $H^3(X, O_X)$. □

(4.3) Lemma. *In $H^2(X, W_n(O_X))$ we have*

$$RV(H^2(X, W_n(O_X))) = V(H^2(X, W_{n-1}(O_X))).$$

Proof. The commutativity of the diagram

$$\begin{array}{ccc} W_n(O_X) & \xrightarrow{V} & W_{n+1}(O_X) \\ \big\downarrow R & & \big\downarrow R \\ W_{n-1}(O_X) & \xrightarrow{V} & W_n(O_X) \end{array}$$

gives in cohomology a commutative diagram

$$\begin{array}{ccc} H^2(X, W_n(O_X)) & \xrightarrow{V} & H^2(X, W_{n+1}(O_X)) \\ \big\downarrow R & & \big\downarrow R \\ H^2(X, W_{n-1}(O_X)) & \xrightarrow{V} & H^2(X, W_n(O_X)). \end{array}$$

The surjectivity of the left hand R, which follows from the preceding lemma, implies the claim. □

(4.4) Lemma. *Assume that for some $n > 0$ the map $F : H^2(X, W_n(O_X)) \longrightarrow H^2(X, W_n(O_X))$ vanishes. Then for all $0 \le i \le n$ the map $F : H^2(X, W_i(O_X)) \longrightarrow H^2(X, W_i(O_X))$ is zero. Moreover, for all $0 \le i \le n$ the module $H^2(X, W_i(O_X))$ is a vector space over k.*

Proof. The first result follows from the commutativity of the diagram

$$\begin{array}{ccc} H^2(X, W_n(O_X)) & \xrightarrow{R^{n-i}} & H^2(X, W_i(O_X)) \\ \downarrow F & & \downarrow F \\ H^2(X, W_n(O_X)) & \xrightarrow{R^{n-i}} & H^2(X, W_i(O_X)) \end{array}$$

and Lemma (4.2). The second claim follows from $p = FVR$ and $k \cong W_i(k)/pW_i(k)$. ◻

(4.5) Lemma. *Assume that X is a K3 surface. The following two sequences are exact:*

$$0 \to H^2(X, W_{n-1}(O_X)) \xrightarrow{V} H^2(X, W_n(O_X)) \xrightarrow{R^{n-1}} H^2(X, O_X) \to 0,$$

$$0 \to H^2(X, W(O_X)) \xrightarrow{V} H^2(X, W(O_X)) \xrightarrow{R'} H^2(X, O_X) \to 0,$$

where R' is the map induced by $W_n(O_X) \xrightarrow{R^{n-1}} W_1(O_X)$ as $n \to \infty$.

Proof. The first exact sequence follows from the exact sequence

$$0 \to W_{n-1}(O_X) \xrightarrow{V} W_n(O_X) \xrightarrow{R^{n-1}} O_X \to 0$$

and Lemma (4.2). Because the projective system $H^2(X, W_n(O_X))$ satisfies the Mittag-Leffler condition we may take the projective limit. ◻

5. Characterization of the height

Let X be a K3 surface and let Φ_X be its formal Brauer group in the sense of Artin-Mazur. The isomorphism class of this formal group is determined by its height h. The following theorem expresses this height in terms of Witt vector cohomology.

(5.1) Theorem. *The height satisfies $h(\Phi_X) \ge i + 1$ if and only if the Frobenius map $F\colon H^2(X, W_i(O_X)) \to H^2(X, W_i(O_X))$ is the zero map.*

(5.2) Corollary. *We have the following characterization of the height:*

$$h(\Phi_X) = \min\{i \ge 1 \colon [F : H^2(W_i(O_X)) \to H^2(W_i(O_X))] \ne 0\}.$$

Proof of the theorem. "$\Leftarrow$" In case $h(\Phi_X) = \infty$ the implication $\Leftarrow$ is immediate. So we may consider the case where the height of Φ_X is finite. Assume that the map $F\colon H^2(X, W_i(O_X)) \to H^2(X, W_i(O_X))$ is the zero map. We set

$$M = D(\Phi) \cong H^2(X, W(O_X)), \qquad \text{the covariant Dieudonné module.}$$

Since $\dim_k(H^2(X, W(O_X))/VH^2(X, W(O_X)) = 1$ by Lemma (4.5), we have by b) in Sect. 2

$$\dim_k(H^2(X, W(O_X))/FH^2(X, W(O_X)) = h - 1.$$

The surjectivity of the projection $H^2(X, W(O_X)) \longrightarrow H^2(X, W_i(O_X))$ implies the surjectivity of

$$H^2(X, W(O_X))/FH^2(X, W(O_X)) \longrightarrow H^2(X, W_i(O_X))/FH^2(X, W_i(O_X)).$$

By assumption we have $H^2(X, W_i(O_X))/FH^2(X, W_i(O_X)) \cong H^2(X, W_i(O_X))$ and by Lemma (4.5) we have

$$\dim_k H^2(X, W_i(O_X)) = i,$$

i.e. we find $h - 1 \geq i$, or equivalently, $h \geq i + 1$.

Conversely, we now prove "$\Rightarrow$". If $h(\Phi_X) = \infty$ then $\Phi_X = \hat{\mathbb{G}}_a$, the formal additive group of dimension 1. So F acts as zero on $D(\hat{\mathbb{G}}_a) = D(\Phi_X) = H^2(X, W(O_X))$. As in Lemma (4.4) we conclude that F acts on $H^2(X, W_i(O_X))$ as the zero map. Therefore we may assume that $h(\Phi_X) = h < \infty$. We thus assume that $h(\Phi_X) \geq i + 1$. We set

$$H = H^2(X, W(O_X))$$

and have

$$V^{h-1}H \subset \ldots \subset V^2H \subset VH \subset H.$$

Under projection this is mapped surjectively to

$$0 \subset V^{h-2}H^2(O_X) \subset \ldots \subset VH^2(W_{h-2}(O_X)) \subset H^2(X, W_{h-1}(O_X)).$$

All the inclusions are strict because of Lemma (4.5).

Claim. We have $V^{h-1}H^2(X, W(O_X)) = FH^2(X, W(O_X))$.

Proof of the claim. Since our modules are free over W we deduce from Manin's results [M] (but see also [H] because we use the covariant theory):

$$D(\Phi_X) \cong W[F,V]/W[F,V](F - V^{h-1}).$$

Note that $F - V^{h-1}$ is written on the right. But we can transfer it to the left using $FV = p = VF$ as follows:

$$\begin{aligned} F\Big(\sum a_{ij} F^i V^j\Big) &= \sum a_{ij}^{\sigma} F F^i V^j = \sum \big(a_{ij}^{\sigma} F^i V^j\big) F = \\ &= \sum \big(a_{ij}^{\sigma} F^i V^j\big) V^{h-1} = V^{h-1}\Big(\sum a_{ij}^{\sigma^h} F^i V^j\Big). \end{aligned}$$

This together with Theorem (3.2) proves the claim.

We now find $FH^2(W_{h-1}(O_X)) = 0$. By Lemma (4.4) we conclude that F acts on $H^2(W_i(O_X))$ for $i \leq h-1$ as zero. □

(5.3) Corollary. *The height of Φ_X is ∞ if and only if the Frobenius endomorphism $F : H^2(X, W_{10}(O_X)) \to H^2(X, W_{10}(O_X))$ is zero.*

Proof. If the height is finite, then we know by Artin and Mazur (see (3.1)) that we have $h \leq 10$. □

(5.4) Corollary. *Set $H = H^2(X, W_{10}(O_X))$ and consider the filtration*

$$\{0\} \subset R^9V^9H \subset R^8V^8H \subset \ldots \subset R^{h-1}V^{h-1}H \subset \ldots \subset H.$$

If h is the height of Φ_X then $F(H) = R^{h-1}V^{h-1}(H)$.

Proof. The $(h-1)$-th step $V^{h-1}H^2(W(O_X))$ in the filtration

$$V^{10}H^2(W(O_X)) \subset V^9H^2(W(O_X)) \subset \ldots \subset H^2(W(O_X))$$

maps surjectively to the corresponding step $R^{h-1}V^{h-1}H$ of the filtration on H. By our claim we have

$$V^{h-1}H^2(W(O_X)) = FH^2(W(O_X)).$$

This implies the assertion. □

(5.5) Corollary. *If $h(\Phi_X) = h < \infty$ and if $\{\omega, V\omega, V^2\omega, \ldots, V^{h-1}\omega\}$ is a W-basis of $H^2(X, W(O_X))$ then F acts as zero on $H^2(X, W_i(O_X))$ if and only if $F(\bar{\omega}) = 0$, with $\bar{\omega}$ the image of ω in $H^2(X, W_i(O_X))$.*

(5.6) Corollary. *If $h(\Phi_X) = h < \infty$, then* $\dim_k \ker[F : H^2(W_i) \to H^2(W_i)] = \min\{i, h-1\}$.

Proof. By Lemma (4.5) and Corollary (5.2), we have $\dim_k \ker[F : H^2(W_i) \to H^2(W_i)] = i$ if $i \leq h-1$. Assume $i \geq h$. Using the notation in Corollary (5.5), we know that $\langle V^{i-h+1}R^{i-h+1}\bar{\omega}, V^{i-h+2}R^{i-h+2}\bar{\omega}, V^{i-h+3}R^{i-h+3}\bar{\omega}, \ldots, V^{i-1}R^{i-1}\bar{\omega}\rangle$ is a basis of $\ker F$. □

The case $h \geq 2$ is characterized by the vanishing of Frobenius on $H^2(O_X)$. We now formulate in an inductive way a similar characterization of the condition $h \geq n+1$. If for a K3 surface X one assumes that F is zero on $H^2(W_i)$ $(i = 1, \dots n-1)$ then one has $FH^2(W_n) \subset V^{n-1}H^2(O_X)$ and F vanishes on $VH^2(W_{n-1})$. Since we have a natural $(\sigma^{-1})^{n-1}$-isomorphism $H^2(O_X) \cong V^{n-1}H^2(O_X)$, one has an induced homomorphism

$$\phi_n \colon H^2(O_X) \cong H^2(W_n)/VH^2(W_{n-1}) \to V^{n-1}H^2(O_X) \cong H^2(O_X). \quad (2)$$

This map is σ^n-linear. The following theorem is clear by the construction of ϕ_n.

