Tables for the function $\mathrm{N}_q(\mathrm{g})$

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In the two main tables 'Table $p=2$' and 'Table $p=3$' we present a list of values of the function $N_q(g)$ or an interval in which $N_q(g)$ lies. Here $N_q(g)$ is the maximum number of rational points on an irreducible smooth curve of genus g over the finite field $\mathbb{F}_q$ with q elements. If the precise value of $N_q(g)$ is not known we either give an interval $[a,b]$ or nothing. The meaning of the interval $[a,b]$ is: we know that there exists a curve with *at least* a rational points over $\mathbb{F}_q$ and the best upper bound by Serre, Ihara, Oesterlé or other means says $N_q(g) \leq b$; we enter nothing if $a < [b/\sqrt{2}]$. (Note that this is a slight change from the earlier definition of a where the existence of a curve with exactly a points was required.) For further information concerning this upper bound we refer to the paper [G-V4] of the bibliography. (The tex file of this paper can be found at the address: `http://xxx.lanl.gov/abs/alg-geom/9511005`, but the tables there are not up-dated.) For the lower bound we use the literature as far as it is known to us.

In the tables *Table $q=x$: sources* we list the sources for the (lower) value in the two main tables. We tried to give the source where the value occurs first, but sometimes added a second reference where further details or an explicit construction is given or we provided a construction ourselves.

We tried to give credit to whom it is due, but may have failed due to ignorance. If the reader knows an improvement of an entry in this table we shall appreciate if he/she let us know so that we can update or correct the table.

We hope that the entries are to a large extent self-explanatory. The term 'FP' stands for fibre product of Artin-Schreier curves $y^p - y = f$; e.g. for genus $g=11$ in *Table $q=8$: sources* the entry $\mathrm{FP}\langle x^3+1/x, x+1/(x+1), x+1/x+1\rangle$; [G-V5, Aug. '97] means that the fibre product of the Artin-Schreier covers corresponding to $f=x^3+1/x$, $f=x+1/(x+1)$, and $f=x+1/x+1$ over $\mathbb{P}^1$ gives a curve of genus 11 with 48 rational points over $\mathbb{F}_8$ and that it first appeared in the update of this table of August '97. The entry FP $\langle ax^5+a^2x^3, bx^7 : \dim(a)=2, \mathrm{Tr}(b)=0\rangle$ for $g=18$ in the same table indicates that one has to take the fibre product of three curves $y^2+y=a_1x^5+a_1^2x^3$, $y^2+y=a_2x^5+a_2^2x^3$ with a_1 and a_2 linearly independent over $\mathbb{F}_2$ and $y^2+y=bx^7$,

These tables are found at: `http://www.wins.uva.nl/ ~geer` where the trace of b satisfies $\mathrm{Tr}_{\mathbb{F}_8/\mathbb{F}_2}(b) = 0$. The fibre product of Kummer extensions is denoted by $\otimes$.

In the Appendices we list building blocks and their properties and provide clarifying remarks on some constructions that are used in the tables *Table $q = x$: sources*. Finally, we give two bibliographies: the first contains the sources for items in the tables, while the second gives further papers related to the topic.

A closer look at the tables will convince the reader that there is still ample room for improvement, e.g. with the methods we used in [G-V4] whose applications are by no means exhausted. The tables should be seen as an attempt to record the state of the art. In the January 1998 update of the tables among others many new examples by V. Shabat are listed. In the March 1998 update we recorded new results by Auer, Niederreiter, Shabat and Xing.

* *Last update* Table $p = 2$: March 30, 1998

* *Last update* Table $p = 3$: March 30, 1998.

* *Last update of the source-tables*: August 27, 1997.

* *New update of the all tables is planned for April 1998.*

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Table p=2.