(5.7) Theorem. *Suppose F is zero on $H^2(W_i)$ for $i = 1, \dots n-1$. Then F vanishes on $H^2(W_n)$ if and only if $\phi_n : H^2(O_X) \to H^2(O_X)$ vanishes.*

6. Closed differential forms

Let $F : X \to X^{(p)}$ be the relative Frobenius morphism of a K3 surface X. By means of the Cartier operator $C : \Omega^{\bullet}_{X,\mathrm{closed}} \to \Omega^{\bullet}_X$ we can define sheaves $B_i\Omega^1_X$ of rings inductively by $B_0\Omega^1_X = 0$, $B_1\Omega^1_X = dO_X$ and $C^{-1}(B_i\Omega^1_X) = B_{i+1}\Omega^1_X$. Similarly, we define sheaves $Z_i\Omega^1_X$ inductively by $Z_0\Omega^1_X = \Omega^1_X$, $Z_1\Omega^1_X = \Omega^1_{X,\mathrm{closed}}$, the sheaf of d-closed forms and by setting

$$Z_{i+1}\Omega^1_X := C^{-1}\big(Z_i\Omega^1_X\big).$$

Usually we simply write B_i and Z_i. The sheaves B_i and Z_i can be viewed as locally free subsheaves of $(F^i)_*\Omega^1_X$ on $X^{(p^i)}$. They were introduced by Illusie in [Il] and can be used to provide de Rham-cohomology with a rich structure. The inverse Cartier operator gives rise to an isomorphism

$$C^{-i} : \Omega^1_{X^{(p^i)}} \xrightarrow{\simeq} Z_i/B_i$$

or a σ^{-i}-linear isomorphism $\Omega^1_X \cong Z_i/B_i$. Note that we have the inclusions

$$0 = B_0 \subset B_1 \subset \dots \subset B_i \subset \dots \subset Z_i \subset \dots \subset Z_1 \subset Z_0 = \Omega^1_X.$$

We also have an exact sequence

$$0 \to Z_{i+1} \longrightarrow Z_i \xrightarrow{dC^i} d\Omega^1_X \to 0. \quad (3)$$

(6.1) Lemma. *If X is a K3 surface X we have* i) $H^0(B_i) = 0$ *for all $i \geq 0$;* ii) *the natural inclusion $B_i \to B_{i+1}$ induces an injective homomorphism $H^1(B_i) \to H^1(B_{i+1})$.*

Proof. i) The natural injection $B_i \to \Omega^1_X$ induces an injection $H^0(B_i) \to H^0(\Omega^1_X)$ and we know $H^0(\Omega^1_X) = 0$. ii) This follows from i) and the exact sequence

$$0 \to B_i \longrightarrow B_{i+1} \xrightarrow{C^i} B_1 \to 0$$

with C the Cartier operator. □

There is a close relationship between the Witt vector cohomology and the cohomology of B_i as follows. Serre introduced in [S] a map $D_i : W_i(O_X) \longrightarrow \Omega^1_X$ of sheaves in the following way:

$$D_i(a_0, a_1, \ldots, a_{i-1}) = a_0^{p^{i-1}-1} da_0 + \ldots + a_{i-2}^{p-1} da_{i-2} + da_{i-1}.$$

It satisfies $D_{i+1}V = D_i$, and Serre showed that this induces an injective map of sheaves of additive groups

$$D_i : W_i(O_X)/FW_i(O_X) \longrightarrow \Omega^1_X$$

inducing an isomorphism

$$D_i : W_i(O_X)/FW_i(O_X) \xrightarrow{\sim} B_i\Omega^1_X. \tag{4}$$

The exact sequence $0 \to W_i \xrightarrow{F} W_i \longrightarrow W_i/FW_i \to 0$ gives rise to the exact sequence

$$0 \to H^1(W_i/FW_i) \to H^2(W_i) \xrightarrow{F} H^2(W_i) \to H^2(W_i/FW_i) \to 0 \tag{5}$$

and we thus have an isomorphism $H^1(W_i/FW_i) \cong \ker[F : H^2(W_i) \to H^2(W_i)]$. Combining the result on the dimension of the kernel of F on $H^2(W_i)$ from Sect. 5 with (4) we get an interpretation of the height h in terms of the groups $H^1(B_i)$.

(6.2) Theorem. *We have*

$$\dim H^1(B_i) = \begin{cases} \min\{i, h-1\} & \textit{if} \quad h \neq \infty, \\ i & \textit{if} \quad h = \infty. \end{cases}$$

The Verschiebung induces an exact sequence

$$0 \to W_i/FW_i \xrightarrow{V} W_{i+1}/FW_{i+1} \to O_X/FO_X \to 0$$

and this gives rise to

$$0 \to H^1(W_i/FW_i) \xrightarrow{V} H^1(W_{i+1}/FW_{i+1}) \longrightarrow H^1(O_X/FO_X) \to \ldots$$

i.e., Verschiebung induces for all i an *injective map*. Moreover, it is surjective if and only if $h \neq \infty$ and $i \geq h-1$.

We have a commutative diagram (with β_i the natural map induced by $B_i \subset \Omega^1_X$)

$$\begin{array}{ccc} H^1(W_i/FW_i) & \xrightarrow{D_i} & H^1\big(\Omega^1_X\big) \\ \big\downarrow \cong & & \big\downarrow = \\ H^1\big(B_i\Omega^1_X\big) & \xrightarrow{\beta_i} & H^1\big(\Omega^1_X\big) \end{array}$$

We study the kernel of D_i, equivalently the kernel of the natural map $\beta_i : H^1(B_i\Omega^1_X) \to H^1(\Omega^1_X)$ in Sects. 9–11.

(6.3) Lemma. *The Euler-Poincaré characteristics of B_i and Z_i are given by $\chi(B_i) = 0$ and $\chi(Z_i) = -20$.*

Proof. Since the kernel and the cokernel of F on $H^2(W_i)$ have the same dimension by (5) the result for B_i follows from (4) and (5). The identity $\chi(B_i) + \chi(\Omega^1_X) = \chi(Z_i)$ resulting from the isomorphism $Z_i/B_i \cong \Omega^1_X$ implies the result. □

7. De Rham cohomology

The de Rham cohomology of a K3 surface is the hypercohomology of the complex $(\Omega^\bullet_X, d)$. The dimensions $h^{p,q}$ of the graded pieces are given by the Hodge diamond.

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

On H^2_{dR} we have a perfect pairing $\langle\, ,\, \rangle$ given by Poincaré duality; cf. [D].

The Hodge spectral sequence with $E_1^{ij} = H^j(X, \Omega^i_X)$ converges to $H^*_{dR}(X)$. The second spectral sequence of hypercohomology has as E_2-term $E_2^{ij} = H^i(\mathcal{H}^j(\Omega^\bullet_X))$ abutting to $H^{i+j}_{\mathrm{dR}}(X/k)$. But the Cartier operator yields an isomorphism of sheaves

$$C^{-1} : \Omega^i_{X^{(p)}} \xrightarrow{\sim} \mathcal{H}^i\big(F_*(\Omega^\bullet_{X/k})\big),$$

so that we can rewrite this as

$$E_2^{ij} = H^i(X', \mathcal{H}^j(\Omega^\bullet)) \cong H^i\big(X', \Omega^j_{X'}\big) \Rightarrow H^*_{dR}(X),$$

where $X' = X^{(p)}$ is the base change of X under Frobenius. We thus get two filtrations on the de Rham cohomology: the Hodge filtration

$$(0) \subset F^2 \subset F^1 \subset H^2_{dR},$$

and the so-called *conjugate filtration*

$$(0) \subset G_1 \subset G_2 \subset H^2_{dR}.$$

We have $\mathrm{rank}(\mathrm{F}^1) = \mathrm{rank}(\mathrm{G}_2) = 21$, $\mathrm{rank}(\mathrm{F}^2) = \mathrm{rank}(\mathrm{G}_1) = 1$ and

$$(F^1)^{\perp} = F^2 \qquad \text{and} \qquad G_1^{\perp} = G_2.$$

We have also

$$F^1/F^2 \cong H^1\big(X, \Omega^1_X\big), \quad G_2/G_1 \cong H^1\big(X, \Omega^1_X\big),$$

cf. [D]. Moreover, from the description with the second spectral sequence it follows that G_1 is the image under Frobenius of H^2_{dR} and also of $H^2(X, O_X)$. The conjugate filtration is an analogue of the complex conjugate of the Hodge filtration in characteristic zero.

The relative position of these two filtrations is an interesting invariant of a K3 surface. We have the three cases

a) $F^1 \cap G_1 = \{0\}$;
b) $G_1 \subset F^1$; $G_1 \neq F^2$;
c) $G_1 = F^2$.

The first case happens if and only if $F : H^2(X, O_X) \to H^2_{\mathrm{dR}}(X) \to H^2(X, O_X)$ is not zero, i.e. if $h = 1$. Such X are called *ordinary*. The second case happens if $h \geq 2$, while the last case is by definition the *superspecial* case. In this case the two filtrations coincide. It is known that two superspecial K3 surfaces are isomorphic (as unpolarized varieties)(cf. [O]).

We have the following result of Ogus (cf. [O]) which provides us with an interpretation of $H^1(Z_1)$.

(7.1) Proposition. *We have an isomorphism* $F^1 \cap G_2 \cong H^1(X, Z_1)$.

Proof. The map $z_1 : H^1(Z_1) \to H^2_{\mathrm{dR}}$ given by $\{f_{ij}\} \mapsto (0, \{f_{ij}\}, 0)$ is injective. Indeed, if $\{f_{ij}\}$ represents an element in the kernel, then it is of the form $(\delta h_{ij}, dh_{ij} + \omega_j - \omega_i, d\omega_i)$ for a $h_{ij} \in C_1(O_X)$, $\omega_i \in C_0(\Omega^1_X)$. Then the ω_i are closed and h_{ij} defines a cocycle. Since $H^1(O_X) = 0$ we can write $dh_{ij} = \eta_j - \eta_i$ and f_{ij} is a coboundary. The image is contained in F^1 and is orthogonal to the image G_1 of Frobenius. Indeed, take a class $F(a)$ and consider the cupproduct $\langle F(a), z_1(f_{ij})\rangle$. Applying the Cartier operator we see that it is zero. But $C : H^4_{\mathrm{dR}} \to H^4_{\mathrm{dR}}$ is a bijection. Hence the image lies in $F^1 \cap G_2$. This implies that $\dim H^1(Z_1) \leq 20$ if X is not superspecial and ≤ 21 for superspecial X. The exact sequence

$$0 \to H^1(B_1) \to H^1(Z_1) \xrightarrow{C} H^1\big(\Omega^1_X\big) \to H^2(B_1) \to H^2(Z_1) \to 0$$

implies together with the value of $h^1(B_1) = h^2(B_1)$ and $\chi(Z_1) = -20$ that $h^1(Z_1) = 20$ unless X is superspecial. But if X is superspecial then because of $F^2 = G_1$ the Cartier operator gives an isomorphism $C : H^1(Z_1)/H^1(B_1) \cong H^1(\Omega^1_X)$ implying that $h^1(Z_1) = 21$. □

8. An extension of de Rham cohomology

We define an extension of de Rham cohomology by considering an enlarged complex. (It captures the [0, 1)-part of crystalline cohomology.) It is defined as follows. We denote by $H^i_{\mathrm{dRW_n}}(X/S)$ the cohomology of the double complex CW_n of additive groups which is defined by the commutative diagram:

$$\begin{array}{ccccc}
\partial\uparrow & & \partial\uparrow & & \partial\uparrow \\
C_2(W_n(O_{X/S})) & \xrightarrow{D_n} & C_2\big(\Omega^1_{X/S}\big) & \xrightarrow{d} & C_2\big(\Omega^2_{X/S}\big) \\
\partial\uparrow & & \partial\uparrow & & \partial\uparrow \\
C_1(W_n(O_{X/S})) & \xrightarrow{D_n} & C_1\big(\Omega^1_{X/S}\big) & \xrightarrow{d} & C_1\big(\Omega^2_{X/S}\big) \\
\partial\uparrow & & \partial\uparrow & & \partial\uparrow \\
C_0(W_n(O_{X/S})) & \xrightarrow{D_n} & C_0\big(\Omega^1_{X/S}\big) & \xrightarrow{d} & C_0\big(\Omega^2_{X/S}\big),
\end{array}$$

where C_i are the i-th Čech cochains, D_n are the maps induced by the differential of Serre given in Sect. 6, the differentials d are defined by the exterior differentiation of differential forms and the vertical differentials are taken in the Čech sense. As usual, we denote by δ the differential of the single complex associated with CW_n. An element of $H^2_{\mathrm{dRW_n}}(X/S)$ is represented by a triple $(\alpha_0, \alpha_1, \alpha_2) \in C_2(W_n(O_{X/S})) \oplus C_1(\Omega^1_{X/S}) \oplus C_0(\Omega^2_{X/S})$. In case $n = 1$, $H^2_{\mathrm{dRW_1}}(X/S)$ is nothing but the de Rham cohomology $H^2_{\mathrm{dR}}(X/S)$. On $H^2_{\mathrm{dRW_n}}(X/S)$ we have the Hodge filtration

$$0 \subset F^2 \subset F^1 \subset H^2_{\mathrm{dRW_n}}(X/S).$$

Here the F^i (for $i \neq 0$) is naturally isomorphic to the F^i-part in the Hodge filtration of $H^2_{\mathrm{dR}}(X/S)$. We have a natural isomorphism

$$H^2_{\mathrm{dRW_n}}(X/S)/F^1 \cong H^2(X, W_n(O_{X/S})).$$

Since the Frobenius morphism F is a zero map on F^1, we have an induced homomorphism

$$F : H^2(X, W_n(O_{X/S})) \longrightarrow H^2_{\mathrm{dRW_n}}(X/S).$$

The map $V^{n-1} : O_{X/S} \longrightarrow W_n(O_{X/S})$ gives rise to a homomorphism

$$V^{n-1} : C_i(O_{X/S}) \longrightarrow C_i(W_n(O_{X/S})).$$

Using this homomorphism and taking the identity mapping from $C_i(\Omega^j_{X/S})$ to $C_i(\Omega^j_{X/S})$, we have a homomorphism of complexes of additive groups $CW_1 \longrightarrow CW_n$. Therefore, we have a homomorphism of additive groups:

$$V^{n-1} : H^2_{\mathrm{dR}}(X/S) \longrightarrow H^2_{\mathrm{dRW_n}}(X/S).$$

Let X_0 be a K3 surface over a field k and assume now that $F : H^2(W_i(O_{X_0})) \longrightarrow H^2(W_i(O_{X_0}))$ is zero for $i = 1, \ldots, n-1$. Then, by the same argument as in Sect. 5, we have $FH^2_{\mathrm{dRW_n}}(X_0) \subset V^{n-1}H^2_{\mathrm{dR}}(X_0)$. Therefore, using the inverse of the natural isomorphism of additive groups $H^2_{\mathrm{dR}}(X_0) \cong V^{n-1}H^2_{\mathrm{dR}}(X_0) \subset H^2_{\mathrm{dRW_n}}(X_0)$, we have a homomorphism

$$\Phi_n : H^2(W_n(O_{X_0})) \xrightarrow{F} V^{n-1}H^2_{DR}(X_0) \cong H^2_{DR}(X_0).$$

Since we have $H^2(W_n(O_{X_0}))/VH^2(W_{n-1}(O_{X_0})) \cong H^2(O_{X_0}) \cong H^2_{DR}(X_0)/F^1$, and since Φ_n maps $VH^2(W_{n-1}(O_{X_0}))$ to F^1, the map Φ_n induces a homomorphism from $H^2(O_{X_0})$ to $H^2(O_{X_0})$. This homomorphism coincides with the map ϕ_n which was constructed in Sect. 5.

We now take a basis ω_0 of $H^0(X_0, \Omega^2_{X_0})$ and take the dual basis ζ_0 of $H^2(X_0, O_{X_0})$. Via the Hodge filtration of $H^2_{\mathrm{dR}}(X_0)$, we can naturally regard $H^0(X_0, \Omega^2_{X_0})$ as a subspace of $H^2_{\mathrm{dR}}(X_0)$. Therefore, we may assume ω_0 is an element of $H^2_{\mathrm{dR}}(X_0)$.

Since $R^{n-1} : H^2(W_n(O_{X_0})) \to H^2(O_{X_0})$ is surjective, there exists an element $\alpha_0 \in H^2(W_n(O_{X_0}))$ such that $R^{n-1}(\alpha_0) = \zeta_0$. Then, by Theorems (5.1) and (5.7) we have the following proposition.

(8.1) Proposition. *Suppose that for a K3 surface X_0 the map $F : H^2(W_i(O_{X_0})) \to H^2(W_i(O_{X_0}))$ is zero for $i = 1, \ldots, n-1$. Then, with the notation introduced above, $h(\Phi_{X_0}) \geq n+1$ if and only if $\langle \Phi_n(\alpha_0), \omega_0 \rangle = 0$.*

9. The dimensions of the spaces of closed forms

We study the dimensions of the spaces $H^1(X, B_n)$ and $H^1(X, Z_n)$. We also consider their images in $H^2_{\mathrm{dR}}(X)$ and this gives a finer structure on these de Rham cohomology groups.

Let us consider the natural map β_n induced in H^1 by the inclusion $B_n \subset \Omega^1_X$:

$$\beta_n : H^1(B_n) \longrightarrow H^1(\Omega^1).$$

(9.1) Proposition. *If β_n is not injective then β_m is not injective for every $m \geq n$ and* $\dim H^1(B_m) < \dim H^1(B_{m+1})$.

Proof. The maps β_n are compatible with the natural maps $H^1(B_n) \to H^1(B_{n+1})$ and by (6.1) these maps $H^1(B_n) \to H^1(B_{n+1})$ are injective. If β_n is not injective it follows that β_{n+1} is not injective. To prove the second statement, we start with the case $n = 1$. If $\beta_1 : H^1(B_1) \to H^1(\Omega^1_X)$ is not injective then there exists a non-trivial cocycle $f_{ij} \in C^1(O_X/FO_X)$ and a 1-cochain ω_i of 1-forms such that $df_{ij} = \omega_j - \omega_i$. Since for affine open sets U the Cartier map $H^0(\Omega^1_{U,\mathrm{closed}}) \to H^0(\Omega^1_U)$ is surjective we can find closed forms $\tilde{\omega}_i$ and regular functions g_{ij} on $U_i \cap U_j$ such that we have a relation

$$f_{ij}^{p-1} d f_{ij} + dg_{ij} = \tilde{\omega}_j - \tilde{\omega}_i. \tag{6}$$

Note that this implies that the map $H^1(B_2) \to H^1(Z_1)$ has a non-trivial kernel. Suppose that the left-hand-side of (6) represents an element in the image of $H^1(B_1) = H^1(dO_X) \to H^1(B_2)$. Then we find a relation $dh_{ij} = \hat{\omega}_j - \hat{\omega}_i$ with $\hat{\omega}_i$ closed. Then since C annihilates dh_{ij} the $C\hat{\omega}_i$ define a global 1-form and this must be zero. Hence we can write $\hat{\omega}_i = d\phi_i$ and this shows that dh_{ij} represents the zero-class in $H^1(dO_X)$, contrary to the assumption. Hence we find a non-trivial element in $H^1(B_2)$ which does not lie in the image of the natural inclusion $H^1(B_1) \to H^1(B_2)$. Carrying out this argument for all n proves the claim. □

(9.2) Corollary. *Assume that $h < \infty$. Then for all $n \geq 1$ the natural map $\beta_n : H^1(B_n) \to H^1(X, \Omega^1_X)$ is injective and the image has dimension* $\min\{n, h-1\}$.

Proof. Since by (6.2) $\dim H^1(B_n)$ stabilizes for $h \neq \infty$, non-injectivity would contradict the preceding proposition. □

Note that the natural map $H^1(B_n) \longrightarrow H^1(\Omega^1_X)$ is not necessarily injective for $h = \infty$ because $\dim H^1(B_n) > 20$ for $n > 20$. In the case of $h \neq \infty$, we often identify $H^1(B_n)$ with the image of the natural inclusion $H^1(B_n) \to H^1(\Omega^1_X)$ in Corollary (9.2).

Let $Z_n \to \Omega^1_X$ be the natural inclusion. We have an induced map

$$z_n : H^1(Z_n) \longrightarrow H^1\big(\Omega^1_X\big).$$

We would like to characterize both the image and the kernel of this map. We often write $\mathrm{Im}(H^1(Z_n))$ for the image of z_n.

(9.3) Lemma. *i) We have* $\mathrm{Im}(H^1(B_n)) \subseteq \mathrm{Im}(H^1(Z_n))^{\perp}$, *in particular,* $\mathrm{Im}(H^1(B_n)) \subseteq \mathrm{Im}(H^1(B_n))^{\perp}$. *ii) Assume that $h < \infty$. If $C : H^1(B_{n+1}) \to H^1(B_n)$ is surjective then we have the equality* $\mathrm{Im}(H^1(Z_n)) = \mathrm{Im}(H^1(X, B_n))^{\perp}$.