$g\backslash q$	2	4	8	16	32	64	128
1	5	9	14	25	44	81	150
2	6	10	18	33	53	97	172
3	7	14	24	38	63–66	113	190–195
4	8	15	25–29	45–47	70–77	129	200–217
5	9	17–18	29–32	49–55	74–88	130–145	227–239
6	10	20	33–36	65	81–99	161	225–261
7	10	21–22	33–39	63–70	89–110	177	241–283
8	11	21–24	34–43	61–76	97–121	169–193	257–305
9	12	26	45–47	72–81	108–132	209	241–327
10	13	27–28	38–50	81–87		225	289–349
11	14	26–30	48–54	80–92	113–154	201–241	
12	14–15	29–31	49–57	68–97	129–165	257	321–393
13	15	33	56–61	97–103	129–176	225–271	
14	15–16	32–35	65	97–108	145–187	241–287	353–437
15	17	33–37	56–68	98–113	158–196	257–303	369–459
16	17–18	36–38	56–71	93–118	147–204		
17	17–18	40	62–74	100–124	154–212		
18	18–19	41–42	65–77	113–129	161–220	281–351	
19	20	37–43	60–80	121–134	172–228		
20	19–21	37–45	68–83	121–140	177–236		
21	21	41–47	72–86	129–145	185–244		
22	21–22	40–48	66–89	129–150		321–415	
23	22–23	41–50	68–92	126–155			
24	21–23	49–52	81–95	129–161	225–267	337–447	513–657
25	24	51–53	84–97	144–166			
26	24–25	55	81–100	150–171		385–479	
27	22–25	49–56	96–103	145–176	209–290	401–495	
28	25–26	51–58	97–106	145–181	257–298	513	577–745
29	25–27	52–60	97–109	161–187	225–306		
30	25–27	53–61	96–112	161–192	273–313	385–536	609–789
31	27–28	60–63		150–197			
32	26–29	57–65					
33	28–29	65–66	92-121	193–207			
34	27–30	57–68		156–213			
35	29–31	64–69	112–127		253–352		
36	30–31	64–71	107–130	185–223			
37	29–32	66–72	121–132	208–228			
38	28–33	64–74	129–135	193–233	291–375	449–627	
39	33	65–75	120–138	194–239			
40	32–34	75–77	103–141	197–244	293–390		
41	33–35	65–78	118–144	216–249	308–398		
42	33–35	68–80	129–147	209–254	307–405		
43	33–36	72–81	116–150	226–259	306–413		
44	33–37	68–83	130–153	226–264	325–420		
45	33–37	80–84	144–156	242–268	304–428		
46	34–38	81–86	129–158	243–273			
47	36–38	73–87	120–161				
48	34–39	77–89	126–164				
49	36–40	81–90	130–167				
50	40	91–92	130–170	225–291		561–762	

Table p=3.

$g\backslash q$	3	9	27	81
1	7	16	38	100
2	8	20	48	118
3	10	28	58	136
4	12	30	64–68	154
5	12–13	32–36	55–78	152–172
6	14–15	35–40	76–88	190
7	16–17	39–43		160–208
8	15–18	38–47	92–108	226
9	19	40–51		244
10	19–21	54–55	91–121	226–262
11	20–22	55–59		
12	22–24	55–63	109–148	298
13	24–25	60–66	136–156	
14	24–26	56–70		
15	28	64–74	136–171	
16	27–29	74–78	136–178	370
17	24–30			
18	26–31	67–85		
19	28–32	84–88		
20	30–34	68–91		
21	32–35	82–95	163–214	
22	30–36	78–98		
23	30–37	92–101		
24	28–38	91–104	190–235	
25	36–40		196–242	
26	36–41	110–111		
27	39–42			
28	37–43	105–117		
29	42–44	104–120		
30	34–46			
31	40–47			
32	38–48			
33	37–49		220–298	
34	44–50	111–136		
35	47–51			
36	46–52	110–142	244–319	730
37	48–54	120–145		
38		105–149		
39	46–56		271–340	
40	54–57		244–346	
41	50–58			
42			280–360	
43	55–60	120–164		
44				
45	48–62			
46	55–63	162–173		
47	47–65	154–177	299–395	
48	55–66	163–180	325–402	
49	63–67	168–183		
50	56–68	182–186	299–416	

Table q=2: sources.

genus	N	source
1	5	[S1,4]
2	6	[S1,4]
3	7	[S1,4]
4	8	[S1,4]
5	9	[S1,4]
6	10	[S1,4]
7	10	[S1,4]
8	11	[S1,4]
9	12	[S1,4]
10	13	Serre; [N-X2]
11	14	Serre; [Sch], [N-X1]
12	14–15	[S2,4]
13	15	Serre; [Sch]
14	15–16	[S2,4]
15	17	[S1,4]
16	16–18	[S2,4]
17	17–18	[S2,4]
18	18–19	[S2,4]
19	20	[S1,4]
20	19–21	[S2,4]
21	21	[S1,4]
22	21–22	[Sch]
23	22–23	[X-N1]
24	20–23	Schoof; [G-V4]
25	24	[X-N1]
26	24–25	Schoof; [G-V4]
27	22–25	Schoof; [G-V4]
28	24–26	Schoof; [G-V4]
29	25–27	[X-N1]
30	24–27	Schoof; [G-V4]
31	27–28	[X-N1]
32	26–29	Schoof;[G-V4]
33	28–29	Schoof; [G-V4]
34	27–30	[X-N1]
35	28–31	Schoof; [G-V4]
36	30–31	[X-N1]
37	28–32	Schoof; [G-V4]
38	28–33	Schoof; [G-V4]
39	33	[S1,4]
40	32–34	Schoof; [G-V4]
41	32–35	[N-X5]
42	30–35	Schoof; [G-V4]
43	33–36	[X-N1]
44	32–37	[X-N1]
45	32–37	[X-N1]
46	34–38	Schoof; [G-V4]
47	36–38	Schoof; [G-V4]
48	34–39	Schoof; [G-V4]
49	36–40	[X-N1]
50	40	[S1,4]

Table q=4: sources. $\mathbb{F}_4 = \mathbb{F}_2(\alpha)$ with $\alpha^2 + \alpha + 1 = 0$.

genus	N	source
1	9	[S2,4]
2	10	[S2,4]
3	14	[S2,4]
4	15	[S3,4]
5	17–18	[St2]
6	20	Schoof; [G-V4]
7	21–22	[N-X3]
8	21–24	[N-X3]
9	26	[N-X3]
10	27–28	modular curve $\Gamma(9)$, [I]; [G-V4]
11	26–30	FP$\langle x+1/x, x+1+1/(x+1), x^3\rangle$; [G-V5, Aug. '97]
12	28–31	Schoof; [G-V4]
13	33	[St2]
14	32–35	FP$\langle x/f, (x^2+1)/f, (\alpha x^2+\alpha^2)/f : f = x^3+x+1\rangle$; [G-V5, Dec.'96]
15	33–37	[N-X3]
16	36–38	Schoof; [G-V4]
17	40	[N-X3]
18	34–42	[N-X4]
19	36–43	Schoof; [G-V4]
20	36–45	[N-X4]
21	41–47	[N-X4]
22	40–48	[N-X4]
23	40–50	[N-X4]
24	42–52	[N-X4]
25	51–53	[N-X4]
26	55	modular curve $\Gamma(11)$, [I]; [G-V4]
27	49–56	FP $\langle x^3, \alpha(x^3+x), x^5+x^2, \alpha(x^5+x^2)\rangle$; [G-V4]
28	44–58	[N-X4]
29	49–60	[N-X4]
30	52–61	[N-X4]
31	60–63	[N-X4]
32	52–65	[N-X4]
33	65–66	[L]
34	57–68	FP $\langle x+1/x, x^3, x^5+x^2, \alpha(x^5+x^2)\rangle$;[G-V4]
35	54–69	[N-X4]
36	64–71	[N-X4]
37	66–72	[N-X4]
38	56–74	[N-X4]
39	56–75	[N-X4]
40	75–77	[N-X4]
41	65–78	FP $\langle x^3, x^5+x^2, x^7+x, \alpha(x^7+x)\rangle$; [G-V4]
42	66–80	[N-X4]
43	72–81	[N-X4]
44	68–83	Schoof; [G-V4]
45	80–84	Schoof; [G-V4]
46	66–86	[N-X4]
47	68–87	[N-X4]
48	77–89	[N-X4]
49	81–90	[N-X4]
50	91–92	modular curve $\Gamma(13)$, [I]; [G-V4]

Table q=8: sources. $\mathbb{F}_8 = \mathbb{F}_2(\alpha)$ with $\alpha^3 + \alpha^2 + 1 = 0$.