Proof. We first show that $\mathrm{Im}(H^1(B_n))$ and $\mathrm{Im}(H^1(Z_n))$ are orthogonal. Let $\alpha \in \mathrm{Im}(H^1(B_1))$ and $\beta \in \mathrm{Im}(H^1(Z_1))$. Then we find an element $\alpha \wedge \beta \in H^2(\Omega^2_X) \cong k$ representing the cup product $\langle \alpha, \beta \rangle$. If we apply Cartier to $\alpha \wedge \beta$ a suitable number of times then it is zero. Now use the exact sequence

$$0 \to d\Omega^1_X \longrightarrow \Omega^2_{X,\mathrm{closed}} \xrightarrow{C} \Omega^2_X \to 0, \tag{7}$$

and the fact that $\Omega^2_{X,\mathrm{closed}} = \Omega^2_X$. Then, we have from the long exact sequence the exact sequence

$$H^2\big(d\Omega^1_X\big) \longrightarrow H^2\big(\Omega^2_X\big) \xrightarrow{C} H^2\big(\Omega^2_X\big) \to 0.$$

The fact that $\dim H^2(\Omega^2_X) = 1$ implies that $C : H^2(\Omega^2_X) \to H^2(\Omega^2_X)$ is an isomorphism as a p-linear mapping. Therefore, for $x \in H^2(\Omega^2_X)$ we have $x = 0$ if and only if $C^n(x) = 0$ for some n. Hence, we conclude $\alpha \wedge \beta = 0$. We now prove equality by induction. For $n = 1$ we have $\mathrm{Im}(H^1(B_1))^\perp = \mathrm{Im}(H^1(Z_1))$ because $\mathrm{Im}(H^1(B_1))$ is the kernel G_1/F^2 of the Cartier operator and $\mathrm{Im}\,(H^1(Z_1))$ is $F^1 \cap G_2 = F^1 \cap G_1^\perp$. Suppose that we have proved that $\mathrm{Im}\,(H^1(B_i))^\perp = \mathrm{Im}(H^1(Z_i))$ for $i \leq n$. If $\beta \in \mathrm{Im}(H^1(Z_1))$ is orthogonal to all $\alpha \in \mathrm{Im}(H^1(B_{n+1}))$ then we have $\langle C\alpha, C\beta\rangle = 0$ and since $C : H^1(B_{n+1}) \to H^1(B_n)$ is surjective this implies that $C\beta \in \mathrm{Im}(H^1(Z_n))$, i.e. $\beta \in \mathrm{Im}(H^1(Z_{n+1}))$. □

(9.4) Lemma. *The Cartier operator $C : H^1(B_n) \to H^1(B_{n-1})$ is surjective for $n \leq h-1$. Moreover, for $n \leq h-1 < \infty$ we have* $\dim \mathrm{Im}(H^1(Z_n)) = 20 - n$.

Proof. Note that we know that $h^1(B_n) = n$ for $n \leq h-1$ and thus the exact sequence $0 \to B_1 \to B_n \to B_{n-1} \to 0$ implies that $C : H^1(B_n) \to H^1(B_{n-1})$ is surjective for $n \leq h-1$. The rest follows from (9.3). □

(9.5) Corollary. *If $h \neq \infty$ we have the following orthogonal filtration in $H^1(\Omega^1)$:*

$$\begin{aligned} 0 \subset H^1(B_1) \subset H^1(B_2) \subset \ldots \subset H^1(B_{h-1}) \subset \mathrm{Im}(H^1(Z_{h-1})) \subset \\ \subset \mathrm{Im}(H^1(Z_{h-2})) \subset \ldots \subset \mathrm{Im}(H^1(Z_1)) \subset H^1\big(\Omega^1_X\big). \end{aligned} \tag{8}$$

The exact sequence (3) gives for $i = 0$ rise to the exact sequence

$$0 \to H^0\big(d\Omega^1_X\big) \longrightarrow H^1(Z_1) \longrightarrow H^1\big(\Omega^1_X\big) \xrightarrow{d} H^1\big(d\Omega^1_X\big) \longrightarrow H^2(Z_1) \to$$

The natural map $H^1(Z_1) \to H^1(\Omega^1)$ is the composition of $H^1(Z_1) \to H^2_{\mathrm{dR}}$ and the projection $H^2_{\mathrm{dR}} \to F^1/F^2$, i.e. by (7.1) it is the map $F^1 \cap G_2 \to F^1/F_2$. This is an isomorphism for $h = 1$ and it has a 1-dimensional kernel otherwise. It follows that

$$\dim H^0(d\Omega^1) = \dim H^1(d\Omega^1) = \begin{cases} 0 & \text{if} \quad h = 1 \\ 1 & \text{if} \quad h \neq 1. \end{cases}$$

From the exact sequence

$$0 \to H^0\big(d\Omega^1_X\big) \longrightarrow H^1(Z_{n+1}) \xrightarrow{\psi_{n+1}} H^1(Z_n) \longrightarrow H^1\big(d\Omega^1_X\big) \to$$

with ψ_{n+1} the map induced by inclusion we deduce that for $h \neq 1$

$$\psi_{n+1} \text{ is surjective} \iff \dim H^1(Z_{n+1}) > \dim H^1(Z_n). \tag{9}$$

(9.6) Lemma. *For $h \neq \infty$ we have* $\dim H^1(Z_n) = 20$.

Proof. If $h = 1$ we have $h^0(d\Omega^1_X) = 0$ and $h^1(d\Omega^1_X) = 0$ hence all ψ_n are isomorphisms. Since we know $h^1(Z_0) = 20$ the result follows for $h = 1$. If $h \neq 1$ then $H^0(d\Omega^1_X) \cong k$. For $n \leq h-1$ we have $\mathrm{Im}(H^1(Z_n)) = 20 - n$ by Lemma (9.3) and $\dim H^1(B_n) = \min\{n, h-1\}$. Suppose there exists an n $(n \leq h-1)$ such that ψ_n is surjective. Take the smallest such n. Then, the image of z_n coincides with the image of z_{n-1}, which contradicts $\dim \mathrm{Im}(H^1(Z_n)) \neq \mathrm{Im}(H^1(Z_{n-1}))$. Hence, $h^1(Z_n) = 20$ for $n \leq h-1$. Consider for $n = h$ the commutative diagram of exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & B_1 \arrow[r, phantom, "="] \arrow[d] & B_1 \arrow[d] & & \\
0 \arrow[r] & B_n \arrow[r] \arrow[d, "C"] & Z_n \arrow[r, "C^n"] \arrow[d, "C"] & \Omega^1_X \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & B_{n-1} \arrow[r] \arrow[d] & Z_{n-1} \arrow[r, "C^{n-1}"] \arrow[d] & \Omega^1_X \arrow[r] & 0 \\
 & 0 & 0 & & .
\end{tikzcd}$$

The diagram shows that $C^h : H^1(Z_h) \longrightarrow H^1(\Omega^1_X)$ factors through the image of C^{h-1}. This implies that $\dim H^1(Z_h) - (h-1) \leq 20 - (h-1)$. Since $h^1(Z_n) \geq 20$ for all $n \geq 1$ we get $h^1(Z_h) = 20$. We can repeat this argument for $H^1(Z_m)$ with $m \geq h$. $\square$

10. Chern classes of line bundles and closed forms

We start with a well-known result due to Ogus [O, Cor. 1.5]. We give here the proof by Shafarevich [Sh] for the reader's convenience.

(10.1) Proposition. *The map $c_1 : NS(X)/pNS(X) \longrightarrow H^2_{\mathrm{dR}}$ is injective and factors through $F^1H^2_{\mathrm{dR}}$.*

Proof. (Shafarevich) We take an affine open covering $\{U_i\}$ of X. A class in H^2_{dR} is represented by a tripel $(a, b, c) \in C^2(O_X) \oplus C^1(\Omega^1_X) \oplus C^0(\Omega^2_X)$. The boundaries are of the form $(\delta h_{ij}, dh_{ij} + \omega_j - \omega_i, d\omega_i)$ with $(h_{ij}, \omega_i) \in C^1(O_X) \oplus C^0(\Omega^1_X)$. So if a Chern class $c_1(L)$, represented by $(0, d\log f_{ij}, 0)$, is zero in H^2_{dR} then there exists $(h_{ij}, \omega_i) \in C^1(O_X) \oplus C^0(\Omega^1_X)$ with $d\omega_i = 0$ and $\delta h_{ij} = 0$ and we have $d\log f_{ij} = \omega_j - \omega_i + dh_{ij}$. By the relation

$\delta(h_{ij}) = 0$ the h_{ij} defines a class in $H^1(X, O_X) = 0$, so we have $h_{ij} = \eta_j - \eta_i$ with η_i regular and we can replace ω_i by $\omega_i + d\eta_i$ and obtain a relation

$$d\log f_{ij} = \omega_j - \omega_i \qquad \text{with} \quad \omega_i \quad \text{closed.} \tag{10}$$

Applying the Cartier operator we find

$$d\log f_{ij} = C\omega_j - C\omega_i. \tag{11}$$

Subtracting (1) from (2) we find $C\omega_i - \omega_i = C\omega_j - \omega_j$. This defines a global 1-form which must be zero. Hence we see $C\omega_i = \omega_i$ and it follows that $\omega_i = d\log\phi_i$ (after shrinking the U_i if necessary). We find

$$d\log f_{ij} = d\log\phi_i\phi_j^{-1},$$

hence

$$f_{ij} = \phi_i\phi_j^{-1}\psi_{ij}^p$$

for some $\psi_{ij} \in O(U_i \cap U_j)$. Thus modulo a p-th power L is trivial. The proof also shows that the image lands in $F^1H^2_{\mathrm{dR}}$. □

(10.2) Proposition. *If $h < \infty$ then we have $\langle c_1(NS(X))\rangle \cap \mathrm{Im}(H^1(B_n)) = \{0\}$ for all n. Moreover, $c_1(NS(X))$ is orthogonal with $\mathrm{Im}(H^1(B_n))$ for all n.*

Proof. First we show that $c_1(NS(X)) \cap \mathrm{Im}(H^1(B_n)) = (0)$ for all $n > 0$. If it is not, then take a minimal n such that $\mathrm{Im}(H^1(B_n))$ contains a Chern class $0 \neq [d\log f_{ij}]$. We can write a (non-trivial) relation as

$$d\log f_{ij} = \beta_{ij} + \omega_j - \omega_i, \tag{12}$$

where the β_{ij} are forms in B_n, but not in B_{n-1}. Apply the inverse Cartier operator as in (9.1) to get a relation

$$d\log f_{ij} = \tilde{\beta}_{ij} + \tilde{\omega}_j - \tilde{\omega}_i \tag{13}$$

where the $\tilde{\omega}_i$ are closed forms with $C(\tilde{\omega}_i) = \omega_i$ and the $\tilde{\beta}_{ij}$ are forms in B_{n+1} with $C(\tilde{\beta}_{ij}) = \beta_{ij}$. Subtracting (12) from (13) shows that $\tilde{\beta}_{ij} - \beta_{ij}$ is a boundary. Since β_{ij} defines a non-zero element of $H^1(B_n)$ which is not in the image of $H^1(B_{n-1})$ the cocycle $\tilde{\beta}_{ij}$ gives an element of $H^1(B_{n+1})$ not in the image of $H^1(B_n)$. Hence the left hand side is not zero in $H^1(B_{n+1})$ and this shows that $H^1(B_{n+1}) \to H^1(\Omega^1)$ is not injective.

Suppose now that $\langle c_1(NS(X))\rangle \cap \mathrm{Im}(H^1(B_n)) \neq 0$. Considering all n which satisfy this condition, we then have a relation with $m \geq 2$ minimal

$$d\log f_{ij}^{(1)} + \sum_{\nu=2}^{m} a_\nu d\log f_{ij}^{(\nu)} = \beta_{ij} + \omega_j - \omega_i.$$

We may assume that $m \geq 2$ and that $a_\nu \notin \mathbb{F}_p$ for all $\nu \geq 2$. Then by applying C^{-1} as before we find

$$d \log f_{ij}^{(1)} + \sum_{\nu=2}^{m} a_\nu^p d \log f_{ij}^{(\nu)} = \tilde{\beta}_{ij} + \tilde{\omega}_j - \tilde{\omega}_i,$$

where the $\tilde{\omega}_i$ are closed and $C\tilde{\omega}_i = \omega_i$. Subtracting the two relations we find a shorter relation (m smaller but with n maybe larger). This contradiction shows that $\langle c_1(NS(X))\rangle \cap \mathrm{Im}(H^1(B_n)) = 0$.

The orthogonality of $\langle c_1(NS(X))\rangle$ and $\mathrm{Im}(H^1(B_n))$ follows from the fact that $\langle c_1(NS(X))\rangle \subset \mathrm{Im}(H^1(Z_n))$ and Lemma (9.3). □

(10.3) Proposition. *Suppose that $h < \infty$. Then the Chern class map*

$$c_1 \otimes k : NS(X)/pNS(X) \otimes k \to H^1(X, \Omega_X^1)$$

is injective.

Proof. Suppose we have a relation $\sum_{\nu=1}^{r} a_\nu c_1(L_\nu) = \omega_j - \omega_i$ for line bundles L_ν and $a_\nu \in k$. We may assume that $a_1 = 1$ and that the relation is the shortest possible (r minimal). Furthermore, we can assume that $a_\nu/a_\mu \notin \mathbb{F}_p$ for $\nu \neq \mu$; otherwise we can easily find a shorter one. Now apply the inverse Cartier operator C^{-1} to the relation as we did before. We find a new relation

$$dg_{ij} + c_1(L_1) + \sum_{\nu=2}^{r} a_\nu^p c_1(L_\nu) - \tilde{\omega}_i + \tilde{\omega}_j = 0,$$

where the g_{ij} are regular on $U_i \cap U_j$. If the cocycle dg_{ij} defines a zero class in $H^1(X, \Omega_X^1)$, we can write $dg_{ij} = \eta_j - \eta_i$, and we can replace the relation by a shorter one by subtracting the two relations contradicting the minimality of r. Hence $\{dg_{ij}\}$ defines a non-zero class in $H^1(X, \Omega_X^1)$ and we find a non-zero element in $\mathrm{Im}(H^1(B_1)) \cap \langle c_1(NS(X))\rangle$. □

As a corollary of (6.2), (10.2) and (10.3) we now find the well-known result of Artin and Mazur on the rank ρ of the Néron-Severi group:

(10.4) Corollary. *For $h \neq \infty$ we have $\rho \leq 22 - 2h$.*

(10.5) Remark. A line bundle L defined by transition functions f_{ij} defines a cocycle $d \log f_{ij}$ with values in $Z_n\Omega_X^1$ for all $n \geq 0$. We thus can view the class $c_1(L)$ as a class in $H^1(Z_n)$ for all $n \geq 0$ as well as in H^2_{dR}. If $h < \infty$ the maps

$$c_1 \otimes k : NS(X)/pNS(X) \otimes k \longrightarrow H^1(Z_n)$$

are injective for all $n \geq 0$.

11. The supersingular case

The map $c_1 : NS/pNS \to H^2_{\mathrm{dR}}$ is injective and factors through $H^1(Z_j)$ for all $j \geq 1$. However, the map $c_1 \otimes k : NS \otimes k \to H^2_{\mathrm{dR}}$ is not necessarily injective. For X supersingular in Shioda's sense, i.e. $\rho = B_2 = 22$, it cannot be injective since $\dim_k H^1(Z_1) = 20$ or 21, the latter if X is superspecial.

We define for $j = 0, 1 \ldots$

$$U_j := \ker\left\{c_1 \otimes k : NS \otimes k \to H^1(Z_j)\right\}$$

and we set

$$\dim U_1 = \sigma_0.$$

Using the natural maps $H^1(Z_i) \to H^1(Z_{i-1})$ we have $U_{j+1} \subseteq U_j$ for $j = 0, 1, 2, \ldots$. We define two bijective operators on $NS \otimes k$

$$\varphi = 1 \otimes F \qquad \text{and} \qquad \gamma = 1 \otimes F^{-1},$$

with the Frobenius action $F : a \mapsto a^p$ on the second factor k.

(11.0) Remark. If we assume that $\rho = B_2 = 22$ (i.e. the truth of the Artin conjecture that $h = \infty$ implies $\rho = 22$) then one can show that the invariant σ_0 just introduced equals the Artin invariant σ_0, i.e. the intersection form on the lattice $NS(X)$ has discriminant

$$\mathrm{disc}(NS(X)) = -p^{2\sigma_0}.$$

(11.1) Lemma. *We have $\gamma(U_{j+1}) \subseteq U_j$; equivalently, we have $U_{j+1} \subseteq \varphi(U_j)$. Moreover, we have $U_{j+1} \subseteq U_j \cap \varphi(U_j)$.*

Proof. This follows from the commutativity of the diagram

$$\begin{tikzcd} NS \otimes k \arrow[r, "\gamma"] \arrow[d, "c_1 \otimes k"] & NS \otimes k \arrow[d, "c_1 \otimes k"] \\ H^1(Z_{j+1}) \arrow[r, "C"] & H^1(Z_j) \end{tikzcd}$$

with C the Cartier operator. The second result follows from this and the inclusion $U_{j+1} \subset U_j$. $\square$

Now choose an element $u_{\min} = u^{(j)}_{\min} \neq 0$ of minimal length in U_j under the assumption that U_j is non-zero, i.e. write $u_{\min} = \sum_{i=1}^m a_i[L_i]$ and require $m \geq 2$ to be minimal. We also may assume – and we shall – that $a_1 = 1$.

(11.2) Lemma. *For $j \geq 1$ we have $u_{\min} \notin \varphi(U_j)$. Similarly we have $u_{\min} \notin \gamma(U_j)$. If X is not superspecial the conclusion holds also for $j = 0$.*

Proof. If $u_{\min} \in \varphi(U_j)$ is such a minimal element with $a_1 = 1$ then $\gamma(u_{\min}) - u_{\min}$ would be a shorter element or zero. If it is zero, then $u_{\min} \in NS \otimes \mathbf{F}_p \cap U_1 = \{0\}$. For $j = 0$ the argument is similar. Note that $NS \otimes \mathbf{F}_p \cap U_0 \neq \{0\}$ if and only if X is superspecial, cf. [O 1], Cor. 1.4.

(11.3) Lemma. *The map $c_1 \otimes k : \varphi(U_j) \to H^1(Z_{j+1})$ factors via $H^1(B_1) \to H^1(Z_{j+1})$ and the induced map $\varphi(U_j) \to H^1(B_1)$ is surjective if $U_j \neq \{0\}$.*

Proof. If $u \in U_j$ there exist closed forms $\zeta_x \in Z_j(V_x)$ for some open covering V_x such that $(c_1 \otimes k)(u)$ is a coboundary: $\zeta_\beta - \zeta_\alpha$. Now use the local surjectivity of C to write

$$(c_1 \otimes k)(\varphi(u)) = \tilde{\zeta}_\beta - \tilde{\zeta}_\alpha + \phi_{\alpha\beta}$$

with $\tilde{\zeta}_x \in Z_{j+1}$, $C\tilde{\zeta}_x = \zeta_x$, $\phi_{\alpha\beta} \in B_1$ on a suitable open covering. Then this $\phi_{\alpha\beta}$ defines a cocycle, thus an element in $H^1(B_1) \subset H^1(Z_{j+1})$.

To prove the surjectivity, choose a non-zero element $u_{\min} \in U_j$. Suppose that $\phi_{\alpha\beta} = \eta_\beta - \eta_\alpha$ with $\eta \in B_1$. Then $\varphi(u_{\min}) \in U_j$, hence $U_{\min} \in \gamma(U_j)$ which contradicts Lemma (11.2).

(11.4) Corollary. *We have $U_{j+1} = U_j \cap \varphi(U_j)$ and $\dim(U_{j+1}) = \max\{\dim(U_j) - 1, 0\}$.*

Proof. The kernel of $c_1 \otimes k : \varphi(U_j) \to H^1(Z_{j+1})$ equals U_{j+1} by (11.1) and has codimension 1 by (11.3). Since $U_j \neq \varphi(U_j)$, and since their intersection contains U_{j+1} we must have $U_{j+1} = U_j \cap \varphi(U_j)$. The statement about dimensions follows.

If we assume that $\sigma_0 \geq 1$ then we have a strictly increasing sequence

$$\{0\} = U_{\sigma_0+1} \subset U_{\sigma_0} \subset \ldots \subset U_2 \subset U_1 \tag{14}$$

and this implies:

(11.5) Proposition. *The map $c_1 \otimes k$ factors through an* injection

$$NS(X)/pNS(X) \otimes k \to H^1(Z_{\sigma_0+1}).$$

We can generalize the result of Corollary (11.4).

(11.6) Lemma. *We have $\varphi^k(U_j) \cap U_j = U_{j+k}$. In particular $\varphi^{\sigma_0}(U_1) \cap U_1 = \{0\}$.*

Proof. We prove this by induction on k, the case $k = 1$ was proved in (11.4). Suppose it holds for k. Then

$$\varphi^{k+1}(U_j) \cap U_j \subset \varphi[\varphi^k(U_{j-1}) \cap U_{j-1}] \subset \varphi(U_{j+k-1}).$$

On the other hand we have

$$\varphi(U_{j+k-1}) \cap \varphi^{k+1}(U_j) \subset \varphi(U_j \cap \varphi^k(U_j)) = \varphi(U_{j+k}).$$

But by an easy induction one has

$$\varphi(U_{j+k}) \cap U_j \subset U_{j+k+1}.$$

In view of $\dim(\varphi^{k+1}(U_j) \cap U_j) \geq \dim(U_j) - (k+1)$ the result follows.

(11.7) Lemma. *Suppose that* $U_1 \neq \{0\}$ *and let* $u_{\min} \in U_1$. *Then* $\gamma(u_{\min}) \in U_0 \backslash U_1$. *In particular,* $(c_1 \otimes k)(\gamma(u_{\min})) \in H^0(\Omega^2) \subset H^1(Z_1) \subset H^2_{\mathrm{dR}}$.

Proof. Since $\gamma(u_{\min})$ does not lie in U_1, but lies in U_0 we see that $(c_1 \otimes k)$ $(\gamma(u_{\min}))$ must lie in the kernel of $H^1(Z_1) \to H^1(\Omega^1)$, which is $H^0(\Omega^2_X)$.