g	N	source
1	14	[S2,4]
2	18	[S2,4]
3	24	[S2,4]
4	25–29	FP $\langle x + 1/x + 1, x^3 + 1/x \rangle$; [G-V5, May '96]
5	29–32	FP $\langle x^3 + \alpha x^2, x^5 + x^3 \rangle$; [G-V4]
6	33–36	[St2]
7	33–39	[G-V1]
8	33–43	FP $\langle x^5 + x^3, \alpha^3 x^7 \rangle$ [G-V5, Aug. '97]
9	36–47	Schoof; [G-V4]
10	38–50	FP $\langle x^3 + \alpha x^2, \alpha x^3 + \alpha^4 x^2, x^3 + 1/x \rangle$; [G-V5, Aug. '97]
11	48–54	FP$\langle x^3 + 1/x, x + 1/(x+1), x + 1/x + 1 \rangle$; [G-V5, Aug. '97]
12	49–57	FP $\langle x + 1/x + 1, x^3 + 1/x, ax^3 + a^2/x : a \notin \mathbb{F}_2 \rangle$; [G-V5, Aug. '97]
13	50–61	FP $\langle x + 1/x + 1, \alpha x^3 + \alpha^2/x, \alpha^2 x^3 + \alpha^4/x \rangle$; [G-V5, May '96]
14	65	[H-S]
15	54–68	FP$\langle (x+1)^3 + 1/(x+1), x^3 + 1/x, ax^3 + a^2/x : a \notin \mathbb{F}_2 \rangle$; [G-V5, Aug. '97]
16	56–71	Schoof; [G-V4]
17	61–74	FP $\langle x^3 + 1/x, ax^5 + a^2x^3 : \dim(a) = 2 \rangle$; [G-V5, May '96]
18	65–77	FP $\langle ax^5 + a^2x^3, bx^7 : \dim(a) = 2, \mathrm{Tr}(b) = 0 \rangle$; [G-V5, Aug. '97]
19		
20	65–83	FP $\langle x^5 + x^3, ax^7 : \mathrm{Tr}(a) = 0, \dim(a) = 2 \rangle$; [G-V5, Aug. '97]
21	72–86	see App. $q = 8$;[G-V5, Aug. '97]
22	66–89	Schoof; [G-V4]
23	68–92	Schoof; [G-V4]
24		
25		
26		
27	81–103	FP $\langle x^3 + \alpha x^2, \alpha x^3 + \alpha^4 x^2, ax^5 + a^2x^3 : \dim(a) = 2 \rangle$; [G-V4]
28	97–106	FP $\langle ax^3 + a^2/x, x + 1/x + 1 : \dim(a) = 3 \rangle$; [G-V5; May '96]
29	97–109	FP $\langle ax^5 + a^2x^3, x^3 + \alpha x^2 : \dim(a) = 3 \rangle$; [G-V4]
30		
31		
32		
33		
34		
35	106–127	FP$\langle (x+1)^3 + 1/(x+1), ax^3 + a^2/x : \dim(a) = 3 \rangle$; [G-V5, Aug. '97]
36	105–130	FP $\langle ax^5 + a^2x^3, x + 1/x + 1 : \dim(a) = 3 \rangle$; [G-V4]
37	121–132	FP $\langle ax^5 + a^2x^3, x^3 + 1/x : \dim(a) = 3 \rangle$; [G-V5; May '96]
38	129–135	FP$\langle ax^5 + a^2x^3, bx^7 : \dim(a) = 3, \mathrm{Tr}(b) = 0 \rangle$; [G-V5, Aug. '97]
39	117–138	FP $\langle ax^5 + a^2x^3, bx^3 + b^2/x : \dim(a) = \dim(b) = 2 \rangle$; [G-V5, May '96]
40		
41	112–144	FP$\langle x^3 + 1/x, (x+1)^3 + 1/(x+1), x^5 + x^3, \alpha x^5 + \alpha^2 x^3 \rangle$;[G-V5, Aug.'97]
42	129–147	FP $\langle ax^5 + a^2x^3, bx^7 : \mathrm{Tr}(b) = 0, \dim(a) = \dim(b) = 2 \rangle$; [G-V5, Aug. '97]
43		
44	129–153	FP, see App. $q = 8$; [G-V5, Aug. '97]
45	144–156	Schoof; [G-V4]
46	129–158	FP $\langle ax^5 + a^2x^3, x^9 + x^2 : \dim(a) = 3 \rangle$; [G-V4]
47	116–161	Schoof; [G-V4]
48	117–164	FP $\langle ax^5 + a^2x^3, ax + a^4/x^3 : \dim(a) = 2 \rangle$; [G-V5, May '96]
49		
50	129–170	FP, see App. $q = 8$; [G-V5, Aug. '97]

Table q=16: sources. $\mathbb{F}_{16} = \mathbb{F}_2(\alpha)$ with $\alpha^4 + \alpha^3 + 1 = 0$.

genus	N	source
1	25	[S1,4]
2	33	[S2,4]
3	38	[S3,4]
4	45–48	[M-Z-Z]
5	49–56	FP $\langle x^3+x^2, x^5\rangle$; [G-V4]
6	65	[Seg];[G1]
7	58–70	[G-V1]
8	61–76	FP $\langle x+1/(x^2+x), x^5\rangle$; [G-V4]
9	65–81	FP $\langle x^5, f_1\rangle$, see App. $q=16$; [G-V5, Aug. '97]
10	65–87	FP $\langle x^5, x^9+x^3\rangle$;[G-V4]
11	75–92	FP $\langle(\alpha^3+\alpha)x+1/x, (\alpha^8+\alpha^6)x+\alpha^{10}/x, x^5\rangle$;[G-V4]
12	68–97	See App. $q=16$; [G-V5, Aug. '97]
13	97–103	FP $\langle x^3+x^2, x^5, \alpha^5x^5\rangle$; [G-V4]
14	97–108	FP $\langle x^5+x^3+x^2, x^5, \alpha^5x^5\rangle$; [G-V4]
15	98–113	[G-V1]
16	93–118	FP $\langle(\alpha^3+\alpha)x+1/x, x^5, \alpha^5x^5\rangle$; [G-V4]
17		
18	93–129	FP $\langle x+1/(x^2+x), \alpha^{12}x^3+\alpha^{13}x^2, x^5\rangle$; [G-V4]
19		
20	121–140	FP $\langle x+1/(x^2+x), x^5, \alpha^5x^5\rangle$; [G-V4]
21	129–145	FP $\langle x^5, \alpha^5x^5, f_1\rangle$, see App. $q=16$;[G-V5, Aug. '97]
22	129–150	[St2]
23	123–155	FP $\langle x^5, ax^3+a^4/x^3 : \dim(a)=2\rangle$;[G-V5, May '96]
24	121–161	FP $\langle x^5, x^7+1/x, (x+1)^7+1/(x+1)\rangle$; [G-V5, Aug. '97]
25		
26	129–171	FP $\langle x^5, ax^9+a^2x^3 : \dim(a)=2\rangle$;[G-V4]
27		
28	136–181	See App. $q=16$; [G-V5, Aug. '97]
29		
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31		
32		
33		
34		
35		
36		
37		
38		
39		
40		
41		
42	179–254	FP $\langle x^3+x^2, x^3+1/x^3, ax^3+a^4/x^3 : \dim(a)=2\rangle$; [G-V5, Aug. '97]
43		
44		
45	242–268	FP $\langle ax^3+a^4/x^3 : \dim(a)=4\rangle$;[G-V5, May '96]
46		
47		
48		
49		
50		

Table q=32: sources.

genus	N	source
1	44	[S1,4]
2	53	[S2,4]
3	63–66	[M-Z-Z]
4	70–77	[M-Z-Z]
5	73–88	[M-Z-Z]
6	81–99	[G-V1]
7	89–110	[G-V1]
8	97–121	[Sh]
9		
10		
11		
12	129–165	[G-V1]
13		
14	145–187	[G-V1]
15		
16		
17		
18		
19		
20		
21		
22		
23		
24	225–267	[Sh]
25		
26		
27		
28	257–298	[G-V1]
29		
30	273–313	[G-V1]
31		
32		
33		
34		
35	253–352	FP $\langle ax^9 + a^4x^5, x^7 + 1/x^3 : \dim(a) = 2\rangle$; [G-V5, May '96]
36		
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49		
50		

Table q=64: sources. $\mathbb{F}_{64} = \mathbb{F}_2(\alpha)$ with $\alpha^6 + \alpha^5 + 1 = 0$.

genus	N	source
1	81	[S1,4]
2	97	[S2,4]
3	113	[Wi]
4	129	[Wo]
5	130–145	[M-Z-Z]
6	161	[G-V3]
7	177	[Wo]
8		
9	173–209	[Wi]
10	225	Elkies; $\mathbb{Z}/3\mathbb{Z}$-quotient of $x^9 + y^9 + z^9 = 0$;[G-V5, Aug. 97]
11	201–241	FP $\langle x^7, x^9 \rangle$; [G-V5, May '96]
12	257	[Wi]
13	199–272	FP $\langle x^7, x^7 + 1/x^7 \rangle$; [G-V5 May '96]
14		
15	257–304	$d_4(BCH(2)^\perp) = 48$; [G-V4]
16		
17		
18		
19		
20		
21		
22		
23		
24		
25		
26		
27	401–496	FP $\langle x^7, x^9, \alpha^9 x^9 \rangle$;[G-V5, May'96]
28	513	[H]
29		
30		
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33		
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36		
37		
38		
39		
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Table q=128: sources.

genus	N	source
1	150	[S1,4]
2	172	[S2,4]
3	190–195	[G-V1]
4	200–217	[Wi]
5	227–239	[M-Z-Z]
6	225–261	[Wi]
7	241–283	[Wi]
8	257–305	[Wi]
9	241–327	[G-V3]
10	289–349	[G-V3]
11		
12	321–393	[G-V1]
13		
14	353–437	[G-V3]
15	369–459	[Wi]
16		
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20		
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22		
23		
24	513–657	[G-V1]
25		
26		
27		
28	577–745	[G-V1]
29		
30	609–789	[G-V3]
31		
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Appendix q=4. $\mathbb{F}_4 = \mathbb{F}_2(\alpha)$ with $\alpha^2+\alpha+1=0$.

The building blocks used are curves $y^2+y=f$ with f, their genus and trace of Frobenius τ as follows:

curve	genus	τ
x^3	1	-4
$x+1/x$	1	-3
$\alpha(x^3+x)$	1	-2
$a(x^5+x^2),\ \ a\in\mathbb{F}_4^*$	2	-4
$x/(x^3+x+1)$	3	-5
$(x^2+1)/(x^3+x+1)$	3	-5
$a(x^7+x),\ a\in\mathbb{F}_4^*$	3	-4
$(\alpha x^2+\alpha^2)/(x^3+x+1)$	3	-3

Appendix q=8. $\mathbb{F}_8 = \mathbb{F}_2(\alpha)$ with $\alpha^3+\alpha^2+1=0$.