(11.8) Lemma. *The Chern class map* $c_1 \otimes k : \varphi^m(U_j) \longrightarrow H^1(Z_{j+m})$ *factors through* $H^1(B_m)$. *For any* $t \geq 1$ *the natural image of* $H^1(B_t)$ *in* $H^1(Z_{\sigma_0+1})$ *is contained in the image of* $NS(X)/pNS(X) \otimes k$ *under* $c_1 \otimes k$.

Proof. As in the proof of (11.3) we can write $(c_1 \otimes k)(u) = \zeta_\beta - \zeta_\alpha$ with $\zeta_x \in Z_j(V_x)$. Now use the local surjectivity of C to write

$$\varphi^m(u) = \tilde{\zeta}_\beta - \tilde{\zeta}_\alpha + \phi_{\alpha\beta}$$

with $\tilde{\zeta} \in Z_{j+m}$, $C^m\tilde{\zeta}_x = \zeta_x$, $\phi_{\alpha\beta} \in B_m$. Then this $\phi_{\alpha\beta}$ defines a cocycle, thus an element in $H^1(B_m) \subset H^1(Z_{j+m})$. This proves the first statement.

We prove the second statement by induction. Note that by (11.3) the image of $H^1(B_1)$ in $H^1(Z_{\sigma_0+1})$ is contained in the image of $NS(X)/pNS(X) \otimes k$ under $c_1 \otimes k$. Let α be an element of the image of $H^1(B_t)$ and $\beta = C\alpha$ in the image of $H^1(B_{t-1})$. Then $\beta = (c_1 \otimes k)(v)$ for some $v \in NS \otimes k$. But then $\alpha - (c_1 \otimes k)(\varphi(v))$ is an element of $H^1(B_1)$. By induction this is in the image of $(c_1 \otimes k)(NS \otimes k)$. Hence α lies in the image of $(c_1 \otimes k)(NS \otimes k)$.

(11.9) Proposition. *Let* $\sigma_0 \geq 1$. *The dimension of the image of* $H^1(B_{\sigma_0})$ *in* $H^1(Z_1)$ *equals* σ_0. *The image in* $H^1(\Omega^1_X)$ *is* $\sigma_0 - 1$*-dimensional.*

Proof. The first statement follows directly from (11.6) and (11.8). Arguing similarly for U_0 we find that $c_1 \otimes k : \varphi^{\sigma_0}(U_0) \to H^1(\Omega^1_X)$ factors through the natural map $H^1(B_{\sigma_0}) \to H^1(\Omega^1_X)$. The intersection $\varphi^{\sigma_0}(U_0) \cap U_0$ has dimension 1.

(11.10) Theorem. *For a K3 surface X with* $B_2 = \rho$ *and Artin invariant* σ_0, *we have* $\dim(\mathrm{Im}\, H^1(Z_{\sigma_0})) = 21 - \sigma_0$ *for the image in* $H^1(\Omega^1_X)$ *and it is generated by Chern classes.*

Proof. Since we have

$$\langle c_1(NS(X)/pNS(X))\rangle \subset \mathrm{Im}\ H^1(Z_{\sigma_0}) \subset (\mathrm{Im}\ H^1(B_{\sigma_0}))^{\perp} \subset H^1\big(\Omega^1_X\big)$$

and $\dim\langle c_1(NS(X)/pNS(X))\rangle = \dim(\mathrm{Im}\ H^1(B_{\sigma_0}))^{\perp} = 20 - (\sigma_0 - 1)$ by (11.9), we have

$$\langle c_1(NS(X)/pNS(X))\rangle = \mathrm{Im}\ H^1(Z_{\sigma_0}) = (\mathrm{Im}\ H^1(B_{\sigma_0}))^{\perp}$$

and so $\dim \mathrm{Im}\ H^1(Z_{\sigma_0}) = 21 - \sigma_0$. □

Since the codimension of Im $H^1(Z_{i+1})$ in Im$H^1(Z_i)$ is at most one, we conclude that

$$\begin{aligned}\mathrm{Im}\ H^1(Z_{\sigma_0}) = \mathrm{Im}\ H^1(Z_{\sigma_0-1}) \subset \mathrm{Im}\ H^1(Z_{\sigma_0-2}) \subset \dots \\ \subset \mathrm{Im}\ H^1(Z_1) \subset H^1\big(\Omega^1_X\big)\end{aligned}$$

and Im $H^1(Z_n)$ = Im $H^1(Z_{\sigma_0})$ for $n \geq \sigma_0$. Here, the inclusions are strict inclusions. Moreover, we see that the injection $c_1 \otimes k : NS(X)/pNS(X) \otimes k \to H^1(Z_{\sigma_0+1})$ is an isomorphism:

$$c_1 \otimes k : NS(X)/pNS(X) \otimes k \cong H^1(Z_{\sigma_0+1}).$$

We now need the following lemma.

(11.11) Lemma. *Let X be a K3 surface X with $B_2 = \rho$ and Artin invariant σ_0. For every $n \geq 0$ the natural map $H^1(Z_{\sigma_0+n+1}) \to H^1(Z_{\sigma_0+n})$ is surjective.*

Proof. By Theorem (11.10) the dimension of the image of $H^1(Z_{\sigma_0})$ in $H^1(\Omega^1)$ is $21-\sigma_0$. By (14) it follows that the image of $H^1(Z_{\sigma_0-1})$ in $H^1(\Omega^1)$ has dimension at least $22 - \sigma_0 - 1$. Since the map $H^1(Z_{\sigma_0+1}) \to H^1(\Omega^1)$ factors through $H^1(Z_{\sigma_0})$ the map $H^1(Z_{\sigma_0+1}) \to H^1(Z_{\sigma_0})$ must be surjective.

We now prove that if the natural mapping $H^1(Z_{n+1}) \to H^1(Z_n)$ is surjective, then so is $H^1(Z_{m+1}) \to H^1(Z_m)$ for any $m \geq n$. Suppose that the natural homomorphism $H^1(A, Z_{n+1}) \to H^1(A, Z_n)$ is surjective. By the diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & B_1 & \longrightarrow & Z_{n+2} & \xrightarrow{C} & Z_{n+1} & \to & 0 \\ & & \big\downarrow{=} & & \big\downarrow{\iota_{n+2}} & & \big\downarrow{\iota_{n+1}} & & \\ 0 & \to & B_1 & \longrightarrow & Z_{n+1} & \xrightarrow{C} & Z_n & \to & 0 \end{array}$$

we have a diagram of exact sequences

$$\begin{array}{ccccccc} \to H^1(X, B_1) & \to & H^1(X, Z_{n+2}) & \xrightarrow{C} & H^1(X, Z_{n+1}) & \to & H^2(X, B_1) \\ \big\downarrow{=} & & \big\downarrow{\iota_{n+2}} & & \big\downarrow{\iota_{n+1}} & & \big\downarrow{=} \\ \to H^1(X, B_1) & \to & H^1(X, Z_{n+1}) & \xrightarrow{C} & H^1(X, Z_n) & \to & H^2(X, B_1). \end{array}$$

From this diagram we see that the natural homomorphism $H^1(X, Z_{n+2}) \to H^1(X, Z_{n+1})$ is also surjective. So this lemma now follows by induction. $\square$

(11.12) Corollary. *Let X be a K3 surface X with $B_2 = \rho$ and Artin invariant σ_0. For $n \geq \sigma_0$ we have* $\mathrm{Im}(H^1(B_n)) = \mathrm{Im}(H^1(Z_n))^{\perp}$ *and* $\dim \mathrm{Im}(H^1(B_n)) = \sigma_0 - 1$.

Proof. By the proof of (11.10), we have $\mathrm{Im}H^1(Z_{\sigma_0})^{\perp} = \mathrm{Im}H^1(B_{\sigma_0})$. Therefore, for $n \geq \sigma_0$, we have

$$\mathrm{Im}H^1(Z_n)^{\perp} = \mathrm{Im}H^1(Z_{\sigma_0})^{\perp} = \mathrm{Im}H^1(B_{\sigma_0}) \subset \mathrm{Im}H^1(B_n).$$

On the other hand, by the proof of (9.3), we have $\mathrm{Im}H^1(Z_n)^{\perp} \supset \mathrm{Im}H^1(B_n)$. Hence, we get the desired results. $\square$

Since $c_1 \otimes k : NS(X)/pNS(X) \otimes k \longrightarrow H^1(Z_i)$ is injective for $i \geq \sigma_0 + 1$, we have the following proposition.

(11.13) Proposition. *For a K3 surface X with $B_2 = \rho$ the following four conditions are equivalent.*

(i) *The natural map $H^1(Z_i) \to H^1(Z_{i-1})$ is surjective.*
(ii) *The Cartier operator $C : H^1(Z_i) \to H^1(Z_{i-1})$ is surjective.*
(iii) $\dim H^1(Z_{10}) \geq 31 - i$.
(iv) $\sigma_0 \leq i$.

12. The Kodaira-Spencer map

Let X_0 be a K3 surface, and let $\pi : X \longrightarrow S$ be the versal formal k-deformation of X_0. Then, as is well-known (cf. [D]), we have $S = \mathrm{Spf}k[[t_1, \ldots, t_{20}]]$ with variables $t_1, \ldots, t_{20}$. We denote by ∇ the Gauss-Manin connection of $H^2_{\mathrm{dR}}(X/S)$:

$$\nabla : H^2_{\mathrm{dR}}(X/S) \longrightarrow \Omega^1_{S/k} \otimes H^2_{\mathrm{dR}}(X/S).$$

We take a basis ω of $H^0(X, \Omega^2_{X/S})$. Then, ∇ composed with cup product with ω gives an isomorphism:

$$\rho_\omega : H^1\big(X, \Omega^1_{X/S}\big) \xrightarrow{\sim} \Omega^1_{S/k}.$$

We denote by m the maximal ideal of the closed point of S. By evaluating ρ_ω at zero we have an isomorphism:

$$\rho_{\omega,0} : H^1\big(X_0, \Omega^1_{X_0/k}\big) \xrightarrow{\sim} m/m^2.$$

(12.1) Remark. Ogus gave an explicit expression of the isomorphism ρ_ω as follows. For an element $\alpha \in H^1(X, \Omega^1_{X/S})$ we choose a lifting $\alpha' \in F^1 H^2_{\mathrm{dR}}(X/S)$ of α. Since $\langle \alpha', \omega \rangle = 0$, we have

$$\rho_\omega(\alpha) = \langle \nabla \alpha', \omega \rangle = -\langle \alpha', \nabla \omega \rangle.$$

For details, see the paper by Deligne/Illusie [D], cf. also Ogus [O].

13. Horizontality

We consider the moduli space $M = M_{2d}$ of K3 surfaces with a polarization of degree $2d$ in characteristic p. Let (X, D) be a polarized K3 surface with a polarization of degree $2d$. The existence of this moduli spaces follows from work of Gieseker. We view these moduli spaces as algebraic stacks. If the Chern class $c_1(D)$ is not zero in the de Rham cohomology of X then the moduli space is formally smooth at $[(X, D)]$.

We shall assume for simplicity that the degree $2d$ of the polarization is prime to p. Let furthermore $\pi : \mathcal{X} \longrightarrow M_{2d}$ be the universal family of polarized K3 surfaces over k. We set

$$M^{(h)} := \{s \in M : h(X_s) \geq h\}.$$

Then, by Artin [A], $M^{(h)}$ is an algebraic subvariety of codimension $\leq h - 1$ in M for $h = 1, \ldots, 10$. We shall show that their codimension is $h - 1$.

The direct image sheaves $R^2\pi_* W_i(O_{\mathcal{X}})$ are coherent sheaves of rings, but not coherent O_M-modules. If there would exist a suitable Grothendieck group of such objects we could calculate Chern classes by using Theorem (5.1). Since we do not know how to do this, we resort to a different method to calculate cycle classes of loci of given height.

Let X_0 be a K3 surface, and assume that the height of the formal Brauer group Φ_{X_0} is greater than or equal to h, i.e., X_0 corresponds to a point in $M^{(h)}$. Then the Frobenius morphism is zero on $H^2(X, W_i(O_{X/S}))$ for $i = 1, \ldots, h-1$. We let S be a formal neighborhood of $M^{(h)}$ at the point, and we also denote by ∇ the Gauss-Manin connection of $H^2_{\mathrm{dR}}(X/S)$. We consider the Hodge filtration $0 \subset F^2 \subset F^1 \subset H^2_{\mathrm{dR}}(X/S)$, and construct, in the same way as in Sect. 8, a homomorphism

$$\Phi_h : H^2(W_h(O_{X/S})) \longrightarrow H^2_{\mathrm{dR}}(X/S).$$

We take a basis ω of $H^0(\Omega^2_{X/S})$ and take the dual basis ζ of $H^2(O_{X/S})$. We take a lifting $\tilde{\zeta} \in H^2_{\mathrm{dR}}(X/S)$ of ζ. Then we have $\langle \tilde{\zeta}, \omega \rangle = 1$. Since $R^{n-1} : H^2(W_n(O_{X/S})) \to H^2(O_{X/S})$ is surjective, we take an element $\alpha \in H^2(W_h(O_{X/S}))$ such that $R^{h-1}(\alpha) = \zeta$. We set

$$g_h = \langle \Phi_h(\alpha), \omega \rangle.$$

Since $\Phi_h(\alpha) - g_h\tilde{\zeta}$ is orthogonal to ω, it follows that $\Phi_h(\alpha) - g_h\tilde{\zeta}$ is contained in the F^1-step of the Hodge filtration. Therefore, using the natural isomorphism $H^2_{\mathrm{dR}}/F^1 \cong H^2(O_{X/S})$, we conclude that

$$\phi_h(\zeta) = g_h\zeta \quad \text{in } H^2(O_{X/S}),$$

where ϕ_h was defined in Sect. 5. This means that the equation $g_h = 0$ gives the scheme theoretic locus of zero of ϕ_h, and by Proposition (8.1), the support of this locus in $M^{(h)}$ coincides with $M^{(h+1)}$.

(13.1) Proposition. *Under the notation and assumptions made above, the image* Im Φ_h *is horizontal with respect to the Gauss-Manin connection.*

Proof. It suffices to prove $\nabla(\Phi_h(\alpha)) = 0$. The element α is represented by a cocycle $\alpha_{ijk} = (\alpha^{(0)}_{ijk}, \dots, \alpha^{(h-1)}_{ijk})$ with respect to a suitable affine open covering $\{U_i\}$ of X/S. Since the Frobenius morphism is zero on $H^2(W_{h-1}(O_{X/S}))$, there exists a cochain $\gamma_{ij} \in \Gamma(U_i \cap U_j, W_{h-1}(O_X/S))$ such that $FR(\alpha) = \partial\{\gamma_{ij}\} = \{\gamma_{jk} - \gamma_{ik} + \gamma_{ij}\}(\in C_2(W_{h-1}))$. Hence we have

$$F(\alpha) - \partial(\{(\gamma_{ij}, 0)\}) = \{(0, \dots, 0, g_{ijk})\}. \tag{15}$$

Put $\tilde{\gamma}_{ij} = (\gamma_{ij}, 0)$, an element in $\Gamma(U_i \cap U_j, W_h(O_{X/S}))$. Then $\phi_h(\zeta) = \{g_{ijk}\}$ and

$$\Phi_h(\alpha) = (g_{ijk}, -D_h(\tilde{\gamma}_{ij}), 0) \in C_2(O_{X/S}) \oplus C_1\big(\Omega^1_{X/S}\big) \oplus C_0\big(\Omega^2_{X/S}\big).$$

We write this as

$$\Phi_h(\alpha) = \{(g_{ijk}, b_{ij}, 0)\}.$$

We have to calculate $\nabla(\Phi_h(\alpha))$. We use the explicit description of the Gauss-Manin connection. Katz and Oda define in [K-O] two operators

$$L_S : C_q(\Omega^p) \to C_q(\Omega^{p+1}), \quad L_S((\beta)(i_0, \dots, i_q)) = d^i_S(\beta(i_0, \dots, i_q))$$

and

$$\begin{aligned} \lambda : C_q(\Omega^p) \to C_{q+1}(\Omega^p), \quad & \lambda(\beta)(i_0, \dots, i_{q+1}) \\ & = (-1)^p(I^{i_0} - I^{i_1})(\beta(i_1, \dots, i_{q+1})). \end{aligned}$$

Here we follow the notation of loc. cit. The (substitution) operator I^{i_0} is given by $\sum_{t=1}^p \mathrm{subs}(dx_t \mapsto d^{i_0}_S)$ and is zero for $p = 0$. In our case this gives $L_S(g_{ijk}) = d^i_S(g_{ijk}) \in C_2(\Omega^1)$, $\lambda(g_{ijk}) = 0$ and $L_S(b_{ij}) = d^i_S(b_{ij}) \in C_1(\Omega^2)$, $\lambda(b)(ijk) = -(I^i - I^j)(b_{jk}) \in C_2(\Omega^1)$. So we find

$$\nabla(\Phi_h(\alpha)) = d^i_S b_{ij} + d^i_S g_{ijk} - I^i b_{jk} + I^j b_{jk}. \tag{16}$$

Here the first term lies in $C_1(\Omega^2)$.

Using d_S^i instead of d we can make an operator $D_{h,S}^i$ similar to the operator D_h defined by Serre. It is zero on the image of Frobenius and so the relation (15) gives

$$-D_{h,S}^i(\partial\{\gamma_{ij}\}) = d_S^i(g_{ijk}).$$

This says $d_S^i(g_{ijk}) = I^i(b_{jk} - b_{ik} + b_{ij})$. Collecting the terms we get

$$\nabla(\Phi_h(\alpha)) = d_S^i b_{ij} + I^i(-b_{ik} + b_{ij}) + I^j b_{jk}.$$

Put $c_{ij} = -D_h(\gamma_{ij})$. Now note that we have

$$d_S^i(D_h(\gamma_{ij})) = d\big(D_{h,S}^i\gamma_{ij}\big).$$

Therefore the right hand side of (16) is a boundary in the total complex. We conclude $\nabla\Phi_h(\alpha) = 0$ in $\Omega^1_{S/k} \otimes H^2_{DR}(X/S)$. □

14. The tangent spaces to the stratification

We denote by D_0 the polarization class of X_0 of degree $2d$ and we shall assume that it is prime to p. Let $M^{(h)}$ be the closed locus of the moduli space $M = M_{2d}$ of polarized K3 surfaces given by the condition height $\geq h$ for $h = 1, \ldots, 10$ and $h = \infty$. We now determine the tangent space of $M^{(h)}$ at the point $x_0 = (X_0, D_0)$. We denote by $\mathrm{Im}\ H^1(X_0, Z_\ell)$ the image of $H^1(X_0, Z_\ell\Omega^1_{X_0})$ in $H^1(X_0, \Omega^1_{X_0})$ induced by the natural inclusion $Z_\ell\Omega^1_{X_0} \to \Omega^1_{X_0}$.

(14.1) Proposition. *Suppose that* (X_0, D_0) *represents a point* x_0 *of* $M^{(h)} - M^{(\infty)}$. *Then for* $1 \leq h \leq 10$ *the tangent space of* $M^{(h)}$ *at* x_0 *is in a natural way isomorphic to* $\mathrm{Im}\ H^1(X_0, Z_{h-1}) \cap c_1(D_0)^\perp$.

Proof. Note that by (9.2) the map $H^1(X_0, B_{h-1}) \to H^1(X_0, \Omega^1_{X_0})$ is injective. Since we have $H^1(X_0, B_{h-1}) \subset c_1(D_0)^\perp$, by Corollary (10.2) and Lemma (9.3), it suffices to prove that $\langle H^1(X_0, B_{h-1}), c_1(D_0)\rangle$ is the normal space of $M^{(h)}$ at x_0. We will show this by induction. Note that we know $\dim H^1(X_0, B_\ell) = \ell$ for $\ell = 0, \ldots, h-1$.

Suppose $h = 1$. Then, we have $H^1(X_0, B_0) = 0$, and by the general theory of moduli spaces the tangent space of $M^{(1)} = M$ at x_0 is given by $c_1(D_0)^\perp \subset H^1(X_0, \Omega^1_{X_0})$.