The building blocks used are curves $y^2+y=f$ with f, their genus and trace of Frobenius τ as follows:

curve	genus	τ
$x+1/x+1$	1	-5
$x^3+\alpha x^2$	1	-4
$\alpha x^3+\alpha^4x^2$	1	-4
$ax^5+a^2x^3,\ \ a\in\mathbb{F}_8^*$	2	-8
$ax^3+a^2/x,\ \ a\in\mathbb{F}_8^*$	2	-7
$ax+a^4/x^3,\ a\in\mathbb{F}_8^*$	2	-7
$\alpha^3(x+1/(x^2+x))$	2	-6
$ax^7,\ \ a\in\mathbb{F}_8^*, \mathrm{Tr}(a)=0$	3	-8
$ax^9+ax^2,\ \ a\in\mathbb{F}_8^*$	4	-8

For $g=21$ we apply the following construction which involves Artin-Schreier curves of the form

$$y^2+y=(\sum_{i=0}^{3} a_i x^i)/f$$

with $a_i\in\mathbb{F}_8$ and $f\in\mathbb{F}_8[x]$ irreducible of degree 4. The condition that

$$\mathrm{Tr}_{\mathbb{F}_8/\mathbb{F}_2}(\sum_{i=0}^{3} a_i x^i/f)=0 \quad \text{for all } x\in\mathbb{F}_8$$

gives a system of 8 homogeneous linear equations over $\mathbb{F}_2$ in 12 unknowns over $\mathbb{F}_2$ whose space of solutions has dimension at least 4 over $\mathbb{F}_2$. These solutions produce curves of genus 3 over $\mathbb{F}_8$ with 18 $\mathbb{F}_8$-rational points. The fibre product generated by a 3-dimensional subspace of solutions yields a curve of genus 21 with 72 rational points over $\mathbb{F}_8$.

For $g=44$ we construct a fibre product using a 4-dimensional space of curves

$$y^2+y=a_1x^3+a_2x+\frac{a_3x+a_4}{f},$$

with $a_i \in \mathbb{F}_8$ and $f \in \mathbb{F}_8[x]$ irreducible of degree 2 such that the trace of the right hand side vanishes on $\mathbb{F}_8$. For $a_1 \neq 0$ these curves are of genus 3 and possess 17 rational points over $\mathbb{F}_8$, while for $a_1=0$ we find a genus 2 curve with 17 rational points.

For $g=50$ we use a 4-dimensional fibre product with basis consisting of two polynomials of the form $ax^5+a^2x^3$, a rational function of the form $a_1x^3+a_2x+(a_3x+a_4)/f$ as above with $a_1 \neq 0$ and a rational function of that form with $a_1=0$.

Appendix q=16. $\mathbb{F}_{16}=\mathbb{F}_2(\alpha)$ with $\alpha^4+\alpha^3+1=0$.

The building blocks used are of the form $y^2+y=f$ with f, the genus and trace of Frobenius as in the table:

curve	genus	τ
x^3+x^2	1	-8
$\alpha^{12}x^3+\alpha^{13}x^2$	1	-8
$(\alpha^3+\alpha)x+1/x$	1	-7
$(\alpha^8+\alpha^6)x+\alpha^{10}/x$	1	-7
$ax^5, \quad a \in \mathbb{F}_4^*$	2	-16
$x+1/(x^2+x)$	2	-14
$x^5+x^3+x^2$	2	-8
$ax^3+a^4/x^3, \quad a \in \mathbb{F}_{16}^*$	3	-15
$ax^9+a^2x^3, \quad a \in \mathbb{F}_{16}^*$	4	-16
$ax^7+a^2/x, \quad a \in \mathbb{F}_{16}^*$	4	-15

For $g=12$ (resp. $g=28$) we construct in the same way as indicated in the appendix for $q=8$ a 2-dimensional (resp. 3-dimensional) subspace of curves

$$y^2+y=(\sum_{i=0}^{4} a_ix^i)/f$$

with $\sum_{i=0}^4 a_ix^i \in \mathbb{F}_{16}[x]$ and with $f \in \mathbb{F}_{16}[x]$ irreducible of degree 5 such that the trace $\mathrm{Tr}_{\mathbb{F}_{16}/\mathbb{F}_2}((\sum a_ix^i)/f)$ vanishes on $\mathbb{F}_{16}$. These curves have genus 4 and possess 34 rational points over $\mathbb{F}_{16}$.

For $g=9$ and $g=21$ we use the Artin-Schreier curves

$$y^2+y=f_1:=a_1x^3+a_2x+a_3+\frac{a_4x+a_5}{f}$$

with $a_i \in \mathbb{F}_{16}$, $a_1 \neq 0$ and $f \in \mathbb{F}_{16}[x]$ irreducible of degree 2 such that $\mathrm{Tr}(f_1)$ vanishes on $\mathbb{F}_{16}$. This curve has genus 3 and 33 points.

Appendix q=32.

The building blocks used are of the form $y^2 + y = f$ with f, the genus and trace of Frobenius as in the table:

curve	genus	τ
$ax^9 + a^4x^5, \ a \in \mathbb{F}_{32}^*$	4	-32
$ax^7 + a^4/x^3, \ a \in \mathbb{F}_{32}^*$	5	-31

Appendix q=64.

The building blocks used are of the form $y^2 + y = f$ with f, the genus and trace of Frobenius as in the table:

curve	genus	τ
$ax^9, \ a \in \mathbb{F}_8^*$	4	-64
$ax^7 + a^8/x^7, \ a \in \mathbb{F}_{64}^*$	7	-63

Table q=3: sources.

genus	N	source
1	7	[S1,4]
2	8	[S2,4]
3	10	[S2,4]
4	12	[S3,4]
5	12–14	[N-X3]
6	14–15	[N-X3]
7	16–17	[N-X3]
8	15–18	[N-X3]
9	19	FP $\langle x-1/x, x^2-1\rangle$; [G-V4]
10	19–21	FP $\langle x^2-1, x^4-x^2\rangle$; [G-V4]
11	20–22	[N-X3]
12	22–24	[N-X3]
13	24–25	[N-X3]
14	24–26	[N-X3]
15	28	FP $\langle x^4-x^2, x^5-x^3\rangle$; [G-V4]
16	27–29	FP $\langle (x^2-1)/(x^2+1)^2, (x^3-x)/(x^2+1)^2)\rangle$;[G-V4]
17	24–30	[N-X6]
18	26–31	[N-X6]
19	27–32	[N-X6]
20	30–34	FP $\langle (x^3-x)/(x^2+1), (x^3-x)/(x^2+1)^2\rangle$; [G-V4]
21	32–35	[N-X6]
22	28–36	FP $\langle x^5-x^3, x^7-x^5\rangle$; [G-V4]
23	26–37	[N-X6]
24	28–38	FP $\langle x^4-x^2, x^8-x^6\rangle$; [G-V4]
25	36–40	[N-X6]
26	36–41	[N-X6]
27	39–42	[N-X6]
28	37–43	[N-X6]
29	42–44	[N-X6]
30	34–46	[N-X6]
31	40–47	[N-X6]
32	38–48	[N-X6]
33	37–49	[N-X6]
34	44–50	[N-X6]
35	38–51	[N-X6]
36	36–52	[N-X6]
37	48–54	[N-X6]
38		
39	42–56	[N-X6]
40	54–57	[N-X6]
41	50–58	[N-X6]
42		
43	55–60	[N-X5]
44		
45	48–62	[N-X6]
46	55–63	FP $\langle x^2-1, x^4-x^2, x^5-x^3\rangle$; [G-V4]
47	47–65	[N-X6]
48	55–66	FP $\langle x^2-1, x-1/x, x^4-x^2\rangle$; [G-V4]
49	54–67	[N-X6]
50	56–68	[N-X6]

Table q=9: sources. $\mathbb{F}_9 = \mathbb{F}_3(\alpha)$ with $\alpha^2+\alpha+2=0$.

genus	N	source
1	16	[S1,4]
2	20	[S2,4]
3	28	[S2,4]
4	30	$(y^2=x^3+x^2+2)\otimes(y^2=2x^3+x^2+2)$; [G-V5, Dec. '96]
5	32–36	modular curve $\Gamma(8)$, [I]; [G-V4]
6	29–40	[Wi]
7	36–43	$y^2=x(x^4+x^2+2)\otimes y^2=x(x^4+x^3+x^2+x+1)$; [G-V5, Aug. '97]
8	38–47	[G-V2]
9	40–51	FP $\langle x^2, x+1/x\rangle$; [G-V4]
10	54–55	FP $\langle x+1/x, x+1+1/(x+1)\rangle$; [G-V5, Aug. '97]
11	55–59	[G-V2]
12	55–63	[G-V2]
13	60–66	modular curve $\Gamma(10)$, [I];[G-V4]
14	56–70	FP $\langle x+1/x, x^2-1/x^2\rangle$; [G-V5, May '96]
15	58–74	FP $\langle x+1/x, \alpha^2x^4\rangle$; [G-V4]
16	74–78	FP $\langle ax^2-a^3/x^2 : \dim(a)=2\rangle$;[G-V5, May '96]
17		
18		
19	76–88	FP $\langle x^2-1/x^2, \alpha^2x^4\rangle$; [G-V5, May '96]
20		
21	82–95	FP $\langle \alpha^2x^4, x^7-x^5\rangle$; [G-V4]
22		
23		
24	82–104	FP $\langle ax^7-a^3x^5 : \dim(a)=2\rangle$; [G-V4]
25		
26	110–111	modular curve $\Gamma(11)$, [I]; [G-V4]
27		
28		
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36		
37	109–145	FP $\langle x^2+\alpha x, 2x^4+2x^2+x, \alpha^2x^4\rangle$; [G-V4]
38		
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46	138–173	FP $\langle x+1/x, x+1+1/(x+1), x+2+1/(x+2)\rangle$; [G-V5, Aug. '97]
47		
48		
49		
50	182–186	modular curve $\Gamma(13)$, [I]; [G-V4]