Now, we assume that the statement holds until h. We use the notation above. Then, by (8.1) $M^{(h+1)}$ is defined by $g_h = \langle\Phi_h(\alpha), \omega\rangle = 0$ in $M^{(h)}$. Using Proposition (13.1), we have

$$\begin{aligned} dg_h &= \langle\nabla\Phi_h(\alpha), \omega\rangle + \langle\Phi_h(\alpha), \nabla\omega\rangle \\ &= \langle\Phi_h(\alpha), \nabla\omega\rangle\ . \end{aligned}$$

We denote by m (resp. m_0) the maximal ideal which corresponds to the point x_0 in the versal formal moduli space around x_0 (resp. in the formal moduli around x_0 in $M^{(h)}$). Then, under the natural homomorphism

$$H^1\big(X_0, \Omega^1_{X_0}\big) \cong m/m^2 \longrightarrow m_0/m_0^2$$

$-\Phi_h(\alpha)(0)$ corresponds to the cotangent vector g_h by the argument of Ogus [O]. The kernel of this homomorphism is isomorphic to $\langle H^1(X_0, B_{h-1}), c_1(D_0)\rangle$ by induction. We have

$$\begin{aligned} -\Phi_h(\alpha)(0) &= -\{D_h(\tilde{\gamma}_{ij})\} \\ &= -\Big\{\sum_{m=0}^{h-1}(\gamma_{ij}^{(m)})^{p^{h-m}-1} d\log\gamma_{ij}^{(m)}\Big\} \end{aligned}$$

and $D_h : H^2(W_h(O_{X_0})/FW_h(O_{X_0})) \to H^1(X_0, \Omega^1_{X_0})$ is injective by Corollary (9.2). Since $\Phi(\alpha)(0)$ lies in $H^1(X_0, B_h)$ but not in $H^1(X_0, B_{h-1})$, we conclude that $g_h \notin m_0^2$. By induction we thus see that the tangent space to $M^{(h+1)}$ can be identified with $H^1(X_0, Z_h) \cap c_1(D_0)^\perp$. $\square$

This argument does not work for $h = \infty$, but can be made to work for the supersingular points for which the subspace $\langle \mathrm{Im}(H^1(B_h)), c_1(D)\rangle$ of $H^1(\Omega^1)$ has dimension h. In Sect. 12 we gave conditions for this. Under the assumption that $\rho = B_2$ this is the case if the Artin invariant σ_0 of a supersingular K3 surface satisfies $\sigma_0 > h$. We thus find:

(14.2) Theorem. *For $h = 1, \dots, 10$ the open stratum $M^{(h)}$, if not empty, is purely of dimension $(20-h)$ and nonsingular at any point of the stratum $M^{(h)}$ where the subspace $\langle \mathrm{Im}(H^1(B_{h-1})), c_1(D_0)\rangle$ of $H^1(\Omega^1_{X_0})$ has dimension h. In particular, it is non-singular at non-supersingular points and assuming the Artin conjecture at all supersingular points with Artin invariant $\sigma_0 \geq h$ and $c_1(D_0) \notin \mathrm{Im}(H^1(B_h))$.*

We refer here to a forthcoming preprint of Ogus for a description of the singularities of the strata. Ogus proved in [O, Prop. 2.6] that for $p \neq 2$ the stratum $M^{(2)}$ has a quadratic singularity at the superspecial points. A variation of his argument there shows that at a point with Artin invariant $\sigma_0 = h - 1$ the singular locus has multiplicity 2. In particular the stratum $M^{(11)}$ has multiplicity 2 at points with $\sigma_0 = 10$, cf. his forthcoming preprint and the discussion in the next section.

15. The loci of K3 surfaces of given height

We now come to the description of the cycle classes of the strata defined by the height. Let $M^{(h)}$ be the closed stratum of the moduli space $M = M_{2d}$

where the height of the formal group Φ_X is at least h, with the convention that $M^{(11)} = M^{(\infty)}$. For simplicity we shall assume that p does not divide $2d$. By our characterization of h these strata can be given a natural scheme structure and these are reduced for $h \neq \infty$ by our results in Sect. 14. It is known by Artin that the strata $M^{(h)}$ for $h = 1, \dots, 11$ have codimension $\leq h - 1$ in M_{2d}, see [A].

Define a line bundle V on M by $V = \pi_*(\Omega^2_{\mathcal{X}/M})$ and let the first Chern class be v.

(15.1) Theorem. *Let $M = M_{2d}$ be the moduli stack of polarized K3 surfaces over k with a polarization of degree $2d$ prime to p. Then for $h = 1, \dots, 10, 11$ the scheme-theoretic locus $M^{(h)}$ of surfaces with height $\geq h$, if not empty, is of codimension $h - 1$ and for $h \neq 11$ it is a local complete intersection. The class of $M^{(h)}$ in the Chow group $CH^{h-1}_{\mathbb{Q}}(M)$ is given by*

$$(p - 1)(p^2 - 1) \dots (p^{h-1} - 1)v^{h-1}.$$

Proof. We prove this by induction. Let M be the moduli space of polarized K3 surfaces of degree $2d$ as above. We know that the generic K3 surface has height 1, and so for $h = 1$ the formula is correct. The codimension of $M^{(h)}$ is $\leq h - 1$ for $1 \leq h \leq 10$ as follows from (5.7). For $h = 2$ the locus $M^{(2)}$ is the non-ordinary locus. This locus is characterized by the fact that the Frobenius map $H^2(X, O_X) \to H^2(X, O_X)$ vanishes. This is a p-linear map and the corresponding O_M-linear map is $(R^2\pi_*O_{\mathcal{X}})^{(p)} \to R^2\pi_*O_{\mathcal{X}}$ with associate cycle class $(p - 1)v$. Locally, at a point of $M^{(2)}$ an equation is given by $g_1 = 0$, see the proof of (14.1) and $dg_1 \neq 0$. So if $M^{(2)}$ is not empty then it is purely 18-dimensional.

Suppose now that the class of $M^{(h-1)}$ is given by the class in the formula. By Proposition (5.7) the locus in $M^{(h-1)}$ where the height increases is given by the vanishing of the map $\phi_{h-1} : (R^2\pi_*O_{\mathcal{X}})^{(p^{h-1})} \to R^2\pi_*(O_{\mathcal{X}})$, equivalently, by the vanishing of a section of $V^{p^{h-1}-1}$. By (14.1) it follows that for a local equation $g_h = 0$ we have $dg_h \neq 0$. Hence the locus is reduced for $h \neq \infty$ and the class on $M^{(h-1)}$ is given by $(p^{h-1} - 1)v$.

Let $j_h : M^{(h)} \to M^{(h-1)}$ and $j : M^{(h-1)} \to M$ be the natural inclusions. Then the class of the locus $M^{(h)}$ in $CH^{h-1}_{\mathbb{Q}}(M)$ is given by

$$j_* j_{h*}[M^{(h)}] = j_*([M^{(h-1)}] \cdot j^*_{h-1}(p^{h-1} - 1)v) = (p^{h-1} - 1)v \cdot (j_*[M^{(h-1)}])$$

by the projection formula. □

The locus $M^{(11)}$ comes with a multiplicity in the formula because of (14.2). For $p \neq 2$ the multiplicity is 2. It makes sense to call the reduced locus $M^{(11)}_{\mathrm{red}}$ the supersingular locus.

(15.2) Remark. In [G] a formula for the class of the supersingular locus on the moduli space of principally polarized abelian surfaces was given. Comparison with Kummer surfaces shows that this is compatible with multiplicity 2 along the supersingular locus, cf. [G-K].

We shall now assume that the line bundle $V = \pi_*(\Omega^2_{\mathcal{X}/M})$ is ample on the moduli space. It is known by the theory of Baily and Borel (see [B-B]) that V is ample on the moduli spaces in characteristic 0; indeed, modular forms of sufficiently high weight define an embedding.

(15.3) Theorem. *Suppose that the class v is ample. Let $\mathcal{X} \to S$ with S complete be a proper smooth family of polarized K3 surfaces with constant $h \neq \infty$. Then this family is isotrivial.*

Proof. It follows from the preceding theorem that the strata $S^{(h)} - S^{(h+1)}$ where the height is constant are quasi-affine for $h = 1, \dots, 10$.

We do not know whether the class v is ample on the moduli spaces M_{2d}, but we expect it to be so.

Suppose that there exists a good Baily-Borel compactification. By this we mean that there exists a projective variety (stack) $\overline{M}_{2d}$ containing M_{2d} such that $\overline{M}_{2d} - M_{2d}$ is 1-dimensional and consists of a configuration of elliptic modular curves. This is the case in characteristic zero, cf. Kondo [Ko]. Then it follows from our theorem that a family of K3 surfaces with $h \geq 3$ does not degenerate. Indeed, it follows from our formula that a class of the form v^m with $m \geq 3$ has zero intersection with the 'boundary components'. This implies that for each boundary component the locus with $h \geq 3$ either has empty intersection with this boundary component or contains it. The boundary components form a connected set and the generic point of each component corresponds to a degenerate K3 surface corresponding to an ordinary elliptic curve. For the degenerate surfaces the height is 1 or 2. Compare the discussion in [R-Z-Sh].

16. An extension for other varieties

Though the theorem in Sect. 5 was formulated for K3 surfaces it holds for a more general class of surfaces.

(16.1) Theorem. *Suppose that X is a smooth algebraic surface such that*

i) $\mathrm{Pic}^0(X)$ *is reduced,*
ii) $\dim H^2(X, O_X) = 1$.

Then Φ^2 is represented by a formal group of dimension 1 and its height satisfies $h(\Phi_X) \geq i + 1$ if and only if the Frobenius map F on $H^2(X, W_i(O_X))$ is the zero map.

(16.2) Corollary. *For such a surface we have the following characterization of the height:*

$$h(\Phi_X) = \min\{i \geq 1 \colon [F : H^2(W_i(O_X)) \to H^2(W_i(O_X))] \neq 0\}.$$

Proof. The proof is analogous to the proof given for K3 surfaces. Instead of the vanishing of $H^1(X, O_X)$ one uses the vanishing of the Bockstein operators. Recall that $H^1(X, W_n(O_X))$ is the subgroup of $k[\epsilon]/(\epsilon^{n+1})$-valued points of the connected component of the Picard scheme P at the origin, cf. [Mu]. A $k[\epsilon]/\epsilon^2$-valued point (tangent vector) is tangent to P_{red} at the origin if and only if it can be lifted to $k[\epsilon]/(\epsilon^n)$-valued point for all n. That is, these correspond precisely to the elements of $H^1(X, O_X)$ that can be lifted to $H^1(X, W_n(O_X))$ for all n. So if $P = P_{\mathrm{red}}$ then all elements of $H^1(X, O_X)$ can be lifted and this implies the analogues of Lemmas (4.2) and (4.5) that we need.

(16.3) Example. 1) An abelian surface satisfies the assumptions. 2) A surface of general type with $H^1(O_X) = 0$ and $p_g = 1$. Examples of such surfaces are surfaces with $K^2 = p_g = 1$. These have $h^1(X, O_X) = 0$ and are resolutions of surfaces of type (6,6) in weighted projective space $\mathbb{P}(1, 2, 2, 3, 3)$, cf. [C].

A nonsingular complete algebraic variety X of dimension n is called a Calabi-Yau variety if the canonical invertible sheaf ω_X is trivial and $H^i(X, O_X) = 0$ for $1 \leq i \leq n-1$. By a criterion of Artin-Mazur [A-M], the Artin-Mazur formal group Φ^n is pro-representable by a one-dimensional formal Lie group for such a variety. In the same way as in Sect. 5, we have also a characterization of the height of the formal group Φ^n.

(16.4) Proposition. *For a Calabi-Yau variety X of dimension n we have the following characterization of the height:*

$$h\left(\Phi_X^n\right) = \min\{i \geq 1 \colon [F : H^n(W_i(O_X)) \to H^n(W_i(O_X))] \neq 0\}.$$

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