Table q=27: sources. $\mathbb{F}_{27} = \mathbb{F}_3(\alpha)$ with $\alpha^3 + 2\alpha^2 + 1 = 0$.

genus	N	source
1	38	[S1,4]
2	48	[S2,4]
3	58	$(y^2 = x^3 + 2x^2 + 2x) \otimes (y^2 = 2x^3 + 2\alpha^4 x^2 + \alpha^8 x)$;[G-V5, Dec. '96]
4	64–68	[G-V2]
5	55–78	[Wi]
6	76–88	[G-V2]
7		
8	92–108	FP $\langle x + 1/x, \alpha^{15} x + \alpha^2/x \rangle$; [GV5, Aug. '97]
9		
10		
11		
12	109–148	[G-V2]
13	136–156	[G-V2]
14		
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23		
24	166–235	FP $\langle x + 1/x, \alpha^{15} x + \alpha^2/x, \alpha^{23} x \rangle$; [GV5, Aug. '97]
25		
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36	244–319	[G-V2]
37		
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39	244–340	FP $\langle x^{10} - x^4, x^{11} - x^7 \rangle$; [G-V4]
40	244–346	FP $\langle a x^{11} - a^3 x^7 : \dim(a) = 2 \rangle$; [G-V5, May '96]
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Table q=81: sources.

genus	N	source
1	100	[S1,4]
2	118	[S2,4]
3	136	[Wi]
4	154	[H]
5	152–172	[Wi]
6	190	[Seg];[G1]
7	160–208	[Wi]
8	226	Elkies; $\mathbb{Z}/5\mathbb{Z}$-quotient of $x^{10}+y^{10}+z^{10}=0$; [G-V5, Aug. '97]
9	244	[Wo]
10	226–262	[Wi]
11		
12	298	[G-V2]
13		
14		
15		
16	370	[H]
17		
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30		
31		
32		
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35		
36	730	[St1]
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Appendix q=3.

The building blocks used are of the form $y^3 - y = f$ with f, the genus and trace of Frobenius as in the table:

curve	genus	τ
$x^2 - 1$	1	-3
$x - 1/x$	2	-4
$x^4 - x^2$	3	-6
$(x^3 - x)/(x^2 + 1)^2$	4	-8
$x^5 - x^3$	4	-6
$(x^3 - x)/(x^2 + 1)$	4	-6
$(x^2 - 1)/(x^2 + 1)^2$	4	-5
$x^7 - x^5$	6	-6
$x^8 - x^6$	7	-6

Appendix q=9. $\mathbb{F}_9 = \mathbb{F}_3(\alpha)$ with $\alpha^2 + \alpha + 2 = 0$.

The building blocks used are of the form $y^3 - y = f$ with f, the genus and trace of Frobenius as in the table:

curve	genus	τ
x^2	1	-6
$x^2 + \alpha x$	1	-6
$x + 1/x$	2	-10
$ax^4, \quad a = \alpha^2, \alpha^6$	3	-18
$2x^4 + x^2 + x$	3	-9
$ax^2 - a^3/x^2, \quad a \in \mathbb{F}_9^*$	4	-16
$ax^7 - a^3x^5, \quad a \in \mathbb{F}_9^*$	6	-18

We also use

$$\begin{cases} y^2 = x^3 + x^2 + 2, & g = 1, \tau = -5 \\ y^2 = 2x^3 + x^2 + 2, & g = 1, \tau = -5 \\ y^2 = x(x^4 + x^2 + 2), & g = 2, \tau = -8 \\ y^2 = x(x^4 + x^3 + x^2 + x + 1), & g = 2, \tau = -8. \end{cases}$$

Appendix q=27. $\mathbb{F}_{27} = \mathbb{F}_3(\alpha)$ with $\alpha^3 + 2\alpha^2 + 1 = 0$.

The building blocks used are of the form $y^3 - y = f$ with f, the genus and trace of Frobenius as in the table:

curve	genus	τ
$x + 1/x$	2	-16
$\alpha^{15}x + \alpha^2/x$	2	-16
$ax^{10} - a^3x^4, a \in \mathbb{F}_{27}^*$	9	-54
$ax^{11} - a^3x^7, \quad a \in \mathbb{F}_{27}^*$	10	-54

We also use

$$\begin{cases} y^2 = x^3 + 2x^2 + 2x, & g = 1, \tau = -10 \\ y^2 = 2x^3 + 2\alpha^4 x^2 + \alpha^8 x, & g = 1, \tau = -10. \end{cases}$$

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